Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials

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Abstract

In this paper, we derive some interesting identities on Bernoulli and Euler polynomials by using the orthogonal property of Laguerre polynomials.

1 Introduction

The generalized Laguerre polynomials are defined by

\[
\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_\alpha^n(x)t^n \quad (\alpha \in \mathbb{Q} \text{ with } \alpha > -1).
\]

From (1.1), we note that

\[
L_\alpha^n(x) = \sum_{r=0}^{n} \frac{(-1)^r (n+r)_{\alpha}}{r!} x^r \quad \text{(see [1–3]).}
\]

By (1.2), we see that \(L_\alpha^n(x)\) is a polynomial with degree \(n\). It is well known that Rodrigues’ formula for \(L_\alpha^n(x)\) is given by

\[
L_\alpha^n(x) = x^{-\alpha} e^x \frac{d^n}{dx^n} \left(e^{-x} x^n e^{-\alpha x}\right) \quad \text{(see [1–3]).}
\]

From (1.3) and a part of integration, we note that

\[
\int_0^{\infty} x^n e^{-x} L_m^{\alpha}(x) L_n^{\alpha}(x) \, dx = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{m,n},
\]

where \(\delta_{m,n}\) is a Kronecker symbol. As is well known, Bernoulli polynomials are defined by the generating function to be

\[
\frac{t}{e^t - 1} e^{\beta(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{(see [1–29]),}
\]

with the usual convention about replacing \(B^n(x)\) by \(B_n(x)\).
In the special case, \( x = 0 \), \( B_n(0) = B_n \) are called the \( n \)th Bernoulli numbers. By (1.5), we get

\[
B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l \quad \text{(see [1–29]).} \tag{1.6}
\]

The Euler numbers are defined by

\[
E_0 = 1, \quad (E + 1)^n + E_n = 2\delta_{0,n} \quad \text{(see [27, 28])}, \tag{1.7}
\]

with the usual convention about replacing \( E_n \) by \( E_n \).

In the viewpoint of (1.6), the Euler polynomials are also defined by

\[
E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l \quad \text{(see [11–24])}. \tag{1.8}
\]

From (1.7) and (1.8), we note that the generating function of the Euler polynomial is given by

\[
\frac{2}{e^t + 1} e^{xt} = e^{E(tx)} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{(see [15–29])}. \tag{1.9}
\]

By (1.5) and (1.9), we get

\[
\frac{2}{e^t + 1} e^{xt} = \frac{1}{t} \left( 2 - 2 \frac{2}{e^t + 1} \right) \left( te^{xt} \right) = -2 \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{E_{l+1}}{l+1} \binom{n}{l} B_{n-l}(x) \right) \frac{t^n}{n!}. \tag{1.10}
\]

Thus, by (1.10), we see that

\[
E_n(x) = -2 \sum_{l=0}^{n} \binom{n}{l} \frac{E_{l+1}}{l+1} B_{n-l}(x). \tag{1.11}
\]

By (1.7) and (1.8), we easily get

\[
\frac{t}{e^t - 1} e^{xt} = \frac{t}{2} \left( \frac{2e^{xt}}{e^t + 1} \right) + \left( \frac{t}{e^t - 1} \right) \left( \frac{2e^{xt}}{e^t + 1} \right). \tag{1.12}
\]

Thus, by (1.12), we see that

\[
B_n(x) = \sum_{k=0,k\neq l}^{n} \binom{n}{k} B_k E_{n-k}(x). \tag{1.13}
\]

Throughout this paper, we assume that \( \alpha \in \mathbb{Q} \) with \( \alpha > -1 \). Let \( P_n = \{ p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n \} \) be the inner product space with the inner product

\[
\langle p(x), q(x) \rangle = \int_{0}^{\infty} x^\alpha e^{-x} p(x)q(x) \, dx,
\]
Therefore, by (2.1) and (2.3), we obtain the following theorem.

