Permutations with Restricted Patterns and Dyck Paths

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Received August 15, 2000; accepted December 12, 2000

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We exhibit a bijection between 132-avoiding permutations and Dyck paths. Using this bijection, it is shown that all the recently discovered results on generating functions for 132-avoiding permutations with a given number of occurrences of the pattern 12...k follow directly from old results on the enumeration of Motzkin paths, among which is a continued fraction result due to Flajolet. As a bonus, we use these observations to derive further results and a precise asymptotic estimate for the number of 132-avoiding permutations of \{1, 2, ..., n\} with exactly r occurrences of the pattern 12...k. Second, we exhibit a bijection between 123-avoiding permutations and Dyck paths. When combined with a result of Roblet and Viennot, this bijection allows us to express the generating function for 123-avoiding permutations with a given number of occurrences of the pattern \((k - 1)(k - 2)\ldots 1k\) in the form of a continued fraction and to derive further results for these permutations.

1 INTRODUCTION

In the recent papers [2, 4, 6, 9], 132-avoiding permutations with a prescribed number of occurrences of the pattern 12...k (the most general results being contained in [6]) and 123-avoiding permutations which also avoid the pattern \((k - 1)(k - 2)\ldots 1k\) were considered. (See the end of

1 Research partially supported by the Austrian Science Foundation, FWF, Grant P13190-MAT.
this section for the precise definition of permutations which avoid a certain pattern and of Dyck paths.) It was found that generating functions for these permutations can be expressed in terms of continued fractions and Chebyshev polynomials.

The purpose of this paper is to make a case for the paradigm:

Whenever you encounter generating functions which can be expressed in terms of continued fractions or Chebyshev polynomials, then expect that Dyck or Motzkin paths are at the heart of your problem, and will help to solve it.

Indeed, as I am going to demonstrate in Section 2, there is an obvious bijection between 132-avoiding permutations and Dyck paths. (In particular, as is well known, 132-avoiding permutations are counted by Catalan numbers.) Surprisingly, this bijection seems to be new.2 Known results for generating functions for Motzkin paths (one of which due to Flajolet [3], the other being folklore; Dyck paths being special Motzkin paths) then allow one immediately to express the generating functions in which we are interested in terms of continued fractions and Chebyshev polynomials (thus making the speculation in [2, Sect. 5] precise and explicit). In particular, we recover all the relevant results from [2, 6, 9]. Furthermore, by exploiting the relation between 132-avoiding permutations and Dyck paths further, we are able to find an explicit expression for the generating function for 132-avoiding permutations with exactly $r$ occurrences of the pattern $12\ldots k$, thus extending a result from [6]. This, in turn, allows us to provide a precise asymptotic estimate for the number of these permutations of $\{1, 2, \ldots, n\}$ as $n$ becomes large. All these results can be found in Section 3, as well as generating functions for 132-avoiding permutations with no occurrence or one occurrence of the pattern $23\ldots k1$.

In Section 4 we exhibit a bijection between 123-avoiding permutations and Dyck paths, which also, and again surprisingly, seems to be new.3 (In particular, as is also well known, 123-avoiding permutations are counted by Catalan numbers.) In Section 5 we combine this bijection with a result of Roblet and Viennot [10] on the enumeration of Dyck paths to obtain a continued fraction for the generating function of 123-avoiding permutations with a given number of occurrences of the pattern $(k - 1)(k - 2)\ldots 1k$. Further results on these permutations (which extend another result from [2]) can be found in Section 5 as well, including precise asymptotic estimates. (By combining this bijection between

2Independently, and at the same time, Emeric Deutsch (private communication) discovered this bijection in an equivalent form.

3This bijection was also independently and at the same time discovered by Emeric Deutsch (private communication). It does not translate into the bijection between 321-avoiding permutations and Dyck paths in [1, after the proof of Theorem 2.1].
123-avoiding permutations and Dyck paths with our bijection between the latter and 132-avoiding permutations, we obtain a bijection between 123-avoiding and 132-avoiding permutations. This bijection appears to be new. In particular, it is different from the one by Simion and Schmidt [11, Sect. 6] and the one by West [15], as can be immediately seen by considering the examples given in [11, Examples after Prop. 19] and [15, Example 2.7], respectively.

For the convenience of the reader, we recall the results on Motzkin and Dyck paths, on which we rely so heavily, in an appendix at the end of the paper.

Let us recall the basic definitions.

Let \( \pi = \pi_1 \pi_2 \ldots \pi_n \) be a permutation of \( \{1, 2, \ldots, n\} \) and \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_k \) be a permutation of \( \{1, 2, \ldots, k\} \), \( k \leq n \). We say that the permutation \( \pi \) contains the pattern \( \sigma \), if there are indices \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that \( \pi_{i_1} \pi_{i_2} \ldots \pi_{i_k} \) is in the same relative order as \( \sigma_1 \sigma_2 \ldots \sigma_k \). Otherwise, \( \pi \) is said to avoid the pattern \( \sigma \), or, alternatively, we say that \( \pi \) is \( \sigma \)-avoiding.

A Dyck path is a lattice path in the plane integer lattice \( \mathbb{Z}^2 \) (\( \mathbb{Z} \) denoting the set of integers) consisting of up-steps \((1, 1)\) and down-steps \((1, -1)\), which never passes below the \( x \)-axis. See Fig. 1 for an example. (To be very explicit: The definition of Dyck path that we use in this paper is wider than usually used in that we do not require that a Dyck path starts and ends on the \( x \)-axis. The latter type of path is, though, the type of path that we will encounter most of the time, Fig. 1 being an example.)

2. A BIJECTION BETWEEN 132-AVOIDING PERMUTATIONS AND DYCK PATHS

In this section we define a map \( \Phi \) which maps 132-avoiding permutations to Dyck paths which start at the origin and return to the \( x \)-axis as follows. Let \( \pi = \pi_1 \pi_2 \ldots \pi_n \) be a 132-avoiding permutation. We read the permutation \( \pi \) from left to right and successively generate a Dyck path. When \( \pi_j \) is read, then in the path we adjoin as many up-steps as necessary, followed by a down-step from height \( h_j + 1 \) to height \( h_j \) (measured from the \( x \)-axis), where \( h_j \) is the number of elements in \( \pi_{j+1} \ldots \pi_n \) which are larger than \( \pi_j \).

For example, let \( \pi = 74352681 \). The first element to be read is 7. There is one element in 4352681 which is larger than 7; therefore the path starts with two up-steps followed by a down-step, thus reaching height 1 (see Fig. 1). Next 4 is read. There are three elements in 352681 which are larger than 4; therefore the path continues with three up-steps followed by a down-step, thus reaching height 3, etc. The complete Dyck path \( \Phi(74352681) \) is shown in Fig. 1.
The reader should note that, for the map $\Phi$ to be well defined, it is essential that the permutation $\pi$ to which the map is applied is 132-avoiding. For this guarantees that $h_j - 1 \leq h_{j+1}$ always, so it is always possible to connect the down-step from height $h_j + 1$ to $h_j$ (formed by the definition of $\Phi$ when considering $\pi_j$) by a number of up-steps (this number being possibly zero) to the down-step from height $h_{j+1} + 1$ to $h_{j+1}$. Conversely, given a Dyck path starting at the origin and returning to the $x$-axis, the obvious inverse of $\Phi$ produces a 132-avoiding permutation.

