NOTE

A New Decomposition of Derangements

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We give a new decomposition of derangements, which gives a direct interpretation of a formula for their generating function. This decomposition also works for counting derangements by number of excedances.

1. INTRODUCTION

A permutation $\pi$ of $[n]=\{1,2,\ldots,n\}$ is a derangement, if $\pi(i) \neq i$, for all $i \in [n]$. A value $i \in [n]$ is an excedance of $\pi$ if $i < \pi(i)$. The number of excedances in $\pi$ is denoted by $\text{exc}\, \pi$. Let $D_n$ be the set of derangements of $[n]$, and $d_n(x)$ the polynomial

$$d_n(x) = \sum_{\pi \in D_n} x^{\text{exc}\, \pi}.$$

For example, $d_0(x) = 1$, $d_1(x) = 0$, $d_2(x) = x$, $d_3(x) = x + x^2$, $d_4(x) = x + 7x^2 + x^3$. The generating function of $d_n(x)$ can be written as [2, 5]

$$u_n = \frac{1}{n!} \sum_{n \geq 0} d_n(x) x^n = 1 - \sum_{n \geq 2} (x + x^2 + \cdots + x^{n-1}) x^n/n!.$$  \ \ (1)

Of course (1) can be proved by various methods, but, as pointed out by Gessel [4], it seems difficult to directly interpret (1) (even in the $x=1$ case).
case!) in terms of derangements. In [4] Gessel gave a direct proof of (1) in a different model with $x = 1$. His proof is actually based on a factorization of some D-permutations, and cannot be generalized in a straightforward way to prove (1). Our purpose is to give a decomposition of derangements which interprets (1) directly.

A sequence $\sigma = s_1, s_2, \ldots, s_k$ of $k$ distinct integers $s_1, ..., s_k$ is called a cycle of length $k$ if $s_1 = \min\{s_1, ..., s_k\}$. A cycle $\sigma$ is called unimodal (resp. prime), if there exists $i, 2 \leq i \leq k$, such that $s_1 < \cdots < s_{i-1} < s_i$ and $s_i > s_{i+1} > \cdots > s_k$ if $i < k$ (resp. in addition, $s_{i-1} < s_k$). Hence each unimodal (resp. prime) cycle is of length 2. Considering that $s_1$ is the smallest in our case, this definition is consistent with the usual definition of “unimodal”. Clearly each cycle $\sigma = s_1, ..., s_k$ can be identified with the cyclic permutation $\sigma$ of the set $\{s_1, ..., s_k\}$ by $\sigma(s_i) = s_{i+1}$ for $i \in [k]$, with $s_{k+1} = s_1$. We let $\text{exc}\sigma$ denote the number of excedances of the associated cyclic permutation $\sigma'$.

Let $(l_1, ..., l_m)$ be a composition of $n$. A P-decomposition of type $(l_1, ..., l_m)$ of $[n]$ is a sequence of prime cycles $\tau = (\tau_1, \tau_2, ..., \tau_m)$ such that $\tau_i$ is of length $l_i$ and the underlying sets of $\tau_i$, $i \in [m]$, form a partition of $[n]$.

Define the excedance of $\tau$ as the total number of excedances in its prime cycles, i.e., $\text{exc}\tau = \text{exc}\tau_1 + \cdots + \text{exc}\tau_m$, and weight $\tau$ by $x^{\text{exc}\tau}$. It turns out that the right-hand side of (1) is the excedance generating function of P-decompositions. Indeed, since the weight of prime cycles on any $l$-set is $x + x^2 + \cdots + x^{l-1}$, the generating function of P-decompositions of type $(l_1, ..., l_m)$ is given by

$$\left(\prod_{i=1}^{m} (x + \cdots + x^{l_i-1}) \right) \frac{x^{l_1+\cdots+l_m}}{(l_1+\cdots+l_m)!}.$$ 

Summing on $l_1, ..., l_m \geq 2$ and $m \geq 0$, we obtain the right hand side of (1).

In the next section we give an algorithm (or bijection), which maps each derangement into a P-decomposition with the same number of excedances, and thus prove (1). In Section 3 we will apply a similar decomposition to give a direct interpretation of a generating function of Eulerian polynomials. Finally, in Section 4 we indicate how to extend our algorithm to deal with similar problems in multipermutations.