\[ \begin{align*}
\text{In this paper, we give some interesting identities on Bernoulli and Euler polynomials which can be derived by an orthogonal basis } \{L_n^0(x), L_n^1(x), \ldots, L_n^n(x)\} \text{ for } \mathbb{P}_n. \\
\end{align*} \]

2 Some identities on Bernoulli and Euler polynomials

Let \( p(x) \in \mathbb{P}_n \). Then \( p(x) \) can be generated by \( \{L_n^0(x), L_n^1(x), \ldots, L_n^n(x)\} \) in \( \mathbb{P}_n \) to be

\[ p(x) = \sum_{k=0}^{n} C_k L_k^n(x), \quad (2.1) \]

where

\[ \begin{align*}
\langle p(x), L_k^n(x) \rangle &= C_k \{L_k^n(x), L_k^n(x)\} \\
&= C_k \int_{0}^{\infty} x^\alpha e^{-x} L_k^n(x) L_k^n(x) \, dx \\
&= C_k \frac{\Gamma(\alpha + k + 1)}{k!}. \quad (2.2)
\end{align*} \]

From (2.2), we note that

\[ \begin{align*}
C_k &= \frac{k!}{\Gamma(\alpha + k + 1)} \langle p(x), L_k^n(x) \rangle \\
&= \frac{k!}{\Gamma(\alpha + k + 1)} \frac{1}{k!} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) p(x) \, dx \\
&= \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) p(x) \, dx. \quad (2.3)
\end{align*} \]

Let us take \( p(x) = \sum_{m=0}^{n} \binom{n}{m} B_m E_n - m(x) \in \mathbb{P}_n \). Then, from (2.3), we have

\[ \begin{align*}
C_k &= \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) \sum_{m=0}^{n} \binom{n}{m} B_m E_{n-m}(x) \, dx \\
&= \frac{(-1)^k}{\Gamma(\alpha + k + 1)} \sum_{m=0}^{n-k} \sum_{l=k}^{n-m} \binom{n}{m} \binom{n-m}{l} B_m E_{n-m-l} \frac{l!}{(l-k)!} \int_{0}^{\infty} x^{l+k-\alpha} e^{-x} \, dx \\
&= \frac{(-1)^k}{\Gamma(\alpha + k + 1)} \sum_{m=0}^{n-k} \sum_{l=k}^{n-m} \binom{n}{m} \binom{n-m}{l} B_m E_{n-m-l} \frac{l!}{(l-k)!} \Gamma(l+\alpha+1) \\
&= (-1)^k \sum_{m=0}^{n-k} \sum_{l=k}^{n-m} \binom{n}{m} \binom{n-m}{l} B_m E_{n-m-l} \frac{l!}{(l-k)!} \frac{(l+\alpha)(l+\alpha-1) \cdots \alpha}{(\alpha+k)(\alpha+k-1) \cdots \alpha} \\
&= (-1)^k n! \sum_{m=0}^{n-k} \sum_{l=k}^{n-m} \frac{B_m}{m!} \frac{E_{n-m-l}}{(n-m-l)!} \frac{(l+\alpha)}{(l-k)}. \quad (2.4)
\end{align*} \]

Therefore, by (2.1) and (2.4), we obtain the following theorem.
Theorem 2.1 For $n \in \mathbb{Z}_+$, we have

$$
\sum_{m=0, m \neq 1}^{n} \binom{n}{m} B_m E_{n-m}(x)
$$

$$
= n! \sum_{k=0}^{n-k} (-1)^k \left( \sum_{m=0, m \neq 1}^{n} \sum_{l=k}^{n-m} B_m \frac{E_n - m - l}{m! (n-m-l)!} \binom{l + \alpha}{l-k} \right) L_k^\alpha(x).
$$

From (1.13), we can derive the following corollary.