In summary, the map $\Phi$ is a bijection between 132-avoiding permutations of $\{1, 2, \ldots, n\}$ and Dyck paths from $(0, 0)$ to $(2n, 0)$. We remark that, in view of the standard bijection between rooted ordered trees and Dyck paths through a depth-first traversal of the trees (cf., e.g., [13, Proposition 6.2.1 (i) and (v), Corollary 6.2.3 (i) and (v)]), this map is equivalent to the bijection between 132-avoiding permutations and rooted ordered trees given by Jani and Rieper [4].

For the sake of completeness, and to show the close relation between the map $\Phi$ and the map $\Psi$ that is to be defined in Section 4, we provide an alternative way to define the map $\Phi$. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a 132-avoiding permutation. In $\pi$, we determine all the left-to-right minima. A left-to-right minimum is an element $\pi_i$ which is smaller than all the elements to its left, i.e., smaller than all $\pi_j$ with $j < i$. For example, the left-to-right minima in the permutation 74352681 are 7, 4, 3, 2, 1.

Let the left-to-right minima in $\pi$ be $m_1, m_2, \ldots, m_r$, so that

$$\pi = m_1w_1m_2w_2\ldots m_rw_r,$$

where $w_i$ is the subword of $\pi$ in between $m_i$ and $m_{i+1}$. Read the decomposition (3.1) from left to right. Any left-to-right minimum $m_i$ is translated into $m_{i-1} - m_i$ up-steps (with the convention $m_0 = n + 1$). Any subword $w_i$ is translated into $|w_i| + 1$ down-steps (where $|w_i|$ denotes the number of elements of $w_i$).
In the lemma below we list two properties of the bijection \( \Phi \), which will be subsequently used in Section 3.

**Lemma \( \Phi \).** Let \( \pi = \pi_1 \pi_2 \ldots \pi_n \) be a 132-avoiding permutation, and let \( P = \Phi(\pi) \) be the corresponding Dyck path. Then,

1. a down-step in \( P \) from height \( i \) to height \( i - 1 \) corresponds in a one-to-one fashion to an element \( \pi_j \) in the permutation which is the first element in an increasing subsequence in \( \pi \) of length \( i \) (i.e., an occurrence of the pattern 12\ldots i) that is maximal with respect to the property that \( \pi_j \) is its first element.

2. a portion of the path \( P \) which starts at height \( h + i - 1 \) and eventually falls down to height \( h \), for some \( h \), followed by an up-step corresponds to an occurrence of the pattern 23\ldots i1 in \( \pi \).

**Proof.** (1) By the definition of \( \Phi \), a down-step from height \( i \) to height \( i - 1 \) in \( P = \Phi(\pi) \) means that we read an element \( \pi_j \) which has the property that there are \( i - 1 \) elements in \( \pi_{j+1} \ldots \pi_n \) which are larger than \( \pi_j \). Since \( \pi \) is 132-avoiding, these \( i - 1 \) elements have to appear in increasing order and thus, together with \( \pi_j \), form an increasing subsequence of length \( i \) that cannot be made longer under the assumption that \( \pi_j \) is the first element in the subsequence.

(2) In a path portion which starts at height \( h + i - 1 \) and eventually falls to height \( h \) we must find a down-step from height \( h + i - 1 \) to \( h + i - 2 \), a down-step from height \( h + i - 2 \) to \( h + i - 3 \), \ldots, and a down-step from height \( h + 1 \) to \( h \). If, in this path portion, we choose the first down-step from height \( h + i - 1 \) to \( h + i - 2 \), the first down-step from height \( h + i - 2 \) to \( h + i - 3 \), \ldots, the first down-step from height \( h + 1 \) to \( h \), then, under the correspondence \( \Phi \), these down-steps correspond to an increasing subsequence \( \pi_j \pi_{j+1} \ldots \pi_{j+i-1} \) of length \( i - 1 \) in \( \pi \). If now the path continues by (at least) one up-step, then the following down-step corresponds to an element \( \pi_{j+i} \), \( j_i > j_{i-1} \), with the property that there are more elements in \( \pi_{j+1} \ldots \pi_n \) that are larger than \( \pi_{j_i} \) than there are elements in \( \pi_{j_i} \ldots \pi_n \) that are larger than \( \pi_{j_{i-1}} \). Evidently, this is only possible if \( \pi_{j_{i-1}} > \pi_{j_i} \). Since \( \pi \) is 132-avoiding, this implies that we have even \( \pi_{j_i} > \pi_{j_i} \). Hence, \( \pi_{j_i} \pi_{j_i+1} \ldots \pi_{j_{i+1}} \pi_{j_{i+1}} \) is an occurrence of the pattern 23\ldots i1 in \( \pi \).  

3. THE ENUMERATION OF 132-AVOIDING PERMUTATIONS WITH A PRESCRIBED NUMBER OF OCCURRENCES OF THE PATTERNS 12\ldots k AND 23\ldots k1

In this section we provide explicit expressions for generating functions for 132-avoiding permutations with a prescribed number of occurrences of
the pattern $12\ldots k$ and for 132-avoiding permutations with a prescribed number of occurrences of the pattern $23\ldots k1$.

First we consider the former permutations. Given a 132-avoiding permutation $\pi$, we denote the number of occurrences of the pattern $12\ldots k$ in $\pi$ by $N(12\ldots k; \pi)$. Given a Dyck path $P$, we assign a weight to it, denoted by $w_1(k; P)$. It is defined as the sum $\sum d(i/d - 1)$, where the sum is over all down-steps $d$ of $P$ and where $i/d$ is the height of the starting point of $d$. For example, the weight $w_1(3; \pi)$ of the Dyck path in Fig. 1 is

$$\begin{align*}
(1/2) + (3/2) + (3/2) + (2/2) + (1/2) + (0/2) + (0/2) = 8.
\end{align*}$$

From Lemma $\Phi(1)$ it is immediate that

$$N(12\ldots k; \pi) = w_1(k; \Phi(\pi)). \quad (3.2)$$

This observation, combined with Flajolet’s continued fraction theorem for the generating function of Motzkin paths (see Theorem A1), allows us to express the generating function which counts 132-avoiding permutations with respect to the number of occurrences of the pattern $12\ldots k$ in the form of a continued fraction. This result was first obtained by Mansour and Vainshtein [6, Theorem 2.1]. In the statement of the theorem, and in the following, we write $|\pi|$ for the number of elements which are permuted by $\pi$. For example, we have $|74352681| = 8$.