### 2. UNIMODAL AND PRIME DECOMPOSITIONS

Given a derangement $\pi$ of $[n]$, we first factorize it into cycles of length $\geq 2$.

$$\pi = (C_1, ..., C_k),$$
sorted in the *decreasing* order of their minima. For each cycle \( \sigma = s_1 s_2 \cdots s_k \) we define the following \( U \)-algorithm to decompose it into a sequence of unimodal cycles. For the algorithm we set \( s_{k+1} = s_1 \).

**U-Algorithm.**

1. If \( \sigma \) is unimodal then \( U(\sigma) = (\sigma) \).

2. Otherwise, let \( i \) be the largest integer such that \( s_i > s_j < s_{i+1} \), let \( j \) be the unique integer greater than \( i \) such that \( s_j > s_{i+1} > s_j \), and set \( U(\sigma) = (U(\sigma_1), \sigma_2) \), where \( \sigma_1 = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k \) and \( \sigma_2 = s_is_{i+1} \cdots s_j \), which is unimodal.

**Example 2.1.** Let \( \sigma = 184 712 14 11 9 13 10 6 3 5 2 \). The \( U \)-algorithm runs as

\[
\begin{align*}
\sigma &\rightarrow (U(184 712 14 11 9 13 10 6 2), 3 5) \\
&\rightarrow (U(184 712 14 11 9 13 10 6 2), 9 13 10, 3 5) \\
&\rightarrow (U(18 2), 4 7 12 14 11 6, 9 13 10, 3 5) \\
&\rightarrow (18 2, 4 7 12 14 11 6, 9 13 10, 3 5).
\end{align*}
\]

We extend \( U \) to \( \pi \) by applying \( U \) to each of its cycles to obtain

\[
U(\pi) = (U(C_1), U(C_2), \ldots, U(C_r)) = (u_1, \ldots, u_m),
\]

which is called the *unimodal decomposition* of \( \pi \).

Note that the first cycle \( C_1 \) of \( \pi \) corresponds to the segment \( (u_1, \ldots, u_i) \), where \( i \) is the smallest integer satisfying \( \min(u_1) > \min(u_{i+1}) \), and the second to a segment of \( (u_{i+1}, \ldots, u_m) \) in the same manner, etc., so that the underlying set of each cycle can be read off from the *unimodal decomposition* of \( \pi \). The following result characterizes all the sequences of unimodal cycles obtained by the \( U \)-algorithm.

**Lemma 2.2.** A sequence of disjoint unimodal cycles, \( u = (u_1, \ldots, u_m) \), is a unimodal decomposition of a derangement in \( S_n \) if and only if the underlying sets of \( u_i, i \in [m] \), form a partition of \([n]\) and \( \max(u_{i+1}) > \min(u_i) \) for each \( i = 2, \ldots, m \).

**Proof.** Clearly it suffices to show the “if” part. Without loss of generality we may assume that \( \min(u_i) < \min(u_{i+1}) \), for each \( i = 2, \ldots, m \). We build \( \pi \) step by step. Let \( \pi^{(1)} = u_1 \). For \( i > 1 \), assume that \( \pi^{(i-1)} \) has been built and that \( \pi^{(i-1)} = s_1 s_2 \cdots s_i \), where \( s_1, \ldots, s_i \) is an appropriate rearrangement of elements in \( u_1, u_2, \ldots, u_{i-1} \). Let \( u_i = r_1 r_2 \cdots r_a \). Since \( \max(u_{i+1}) > \min(u_i) \), there is an integer \( j \) such that \( s_j > \min(u_i) \), let \( j_0 \) be the largest such
integer and set \( \pi^{(i)} = s_1 s_2 \cdots s_j r_1 r_2 \cdots r_m s_{j+1} \cdots s_f \). Let \( \pi = \pi^{(m)} \). Clearly \( U(\pi) = u \).

For each unimodal cycle \( \sigma = s_1 s_2 \cdots s_k \) we define the following \( V \)-algorithm to decompose it into a sequence of prime cycles.

\[ V \text{-Algorithm.} \]

1. If \( \sigma \) is prime then \( V(\sigma) = (\sigma) \).
2. Otherwise, let \( j \) be the smallest integer such that \( s_j > s_j+1 > s_{j-1} \) for some integer \( i \) greater than 1 and set \( V(\sigma) = (V(\sigma_1), \sigma_2) \), where \( \sigma_1 = s_1 \cdots s_{j-1} s_{j+1} \cdots s_k \) and \( \sigma_2 = s_j s_{j+1} \cdots s_k \), which is prime.