**Theorem 1.** The generating function $\sum_\pi y^{N(12\ldots k; \pi)} x^{|\pi|}$, where the sum is over all 132-avoiding permutations, is given by

$$\frac{1}{1 - \frac{xy(k^0_1)}{1 - \frac{xy(k^1_1)}{1 - \frac{xy(k^2_1)}{1 - \ldots}}}}. \quad (3.3)$$

**Proof.** Apply Theorem A1 with $b_i = 0$ and $\lambda_i = xy(k^i_{i-1})$, $i = 0, 1, \ldots$, and use (3.2). \[ \blacksquare \]

We remark that the above proof is essentially equivalent to the one in [4, proof of Corollary 7]. It is obvious that the refinement in [6, expression for $W_k(\ldots)$ after Proposition 2.3] could also easily be derived by using the correspondence $\Phi$ and Flajolet’s continued fraction.

Next we turn our attention to 132-avoiding permutations with a fixed number of occurrences of the pattern $12\ldots k$. Theorem 2 was first obtained by Chow and West [2, Theorem 3.6, second case] in an equivalent form.
Theorem 2. The generating function $\sum_{\pi} x^{\vert \pi \vert}$, where the sum is over all 132-avoiding permutations which also avoid the pattern $12 \ldots k$, is given by

$$\frac{U_{k-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_k(\frac{1}{2\sqrt{x}})},$$

where $U_n(x)$ denotes the $n$th Chebyshev polynomial of the second kind, $U_n(\cos t) = \sin((n + 1)t)/\sin t$.

Proof. By Lemma $\Phi(1)$, the permutations in the statement of the theorem are in bijection with Dyck paths, which start at the origin, return to the $x$-axis, and do not exceed the height $k - 1$. Now apply Theorem A2 with $b_1 = 0, \lambda_i = 1, i = 0, 1, \ldots, K = k - 1, r = s = 0, x$ replaced by $\sqrt{x}$, and use Fact A3.

The next theorem extends a result by Mansour and Vainshtein [6, Theorems 3.1 and 4.1], who proved the special case when $r$ is at most $k(k + 3)/2$.

Theorem 3. Let $r$ and $b$ be positive integers such that $\binom{k+b-1}{k} \leq r < \binom{k+b}{k}$. The generating function $\sum_{\pi} x^{\vert \pi \vert}$, where the sum is over all 132-avoiding permutations with exactly $r$ occurrences of the pattern $12 \ldots k$, is given by

$$\sum \left( \frac{\ell_1 + \ell_2 - 1}{\ell_2} \right) \left( \frac{\ell_2 + \ell_3 - 1}{\ell_3} \right) \cdots \left( \frac{\ell_{b-1} + \ell_b - 1}{\ell_b} \right) \left( \frac{U_{k-1}(\frac{1}{2\sqrt{x}}))^{\ell_1-1}}{(U_k(\frac{1}{2\sqrt{x}}))^{\ell_1+1}} x^{\frac{1}{2}(\ell_1 + (\ell_2 + \ell_3 + \cdots + \ell_b))}, \right)

$$

where the sum is over all nonnegative integers $\ell_1, \ell_2, \ldots, \ell_b$ with

$$\ell_1 \binom{k-1}{k-1} + \ell_2 \binom{k}{k-1} + \ell_3 \binom{k+1}{k-1} + \cdots + \ell_b \binom{k+b-2}{k-1} = r,$$

and where $U_n(x)$ denotes the $n$th Chebyshev polynomial of the second kind.

Remark. The binomials in (3.5) have the effect that if $\ell_t = 0$, then $\ell_j = 0$ for all $j \geq t$ in order to obtain a nonvanishing summand in the sum (3.5). Furthermore, the sum (3.5) reduces to just one term if $r \leq k$, thus recovering [6, Theorem 3.1], and it reduces to a single sum if $k < r \leq k(k + 3)/2$, thus recovering [6, Theorem 4.1].

Proof of Theorem 3. Let $\pi$ be a permutation of the statement of the theorem. We apply $\Phi$ to obtain the corresponding Dyck path $P = \Phi(\pi)$. The Dyck path $P$ has a unique decomposition of the form

$$P_0V_1d_1P_1V_2d_2P_2 \ldots V_sd_sP_s,$$

(3.7)
where $P_0$ is the portion of $P$ from the origin until the first time the height $k - 1$ is reached, where the $d_i$’s are the down steps whose end points have at least the height $k - 1$, where the $P_i$’s, $i = 1, 2, \ldots, s - 1$, are path portions which start and end at height $k - 1$ and never exceed height $k - 1$, where the $V_i$’s are path portions consisting of several subsequent up-steps which fill the gaps in between, and where $P_s$ is the portion of $P$ from the last point at height $k - 1$ until the end of the path. The path portion $P_i$ can only be nonempty if $d_i$ is a down-step from height $k$ to $k - 1$. Clearly, $d_i$ must be a down-step from height $k$ to height $k - 1$.

Now suppose that among the $d_i$’s there are $\ell_1$ down-steps from height $k$ to height $k - 1$, $\ell_2$ down-steps from height $k + 1$ to height $k$, etc. Because of (3.2) the relation (3.6) must hold.

Let us for the moment fix $\ell_1, \ell_2, \ldots$ and ask how many orderings of $\ell_1$ down-steps from height $k$ to height $k - 1$, $\ell_2$ down-steps from height $k + 1$ to height $k$, etc., there are which can come from a decomposition (3.7) when we ignore the $P_i$’s and $V_i$’s. In fact, there are many restrictions to be obeyed: After a down-step from height $h + 1$ to height $h$ there can only follow a down-step of at least that height or at worst from height $h$ to height $h - 1$. Let $t$ be maximal so that $\ell_t$ is nonzero. Then the above observation tells us that after a down-step from height $t + k - 1$ to $t + k - 2$ there can only follow another down-step of this sort (of course, with an up-step in between) or a down-step from height $t + k - 2$ to $t + k - 3$.

Hence, if we just concentrate on these two types of down-steps, of which there are $\ell_1$ down-steps, from height $t + k - 1$ to $t + k - 2$, $\ell_2$ down-steps from height $t + k - 2$ to $t + k - 3$, etc., there are which can come from a decomposition (3.7) when we ignore the $P_i$’s and $V_i$’s. In fact, there are many restrictions to be obeyed: After a down-step from height $h + 1$ to height $h$ there can only follow a down-step of at least that height or at worst from height $h$ to height $h - 1$. Let $t$ be maximal so that $\ell_t$ is nonzero. Then the above observation tells us that after a down-step from height $t + k - 1$ to $t + k - 2$ there can only follow another down-step of this sort (of course, with an up-step in between) or a down-step from height $t + k - 2$ to $t + k - 3$.

Hence, if we just concentrate on these two types of down-steps, of which there are $\ell_1$ down-steps, from height $t + k - 1$ to $t + k - 2$, $\ell_2$ down-steps from height $t + k - 2$ to $t + k - 3$, etc., there are which can come from a decomposition (3.7) when we ignore the $P_i$’s and $V_i$’s. In fact, there are many restrictions to be obeyed: After a down-step from height $h + 1$ to height $h$ there can only follow a down-step of at least that height or at worst from height $h$ to height $h - 1$. Let $t$ be maximal so that $\ell_t$ is nonzero. Then the above observation tells us that after a down-step from height $t + k - 1$ to $t + k - 2$ there can only follow another down-step of this sort (of course, with an up-step in between) or a down-step from height $t + k - 2$ to $t + k - 3$.

To explain the remaining expression, we observe that after any of the $\ell_1$ down-steps, from height $k$ to height $k - 1$, $d_i$, say, except for the last, there follows a (possibly empty) path $P_i$, which is a Dyck path which starts and ends at height $k - 1$ and never exceeds height $k - 1$. By Theorem A2 with $b_i = 0, \lambda_i = 1, i = 0, 1, \ldots, K = k - 1, r = s = k - 1, x$ replaced by $\sqrt{x}$, and use of Fact A3, we conclude that the generating function for these paths is equal to

\[
\frac{x^{(k-1)/2}U_{k-1}\left(\frac{1}{2}\sqrt{x}\right)}{x^{k/2}U_k\left(\frac{1}{2}\sqrt{x}\right)}.
\]
By again applying Theorem A2, this time with \( b_i = 0, \lambda_i = 1, \) \( i = 0, 1, \ldots, K = k - 1, r = 0, s = k - 1, x \) replaced by \( \sqrt{x} \), and using Fact A3, we obtain that the generating function for paths \( P_0 \) from the origin to height \( k - 1 \), never exceeding height \( k - 1 \), is equal to

\[
\frac{x^{(k-1)/2}}{x^{k/2}U_k\left(\frac{1}{\sqrt{x}}\right)}.
\]

(3.8)

Similarly, the generating function for paths \( P_s \) from height \( k - 1 \) to height \( 0 \), never exceeding height \( k - 1 \), is also given by (3.8). If everything is combined, the expression (3.5) results.

Theorem 3 can be readily used to find an asymptotic formula for the number of 132-avoiding permutations with exactly \( r \) occurrences of the pattern \( 12k \). The corresponding result, given in the theorem below, extends [2, Corollary 4.2]. Before we state the theorem, we recall an elementary lemma (cf., e.g., [7, Sect. 9.1]).

**Lemma 4.** Let \( f(x) \) and \( g(x) \) be polynomials. It is assumed that all the zeroes of \( g(x) \) have modulus larger than \( a > 0 \). Consider the expansion

\[
\frac{f(x)}{(x - a)^R g(x)} = \sum_{n=0}^{\infty} c_n x^n.
\]

Then, as \( n \) becomes large, we have

\[
c_n \sim (-1)^R \frac{n^{R-1}}{(R - 1)!} a^{-n-R} \frac{f(a)}{g(a)} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

(3.9)

**Theorem 5.** Let \( r \) and \( k \) be fixed nonnegative integers. Then, as \( n \) becomes large, the number of 132-avoiding permutations with exactly \( r \) occurrences of the pattern \( 12 \ldots k \) is asymptotically

\[
\left(\frac{4 \sin^2 \frac{\pi}{k+1}}{k+1}\right)^{r+1} \frac{n^r}{r!} \left(\frac{4 \cos^2 \frac{\pi}{k+1}}{k+1}\right)^{n-r} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

(3.10)

**Proof.** The zeroes of the Chebyshev polynomials are well known. In particular, the zero with the smallest modulus of the polynomial \( x^{k/2}U_k(1/2\sqrt{x}) \) is \( x = 1/4 \cos^2(\pi/(k + 1)) \).

If \( r = 0 \), then we apply Theorem 2 and Lemma 4 with \( a = 1/4 \cos^2(\pi/(k + 1)), R = 1, f(x) = x^{(k-1)/2}U_k-1(1/2\sqrt{x}), \) and

\[
g(x) = \frac{x^{k/2}U_k\left(\frac{1}{\sqrt{x}}\right)}{x - \frac{1}{4 \cos(\pi/k)}}.
\]

(3.11)
In order to compute the right-hand side of (3.9), we need to substitute 
\( x = \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)} \) in our choices of \( f(x) \) and \( g(x) \). By the definition of the Chebyshev polynomials, we obtain for \( f(x) = x^{(k-1)/2} U_{k-1}(1/2\sqrt{x}) \):

\[
f \left( \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)} \right) = \left( \frac{1}{2 \cos \frac{\pi}{k+1}} \right)^{k-1} \sin \left( k \frac{\pi}{k+1} \right) \frac{\sin \frac{\pi}{k+1}}{\sin \left( k \frac{\pi}{k+1} \right)} = \left( \frac{1}{2 \cos \frac{\pi}{k+1}} \right)^{k-1}.
\]

For our choice (3.11) of \( g(x) \) we obtain

\[
g \left( \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)} \right) = \lim_{x \to \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)}} \frac{x^{k/2} U_k \left( \frac{1}{2 \sqrt{x}} \right)}{x - \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)}}
\]

\[
= \left( \frac{1}{2 \cos \frac{\pi}{k+1}} \right)^k \lim_{\theta \to \frac{\pi}{k+1}} \frac{\sin((k + 1) \theta)}{\sin \theta} \left( \frac{1}{4 \cos^2 \theta} - \frac{1}{4 \cos^2 \left( \frac{\pi}{k+1} \right)} \right).
\]

This limit is easily evaluated by means of de l'Hôpital’s rule.

If \( r \geq 1 \), we start from the generating function given in Theorem 3. It should be observed that, in view of Lemma 4, the summand in (3.5) which asymptotically yields the largest contribution is the one with \( \ell_1 = r \) and all other \( \ell_i \)'s equal to zero. Then an application of Lemma 4 and computations that are very similar to the ones above give finally (3.10).

The next group of results concerns the enumeration of 132-avoiding permutations with a given number of occurrences of the pattern 23\ldots k1. We use again the map \( \Phi \) to translate these permutations into Dyck paths. The property of \( \Phi \) which is important now is given by Lemma \( \Phi(2) \). It says that we can recognize the occurrence of a pattern 23\ldots k1 in a 132-avoiding permutation in the corresponding Dyck path by a portion of the path which starts at height \( h + k - 1 \), eventually falls down to height \( k \), and is then followed by an up-step.

As the first application of our approach we show how to rederive another result due to Chow and West [2, Theorem 3.6, third case], which is reformulated here in an equivalent form.