We extend \( V \)-algorithm to \( U(\pi) \) by applying \( V \) to each of its components to obtain

\[ V \cdot U(\pi) = (V(u_1), V(u_2), \ldots, V(u_m)) = (\tau_1, \ldots, \tau_m), \]

which is called the prime decomposition of \( \pi \).

The structure of the unimodal decomposition of \( \pi \) can be easily obtained from its prime decomposition. The first unimodal cycle in \( U(\pi) \) corresponds to the segment \( (\tau_1, \ldots, \tau_i) \), where \( i \) is the smallest integer satisfying \( \max(\{\tau_i\}) > \min(\{\tau_{i+1}\}) \), and the second to a segment of \( (\tau_{i+1}, \ldots, \tau_m) \) in the same manner, etc.

**Example 2.3.** Let \( \sigma \) be the same as the preceding example, whose unimodal decomposition is \( U(\sigma) = (1 8 2, 4 7 12 14 11 6, 9 13 10, 3 5) \). Note that only the second cycle in \( U(\sigma) \) is not prime. The \( V \)-algorithm applied to the second cycle runs as

\[ 4 7 12 14 11 6 \rightarrow (V(4 7 11 6), 12 14) \rightarrow (4 6, 7 11, 12 14). \]

Therefore \( V \cdot U(\sigma) = (1 8 2, 4 6, 7 11, 12 14, 9 13 10, 3 5) \).

It is clear that the composition \( V \cdot U \) maps any derangement of \( [n] \) into a \( P \)-decomposition of \( [n] \). The following result shows that this mapping is bijective.

**Theorem 2.4.** Any \( P \)-decomposition of \( [n] \) is the prime decomposition of a unique derangement in \( \mathcal{D}_n \).

**Proof.** Let \( \tau = (\tau_1, \tau_2, \ldots, \tau_m) \) be a \( P \)-decomposition of \( [n] \). We first construct a sequence of unimodal cycles as follows: starting from the right, if there is any pair of adjacent \( \tau_i \) and \( \tau_{i+1} \) such that \( \max(\{\tau_i\}) < \min(\{\tau_{i+1}\}) \), then we insert the elements of \( \tau_{i+1} \) in \( \tau_i \) just before the maximum of \( \tau_i \) and obtain a new cycle \( \tau_i^\tau \tau_{i+1} \). Repeat this process with \( (\tau_1, \ldots, \tau_i^\tau \tau_{i+1}, \ldots, \tau_m) \).
until there are no more such pairs. By Lemma 2.2, the resulting sequence \( \sigma \) is a unimodal decomposition of some \( \pi \in S_n \), i.e., \( U(\pi) = \sigma \). It follows that \( V \circ U(\pi) = V(\sigma) = \tau \).

From the \( U \)-algorithm it is clear that the number of excedances in a cycle is the same as the sum of excedances in each unimodal component. Also the prime decomposition has the same property. Thus we have proved (1).

3. APPLICATION TO EULERIAN POLYNOMIALS

If instead of derangements we let \( A_n(x) \) denote the sum of \( x^{\text{exc } \pi} \) for all permutations \( \pi \) of \([n]\), then the polynomials \( xA_n(x) \) are the well-known Eulerian polynomials and have several other combinatorial interpretations in addition to counting permutations by number of excedances \([6]\). By virtue of classical theory of generating functions we see immediately that

\[
A_n(x) = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = e^t \sum_{n \geq 0} x^n \frac{t^n}{n!}.
\]

Hence it follows from (1) that

\[
\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} (x-1)^{n-1} t^n/n!}.
\]

A similar proof can be given for (2), but in this case a weight-preserving sign-reversing involution is needed.

A sequence \( \sigma = a_1 a_2 \ldots a_k \) of \( k \) distinct integers \( a_1, a_2, \ldots, a_k \) is called unimodal if \( k = 1 \) or \( k \geq 2 \) and there exists an integer \( i, 1 \leq i \leq k \), such that \( a_1 < a_2 < \cdots < a_i \) and \( a_i > a_{i+1} > \cdots > a_k \) if \( i < k \). This is the usual definition of "unimodal". We define the weight of the unimodal sequence \( \sigma \) by \( x^{\sigma^{-1}}(-1)^{\sigma-i} \), i.e., an ascent is given \( x \) and a descent \(-1\).

A \( U \)-decomposition (resp. \( I \)-decomposition) of \([n]\) is a sequence of unimodal (resp. increasing) sequences \((\tau_1, \tau_2, \ldots, \tau_m)\) such that the underlying sets of \( \tau_i, i \in [m] \), form a partition of \([n]\) (resp. in addition, for \( i > 1 \), if \( \tau_i \) is a singleton then it is greater than the last entry of \( \tau_{i-1} \)). Hence the right side of (2) is the generating function of \( U \)-decompositions.

We now set up a weight-preserving sign-reversing involution on the \( U \)-decompositions to reduce the above generating function to that of \( I \)-decompositions. Given a \( U \)-decomposition \( \pi = (\pi_1, \pi_2, \ldots, \pi_l) \), we call an integer \( k \) attachable, if \( k \) forms a singleton, i.e., \( \pi_i = k \) for some \( i > 1 \), and \( k \) is smaller than the last entry of \( \pi_{i-1} \); detachable, if there exists \( \pi_j \) whose
last entry is \( k \) and whose penultimate entry is greater than \( k \). The involu-
tion is then defined by detaching or attaching the smallest attachable or
detachable integer (if any). It is clear that \( \pi \) is fixed if and only if \( \pi \) is an
\( I \)-decomposition.

On the other hand, given a permutation \( \pi \) of \([n]\), we can factorize it into
ordered cycles \( \pi = (s_1, ..., s_r, c_1, ..., c_t) \), where \( s_1, ..., s_r \) are the singletons
ordered in increasing order and \( c_1, ..., c_t \) the cycles of length \( \geq 2 \) ordered
in decreasing order of their minima. Applying \( V \cdot U \) algorithm to each cycle
\( c_i \) we obtain
\[
\pi = (s_1, ..., s_r, V \cdot U(c_1), ..., V \cdot U(c_t)) = (\pi_1, ..., \pi_m),
\]
where each \( \pi_i \) is a prime or singleton cycle. Since each prime cycle
\( a_1 \cdot a_{k-1} \cdot a_k \cdot d_1 \) with \( a_1 \leq \cdots \leq a_{k-1} \leq a_k \) is in one-to-
one correspondence with a sequence of increasing segments, \( (a_1 \cdot a_2 \cdot a_{k-1} \cdot a_1,
\]
\( a_{k-1}, a_{k-2}, \cdots, a_2) \), which has no attachable or detachable element, we see
that \( \pi \) is in one-to-one correspondence with an \( I \)-decomposition of \([n]\).

Note that the singletons in \( \pi \) correspond to the singletons to the left of the
first increasing sequence of length greater than one in an \( I \)-decomposition.

Therefore both sides of (2) are the generating functions of \( I \)-decomposi-
tions.

4. REMARKS

Our decompositions work also for permutations of a multiset \([1^n, 2^n, \ldots, m^n]\).
More precisely, let \( w = w_1 w_2 \cdots w_n \) be such a permutation and
\( \delta(w) = p_1 p_2 \cdots p_n \) the nondecreasing rearrangement of the letters in \( w \),
where \( n = n_1 + \cdots + n_m \). Then \( w \) is a multiderangement if \( p_i \neq w_i \) for each
\( i = 1, ..., n \), while the statistic of exceedance of \( w \) is defined by
\( \text{exc } w = \# \{ i : w_i > p_i \} \). Let \( \mathcal{A}(n) \) be the set of all such permutations and define
\[
d_n(x) = \sum_{w \in \mathcal{A}(n)} x^{\text{exc } w}.
\]
Using Foata's factorization of multipermutations (see [3]) we can factorize
each multiderangement as a product of cycles of length at least 2, combing
with our two decompositions we get the following result,
\[
\sum_{n_1, ..., n_m \geq 0} d_n(x) x_1^{n_1} \cdots x_m^{n_m}
\]
\[
= \frac{1}{1 - x \left( e_2 - (x + x^2) e_3 - \cdots - (x + x^2 + \cdots + x^{m-1}) e_m \right)}.
\]
where $e_i (2 \leq i \leq m)$ is the $i$-th elementary symmetric function of $x_1, \ldots, x_m$.
The above result seems to be first proved by Askey and Ismail [1] using MacMahon’s Master Theorem.

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