**Theorem 6.** The generating function \( \sum_{\pi} x^{\vert \pi \vert} \), where the sum is over all 132-avoiding permutations which also avoid the pattern 23\ldots k1, is given by

\[
\frac{U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right)}{\sqrt{x} U_k \left( \frac{1}{2 \sqrt{x}} \right)},
\]

where \( U_n(x) \) denotes the \( n \)th Chebyshev polynomial of the second kind.
Proof. Let \( \pi \) be a permutation of the statement of the theorem. By the observation above the statement of the theorem (which was based on Lemma \( \Phi(2) \)), for the corresponding Dyck path \( P = \Phi(\pi) \) there are two possibilities: Either \( P \) never exceeds the height \( k - 2 \) (and, thus, \( \pi \) does not contain any increasing subsequence of length \( k - 1 \)) or \( P \) can be decomposed as

\[
P_0u_0P_1u_1 \ldots P_su_sP_{s+1}D,
\]

where \( P_0 \) is a path from the origin to height \( k - 2 \) never exceeding height \( k - 2 \), where for \( i = 1, 2, \ldots, s \) the portion \( P_i \) is a path starting and ending at height \( i + k - 2 \), never running below height \( i \), and never exceeding height \( i + k - 2 \), where for \( i = 0, 1, \ldots, s \) the step \( u_i \) is an up-step from height \( i + k - 2 \) to height \( i + k - 1 \), where \( P_{s+1} \) is a path from height \( s + k - 1 \) to \( s + 1 \), never running below height \( s + 1 \), and never exceeding height \( s + k - 1 \), and where \( D \) consists of \( s + 1 \) down-steps, from height \( s + 1 \) to height \( 0 \).

By Theorem A2 with \( b_i = 0, \lambda_i = 1, i = 0, 1, \ldots, K = k - 2, r = s = 0, x \) replaced by \( \sqrt{x} \), and Fact A3, the generating function for the Dyck paths which never exceed height \( k - 2 \) is equal to

\[
\frac{x^{(k-2)/2}U_{k-2}(\frac{1}{\sqrt{x}})}{x^{(k-1)/2}U_{k-1}(\frac{1}{\sqrt{x}})}.
\]

By Theorem A2 with \( b_i = 0, \lambda_i = 1, i = 0, 1, \ldots, K = k - 2, r = 0, s = k - 2, x \) replaced by \( \sqrt{x} \), and Fact A3, the generating function for the possible paths \( P_0 \) in the decomposition (3.13) is equal to

\[
\frac{x^{(k-2)/2}}{x^{(k-1)/2}U_{k-1}(\frac{1}{\sqrt{x}})},
\]

as well as the generating function for the possible paths \( P_{s+1} \). Finally, for any fixed \( j \) between 1 and \( s \), by Theorem A2 with \( b_i = 0, \lambda_i = 1, i = 0, 1, \ldots, K = k - 2, r = s = k - 2, x \) replaced by \( \sqrt{x} \), and Fact A3, the generating function for the possible paths \( P_j \) in the decomposition (3.13) is also given by (3.14).

If everything is combined, then we obtain that the generating function for the permutations of the statement of the theorem is given by

\[
\frac{U_{k-2}(\frac{1}{\sqrt{x}})}{\sqrt{x}U_{k-1}(\frac{1}{\sqrt{x}})} + \sum_{s=0}^{s=0} \frac{1}{\sqrt{x}U_{k-1}(\frac{1}{\sqrt{x}})} \left( \frac{U_{k-2}(\frac{1}{\sqrt{x}})}{\sqrt{x}U_{k-1}(\frac{1}{\sqrt{x}})} \right)^s \frac{1}{\sqrt{x}U_{k-1}(\frac{1}{\sqrt{x}})} x^{s+1}.
\]

The sum is a geometric series and can therefore be evaluated. It is then routine to convert the resulting expression into the expression (3.12) by using standard identities for the Chebyshev polynomials.
It seems difficult to find an explicit expression for the generating function for 132-avoiding permutations with exactly \( r \) occurrences of the pattern \( 23 \) for general \( r \). Yet, as long as \( 1 \leq r \leq k - 1 \) such an explicit expression can be easily derived.

**Theorem 7.** The generating function \( \sum_{\pi} x^{\vert \pi \vert} \), where the sum is over all 132-avoiding permutations with exactly one occurrence of the pattern \( 23 \), is given by

\[
x \frac{1}{U_{k-2} \left( \frac{1}{2\sqrt{x}} \right) U_k \left( \frac{1}{2\sqrt{x}} \right)},
\]

where \( U_n(x) \) denotes the \( n \)th Chebyshev polynomial of the second kind.

More generally, let \( 1 \leq r \leq k - 1 \). Then the generating function \( \sum_{\pi} x^{\vert \pi \vert} \), where the sum is over all 132-avoiding permutations with exactly \( r \) occurrences of the pattern \( 23 \), is given by

\[
\frac{1}{U_{k-3} \left( \frac{1}{2\sqrt{x}} \right) U_k \left( \frac{1}{2\sqrt{x}} \right)} \sum_{\ell} \frac{1}{\ell + 1} \left( \frac{2\ell}{\ell} \right) x^{\ell+\frac{r}{\ell}-\frac{1}{2}} \left( \frac{U_{k-3} \left( \frac{1}{2\sqrt{x}} \right)}{U_{k-2} \left( \frac{1}{2\sqrt{x}} \right)} \right)^{\ell/\ell}
\]

\( (3.16) \)

**Proof.** Let \( \pi \) be a 132-avoiding permutation with exactly \( r \) occurrences of the pattern \( 23 \). Analogously to the argument in the proof of Theorem 6, the Dyck path \( \Phi(\pi) \) corresponding to \( \pi \) can be decomposed as

\[
P_0 u_0 P_1 u_1 \ldots P_{s-1} u_{s-1} P_s u_s d_s P_{s+1} u_{s+1} d_{s+1} P_{s+2} \ldots u_{s+\ell-1} d_{s+\ell-1} P_{s+\ell} d_{s+\ell} P_{s+\ell+1} D,
\]

where for \( i = 0, 1, \ldots, s \) the portions \( P_i \) and steps \( u_i \) are as in (3.13), where the step \( d_i \) is a down-step from height \( s + k - 1 \) to height \( s + k - 2 \), where for \( i = 1, 2, \ldots, \ell - 1 \) the path \( P_{s+i} \) is a path starting and ending at height \( s + k - 2 \), never running below height \( s + 1 \), and never exceeding height \( s + k - 2 \), the step \( u_{s+i} \) is an up-step from height \( s + k - 2 \) to height \( s + k - 1 \), and the step \( d_{s+i} \) is a down-step from height \( s + k - 1 \) to height \( s + k - 2 \), where \( P_{s+\ell} \) is a path from height \( s + k - 2 \) to \( s + 1 \), never running below height \( s + 1 \) and never exceeding height \( s + k - 2 \), where \( d_{s+\ell} \) is a down-step from height \( s + k - 1 \) to height \( s \), where \( P_{s+\ell+1} \) is a path of length \( 2\ell \), starting and ending at height \( s \), and never running below height \( s \), and where \( D \) consists of \( s \) down-steps, from height \( s \) to height 0.

Still following the arguments in the proof of Theorem 6, and taking into account that the number of possible paths \( P_{s+\ell+1} \) is the \((r/\ell)\)th Catalan number, this decomposition implies that the generating function that we
are looking for is given by
\[
\sum_{\ell(r \geq 0)} \frac{1}{\sqrt{xU_{k-1}(\frac{1}{2\sqrt{x}})}} \left( \frac{U_{k-2}(\frac{1}{2\sqrt{x}})}{\sqrt{xU_{k-1}(\frac{1}{2\sqrt{x}})}} \right)^{\ell} \cdot \left( \frac{U_{k-3}(\frac{1}{2\sqrt{x}})}{\sqrt{xU_{k-2}(\frac{1}{2\sqrt{x}})}} \right)^{r-1} \frac{1}{\sqrt{xU_{k-2}(\frac{1}{2\sqrt{x}})}} \frac{1}{r/\ell} + 1 \left( \frac{2r/\ell}{r/\ell} \right)^{x^{r+\ell+1/2}}.
\]

The inner sum is again a geometric series and can therefore be evaluated. A routine calculation, followed by a replacement of \( \ell \) by \( r/\ell \), then transforms the resulting expression into (3.15).

It is obvious that in both cases (that is, for 132-avoiding permutations with no occurrence of the pattern 23\ldots k1, respectively with \( r \leq k - 1 \) occurrences) Lemma 4 could be applied to derive asymptotic formulas for the number of such permutations of \( \{1, 2, \ldots, n\} \), as \( n \) becomes large. We omit the statement of the corresponding formulas for the sake of brevity.

It appears that, for generic \( r \) (i.e., also for \( r \geq k \)), the number of 132-avoiding permutations of \( \{1, 2, \ldots, n\} \) with exactly \( r \) occurrences of the pattern 23\ldots k1 is asymptotically of the order \( \Theta((4\cos^2 \frac{\pi}{k})^n) \), but we are not able to offer a rigorous proof.

4. A BIJECTION BETWEEN 123-AVOIDING PERMUTATIONS AND DYCK PATHS

In this section we define another map, \( \Psi \), between permutations and Dyck paths, which maps 123-avoiding permutations to Dyck paths which start in the origin and return to the \( x \)-axis.

Let \( \pi = \pi_1 \pi_2 \ldots \pi_n \) be a 123-avoiding permutation. In \( \pi \), we determine all the right-to-left maxima. A right-to-left maximum is an element \( \pi_i \) which is larger than all the elements to its right, i.e., larger than all \( \pi_j \) with \( j > i \).

For example, the right-to-left maxima in the permutation 58327641 are 1, 4, 6, 7, 8.

Let the right-to-left maxima in \( \pi \) be \( m_1, m_2, \ldots, m_s \), from right to left, so that
\[
\pi = w_s m_s w_{s-1} m_{s-1} \ldots w_1 m_1,
\]
where \( w_i \) is the subword of \( \pi \) between \( m_{i+1} \) and \( m_i \). Since \( \pi \) is 123-avoiding, for all \( i \) the elements in \( w_i \) must be in decreasing order. Moreover, for all \( i \) all the elements of \( w_i \) are smaller than all the elements of \( w_{i+1} \).

Now we are able to define the map \( \Psi \). Read the decomposition (4.1) from right to left. Any right-to-left maximum \( m_i \) is translated into \( m_i - m_{i-1} \)
up-steps (with the convention $m_0 = 0$). Any subword $w_i$ is translated into $|w_i| + 1$ down-steps (where, again, $|w_i|$ denotes the number of elements of $w_i$). Finally, the resulting path is reflected into a vertical line. (Alternatively, we could have said that we generate the Dyck path from the back to the front.) The Dyck path which corresponds to our special permutation 58327641 is the one in Fig. 1.

It is easy to see that the map $\Psi$ is a bijection between 123-avoiding permutations of $\{1, 2, \ldots, n\}$ and Dyck paths from $(0, 0)$ to $(2n, 0)$. The lemma below states the crucial property of this bijection, which will be subsequently used in Section 5.

**Lemma $\Psi$.** Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a 123-avoiding permutation and let $P = \Psi(\pi)$ be the corresponding Dyck path. Then a peak in $P$ of height $i$ (i.e., an up-step from height $i - 1$ to height $i$ followed by a down-step from height $i$ to height $i - 1$) corresponds in a one-to-one fashion to an element $\pi_j$ in the permutation which is the last element in an occurrence of the pattern $(i - 1)(i - 2)\ldots 1i$ that is maximal with respect to the property that $\pi_j$ is its last element.

**Proof.** By construction of $\Psi$, any peak in the Dyck path corresponds to a right-to-left maximum, $m$ say, in the permutation. Furthermore, by induction one sees that the height of the peak is exactly by 1 larger than the number of elements to the left of $m$ that are smaller than $m$. Clearly, all these elements belong to some $w_j$ in the decomposition (4.1) of the permutation. By the above-mentioned observations, these elements are in decreasing order and, thus, together with $m$ form an occurrence of the pattern $(i - 1)(i - 2)\ldots 1i$ that cannot be made longer under the assumption that $m$ is the last element in the occurrence of the pattern. This proves the assertion of the lemma. 

5. THE ENUMERATION OF 123-AVOIDING PERMUTATIONS WITH A PRESCRIBED NUMBER OF OCCURRENCES OF THE PATTERN $(k - 1)(k - 2)\ldots 1k$

Let $\pi$ be a 123-avoiding permutation. We denote the number of occurrences of the pattern $(k - 1)(k - 2)\ldots 1k$ in $\pi$ by $N((k - 1)(k - 2)\ldots 1k; \pi)$.

Given a Dyck path $P$, we assign a weight to it, denoted by $w_2(k; P)$. It is defined as the sum $\sum_p \binom{\hat{q}(p) - 1}{k - 1}$, where the sum is over all peaks $p$ of $P$ and where $\hat{q}(p)$ is the height of the peak. For example, the weight $w_2(3; \cdot)$ of the Dyck path in Fig. 1 is

$$
\binom{1}{2} + \binom{3}{2} + \binom{3}{2} + \binom{2}{2} + \binom{0}{2} = 7.
$$
From Lemma $\Psi$ it is immediate that
\[
N((k - 1)(k - 2)\ldots 1k; \pi) = w_2(k; \Psi(\pi)). \tag{4.2}
\]

This observation, combined with Roblet and Viennot’s continued fraction theorem for the generating function of Dyck paths (see Theorem A5), allows us to express the generating function which counts 123-avoiding permutations with respect to the number of occurrences of the pattern $(k - 1)(k - 2)\ldots 1k$ in the form of a continued fraction. Again, in the statement of the theorem, we write $|\pi|$ for the number of elements which are permuted by $\pi$.

**Theorem 8.** The generating function $\sum_{\pi} y^{N((k - 1)(k - 2)\ldots 1k; \pi)} x^{|\pi|}$, where the sum is over all 123-avoiding permutations, is given by
\[
\frac{1}{1 - x(y^{(i-1)} - 1) - \frac{x}{1 - x(y^{(i-1)} - 1) - \frac{x}{1 - x(y^{(i-1)} - 1) - \ldots}}}. \tag{4.3}
\]

**Proof.** Apply Theorem A5 with $\lambda_i = x$ and $v_i = xy^{(i-1)}$, $i = 0, 1, \ldots$, and use (4.2). □

Next, similar to Section 3, we study generating functions for 123-avoiding permutations with a fixed number of occurrences of the pattern $(k - 1)(k - 2)\ldots 1k$. The first theorem restates a result due to Chow and West [2, Theorem 3.6, first case]. The proof, however, is different, as it is based on our Dyck path approach.

**Theorem 9.** The generating function $\sum_{\pi} x^{\pi}$, where the sum is over all 123-avoiding permutations which also avoid the pattern $(k - 1)(k - 2)\ldots 1k$, is given by
\[
\frac{U_{k-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_k(\frac{1}{2\sqrt{x}})}, \tag{4.4}
\]
where $U_n(x)$ denotes the $n$th Chebyshev polynomial of the second kind.

**Proof.** By Lemma $\Psi$, the permutations in the statement of the theorem are in bijection with Dyck paths, which start at the origin, return to the $x$-axis, and do not exceed the height $k - 1$. Now we apply Theorem A2 with $b_i = 0$, $\lambda_i = 1$, $i = 0, 1, \ldots$, $K = k - 1$, $r = s = 0$, $x$ replaced by $\sqrt{x}$, and use Fact A3. □
Theorem 10. Let $1 \leq r \leq k - 1$. The generating function $\sum_{\pi} x^{\vert \pi \vert}$, where the sum is over all 123-avoiding permutations with exactly $r$ occurrences of the pattern $(k - 1)(k - 2) \ldots k$, is given by

$$x^{(r-1)/2} \left( \frac{U_{k-1} \left( \frac{1}{\sqrt{x}} \right)}{U_k \left( \frac{1}{\sqrt{x}} \right)} \right)^{r-1},$$

where $U_n(x)$ denotes the $n$th Chebyshev polynomial of the second kind.

Proof. By Lemma $\Psi$, the permutations in the statement of the theorem are in bijection with Dyck paths, which start at the origin, return to the $x$-axis, do not exceed height $k$, and have exactly $r$ peaks at height $k$. (It should be noted that it is here where we use the assumption $r \leq k - 1$. For, a peak of height $i$ corresponds to $(i-1)$ occurrences of the pattern $(k - 1)(k - 2) \ldots 1$, all of which are subsequences of the occurrence of the pattern $(i-1)(i-2) \ldots 1$ guaranteed by Lemma $\Psi$. Therefore, if $r \geq k$ then we would also have to consider paths with peaks of height higher than just $k$.) Such a Dyck path can be decomposed as

$$P_0 u_1 d_1 P_1 u_2 d_2 P_2 \ldots u_r d_r P_r,$$

where $P_0$ is the portion of $P$ from the origin until the first time the height $k - 1$ is reached, where the $u_i$'s are up-steps from height $k - 1$ to height $k$, where the $d_i$'s are down-steps from height $k$ to height $k - 1$, where the $P_i$'s, $i = 1, 2, \ldots, r - 1$, are path portions which start and end at height $k - 1$ and never exceed height $k - 1$, and where $P_r$ is the portion of $P$ from the last point at height $k - 1$ until the end of the path. Application of Theorem A2 and use of Fact A3 implies, by arguments that are more or less identical to those in the proof of Theorem 3, that the generating function for those paths is equal to

$$\frac{1}{\sqrt{x}U_k \left( \frac{1}{\sqrt{x}} \right)} \left( \frac{U_{k-1} \left( \frac{1}{\sqrt{x}} \right)}{\sqrt{x}U_k \left( \frac{1}{\sqrt{x}} \right)} \right)^{r-1} \frac{1}{\sqrt{x}U_k \left( \frac{1}{\sqrt{x}} \right)} x^r,$$

which simplifies to (4.5).

The special case $k = 3$ and $r = 1$ of Theorem 10 appears, in an equivalent form, in [8, Theorem 2].

We could use an idea similar to the one in the proof of Theorem 3 to express, for general $r$, the generating function for 123-avoiding permutations with exactly $r$ occurrences of the pattern $(k - 1)(k - 2) \ldots 1k$ in the form of a sum, taken over all possible ways to arrange the peaks that are at least at height $k$. However, it appears that it is not possible to write the result in a way that is similarly elegant as (3.5). However, for the asymptotics, the same reasoning as in the proof of Theorem 5
remains valid; i.e., in this sum, the summand which, asymptotically, provides the largest contribution, is again the summand (4.5) (which is the same as the summand in (3.5) with \( \ell_1 = r \) and all other \( \ell_i \)'s equal to zero). Therefore an analogue of Theorem 5 in the present context is true. More precisely, the number of 123-avoiding permutations with exactly \( r \) occurrences of the pattern \((k - 1)(k - 2)\ldots 1k\) is asymptotically as large as the number of 132-avoiding permutations with exactly \( r \) occurrences of the pattern \(12\ldots k\). This extends [2, Corollary 4.2].

**Theorem 11.** Let \( r \) and \( k \) be fixed nonnegative integers. Then, as \( n \) becomes large, the number of 123-avoiding permutations with exactly \( r \) occurrences of the pattern \((k - 1)(k - 2)\ldots 1k\) is asymptotically

\[
\left( \frac{4\sin^2 \frac{\pi}{k+1}}{k+1} \right)^{r+1} \frac{n^r}{r!} \left( \frac{4\cos^2 \frac{\pi}{k+1}}{k+1} \right)^{n-r} \left( 1 + O\left( \frac{1}{n} \right) \right). \tag{4.6}
\]

**APPENDIX: GENERATING FUNCTIONS FOR MOTZKIN AND DYCK PATHS**

A Motzkin path is a lattice path in the plane integer lattice \( \mathbb{Z}^2 \) consisting of up-steps \((1, 1)\), level-steps \((1, 0)\), and down-steps \((1, -1)\), which never passes below the x-axis. See Fig. 2 for an example.

Clearly, a Dyck path is just a Motzkin path without level-steps.

Given a Motzkin path \( P \), we denote the length of the path (i.e., the number of steps) by \( \ell(P) \). Furthermore, we define the weights \( w(P) \) of \( P \) to be the product of the weights of all its steps, where the weight of an up-step is 1 (hence, does not contribute anything to the weight), the weight of a level-step at height \( h \) is \( b_h \), and the weight of a down-step from height \( h \) to \( h - 1 \) is \( \lambda_h \). Thus, the weight of the Motzkin path in Fig. 2 is \( b_2\lambda_2b_1\lambda_3\lambda_2 = b_1^2b_2\lambda_2^2\lambda_3 \).

![Fig. 2. A Motzkin path.](image-url)
The theorem below, due to Flajolet, expresses the corresponding generating function for all Motzkin paths which start at the origin and return to the $x$-axis in the form of a continued fraction.

**Theorem A1 (Flajolet [3, Theorem 1]).** With the weight $w$ defined as above, the generating function $\sum_P w(P)$, where the sum is over all Motzkin paths starting at the origin and returning to the $x$-axis, is given by

$$
\frac{1}{1 - b_0 - \frac{\lambda_1}{1 - b_1 - \frac{\lambda_2}{\ddots}}}.
$$

(A.1)

Next we recall the expression, in terms of orthogonal polynomials, for the generating function for Motzkin paths in a strip. Although this is a result in the folklore of combinatorics, probability, and statistics, the only explicit mention that I am able to provide is [14, Chap. V, (27)], which is a volume that is not easily accessible. Therefore I include a sketch of the proof.

**Theorem A2.** Define the sequence $(p_n(x))_{n \geq 0}$ of polynomials by

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + \lambda_n p_{n-1}(x), \quad \text{for } n \geq 1,
$$

(A.2)

with initial conditions $p_0(x) = 1$ and $p_1(x) = x - b_0$. Furthermore, define $(S_p(x))_{n \geq 0}$ to be the sequence of polynomials which arises from the sequence $(p_n(x))$ by replacing $\lambda_i$ by $\lambda_{i+1}$ and $b_i$ by $b_{i+1}$, $i = 0, 1, 2, \ldots$, everywhere in the three-term recurrence (A.2) and in the initial conditions. Finally, given a polynomial $p(x)$ of degree $n$, we denote the corresponding reciprocal polynomial $x^n p(1/x)$ by $p^*(x)$.

With the weight $w$ defined as before, the generating function $\sum_P w(P)s^t(P)$, where the sum is over all Motzkin paths which start at $(0, r)$, terminate at height $s$, and do not pass above the line $y = K$, is given by

$$
\frac{x^{r-t} p^*_r(x) S^{r+1} p^*_K(x)}{p^*_K(x)} \quad r \leq s,
$$

$$
\frac{x^{r-s} p^*_r(x) S^{r+1} p^*_K(x)}{p^*_K(x)} \quad r \geq s.
$$

(A.3)

**Sketch of Proof.** Motzkin paths which never exceed height $K$ correspond in a one-to-one fashion to walks on the path $P_{K+1}$ with loops attached to each vertex (this is the graph on the vertices $v_0, v_1, \ldots, v_K$ where for $i = 0, 1, \ldots, K - 1$ the vertices $v_i$ and $v_{i+1}$ are connected by an edge, and there is a loop for each vertex $v_i$). In this correspondence, an up-step from height $h$ to $h + 1$ in the Motzkin path corresponds to a step from vertex $v_h$ to vertex $v_{h+1}$ in the walk and similarly for level- and down-steps.
It is well known (see e.g. [12, Theorem 4.7.2]) that the generating function for walks from \( v_r \) to \( v_s \) is given by
\[
\frac{(-1)^{r+s} \det(I - xA; s, r)}{\det(I - xA)},
\]
where \( A \) is the (weighted) adjacency matrix of \( P_{K+1} \), where \( I \) is the \((K + 1) \times (K + 1)\) identity matrix, and where \( \det(I - xA; s, r) \) is the minor of \((I - xA)\) with the \(s\)th row and \( r\)th column deleted.

Now, the (weighted) adjacency matrix of \( P_{K+1} \) with the property that the weight of a particular walk would correspond to the weight \( w \) of the corresponding Motzkin path is the tridiagonal matrix
\[
A = \begin{pmatrix}
\lambda_1 & b_1 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & b_2 & 1 & \cdots & \cdots & \vdots \\
0 & 0 & \lambda_3 & b_3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \lambda_{K-2} & b_{K-2} & 1 & 0 \\
\vdots & \ddots & \ddots & 0 & \lambda_{K-1} & b_{K-1} & 1 \\
0 & \cdots & \cdots & \cdots & 0 & \lambda_K & b_K
\end{pmatrix}
\]
It is easily verified that, with this choice of \( A \), we have \( \det(I - xA) = p_{K+1}^*(x) \) (by expanding the determinant with respect to the last row and comparing with the three-term recurrence (A.2)), and, similarly, that the numerator in \( (A.3) \) agrees with \( (-1)^{r+s} \det(I - xA; r, s) \).

The special cases of Theorem A2 in which \( r = s = 0 \), respectively \( r = 0 \) and \( s = K \), appear also in [3, Sect. 3.1].

The following is a well-known and easily verifiable fact:

**Fact A3.** If \( b_i = 0 \) and \( \lambda_i = 1 \) for all \( i \), then the polynomials \( p_n(x) \) defined by the three-term recurrence (A.2) are Chebyshev polynomials of the second kind,
\[
p_n(x) = U_n(x/2).
\]

Fact A3, in combination with Theorem A2, tells that Chebyshev polynomials of the second kind are intimately tied to the enumeration of Dyck paths.

Although we do not make use of it in this paper, we wish to emphasize that the enumeration of Motzkin paths (i.e., also allowing level-steps) is also intimately tied to Chebyshev polynomials of the second kind.

**Fact A4.** If \( b_i = 1 \) and \( \lambda_i = 1 \) for all \( i \), then the polynomials \( p_n(x) \) defined by the three-term recurrence (A.2) are also Chebyshev polynomials of the second kind, namely
\[
p_n(x) = U_n((x - 1)/2).
\]
Now we restrict our attention to Dyck paths. We refine the above defined weight $w$ in the following way, so that in addition it also takes into account peaks: Given a Dyck path $P$, we define the weight $\hat{w}(P)$ of $P$ to be the product of the weights of all its steps, where the weight of an up-step is 1, the weight of a down-step from height $h$ to $h - 1$ which follows immediately after an up-step (thus, together, form a peak of the path) is $\nu_h$, and the weight of a down-step from height $h$ to $h - 1$ which follows after another down-step is $\lambda_h$. Thus, the weight of the Dyck path in Fig. 1 is $\nu_2 \nu_4 \lambda_3 \nu_2 \lambda_2 \nu_1 = \nu_1 \nu_2 \nu_3 \lambda_1 \lambda_2 \lambda_3$.

The theorem below, due to Roblet and Viennot, expresses the corresponding generating function for all Dyck paths which start at the origin and return to the $x$-axis in the form of a continued fraction.

**Theorem A5 (Roblet and Viennot [10, Proposition 1]).** With the weight $\hat{w}$ defined as above, the generating function $\sum_P \hat{w}(P)$, where the sum is over all Dyck paths starting at the origin and returning to the $x$-axis, is given by

\[
\frac{1}{1 - (\nu_1 - \lambda_1) - \frac{\lambda_1}{1 - (\nu_2 - \lambda_2) - \frac{\lambda_2}{1 - (\nu_3 - \lambda_3) - \cdots}}} \quad (A.4)
\]

Finally, we remark that, from a different angle, Katzenbeisser and Panny [5] have undertaken an independent study of the enumeration of Motzkin paths.

**Remark.** I am indebted to the referee for an extremely careful reading of the manuscript.

**REFERENCES**