Corollary 2.2 For $n \in \mathbb{Z}_+$, we have

$$
B_n(x) = n! \sum_{k=0}^{n} (-1)^k \left( \sum_{m=0, m \neq 1}^{n} \sum_{l=k}^{n-m} B_m \frac{E_n - m - l}{m! (n-m-l)!} \binom{l + \alpha}{l-k} \right) L_k^\alpha(x).
$$

Let us take $p(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l+1} B_{n-l}(x)$. By the same method, we get

$$
C_k = \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \frac{d^k}{dx^k} x^{k+\alpha} e^{-x} \left( \sum_{l=0}^{n} \binom{n}{l} \frac{E_{l+1}}{l+1} B_{n-l}(x) \right) dx
$$

$$
= \frac{1}{\Gamma(\alpha + k + 1)} \sum_{l=0}^{n-k} \sum_{m=0}^{n-l} \binom{n}{l} \frac{E_{l+1}}{l+1} B_{n-l-m} \frac{m!}{(m-k)!} \Gamma(m + \alpha + 1)
$$

$$
= (-1)^k \sum_{l=0}^{n-k} \sum_{m=k}^{n-l} \binom{\alpha + m}{m-k} \frac{E_{l+1}}{(l+1)! (n-l-m)!} B_{n-l-m} \frac{m!}{(m-k)!}
$$

$$
= (-1)^k n! \sum_{l=0}^{n-k} \sum_{m=k}^{n-l} \binom{\alpha + m}{m-k} \frac{E_{l+1}}{(l+1)! (n-l-m)!} B_{n-l-m} \frac{m!}{(m-k)!}
$$

(2.5)

Therefore, by (1.11), (2.1), and (2.5), we obtain the following theorem.

Theorem 2.3 For $n \in \mathbb{Z}_+$, we have

$$
-\frac{E_n(x)}{2} = n! \sum_{k=0}^{n} (-1)^k \left( \sum_{l=0}^{n-k} \sum_{m=k}^{n-l} \binom{\alpha + m}{m-k} \frac{E_{m+1}}{(m+1)! (n-m-l)!} B_{n-m-l} \frac{B_{n-l-m}}{(l+1)! (n-l-m)!} \right) L_k^\alpha(x).
$$

For $n \in \mathbb{N}$ with $n \geq 2$ and $m \in \mathbb{Z}_+$ with $n - m \geq 0$, we have

$$
B_{n-m}(x) B_n(x) = \sum_{r} \left[ \binom{n-m}{2r} m + \binom{m}{2r} (n-m) \right] \frac{B_{2r} B_{n-2r}(x)}{n-2r}
$$

$$
+ (-1)^{m+1} \frac{(n-m)! m!}{m!} B_n(x) \quad \text{see [8]}. \hspace{1cm} (2.6)
$$
Let us take \( p(x) = B_{a+m}(x)B_m(x) \in \mathbb{P}_n \). Then \( p(x) \) can be generated by an orthogonal basis \( \{L_0^a(x), L_1^a(x), \ldots, L_n^a(x)\} \) in \( \mathbb{P}_n \) to be

\[
p(x) = \sum_{k=0}^{n} C_k L_k^a(x). \tag{2.7}
\]

From (2.3), (2.6), and (2.7), we note that

\[
C_k = \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) p(x) \, dx
\]

\[
= \frac{1}{\Gamma(\alpha + k + 1)} \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \left\{ \binom{n-m}{2r} m + \binom{m}{2r} (n-m) \right\} \times \frac{B_{2r}}{n-2r} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) B_{n-2r}(x) \, dx
\]

\[
= \frac{1}{\Gamma(\alpha + k + 1)} \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{n-2r} \left\{ \binom{n-m}{2r} m + \binom{m}{2r} (n-m) \right\} \frac{B_{2r}}{n-2r} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha} e^{-x} \right) x^l \, dx
\]

\[
= \frac{(-1)^k}{\Gamma(\alpha + k + 1)} \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{n-2r} \left\{ \binom{n-m}{2r} m + \binom{m}{2r} (n-m) \right\} \frac{B_{2r} B_{n-2r-l}}{(n-2r)(l-k)!} \Gamma(\alpha + l + 1). \tag{2.8}
\]

It is easy to show that

\[
\frac{\Gamma(\alpha + l + 1)}{\Gamma(\alpha + k + 1)(l-k)!} = \frac{(\alpha + l)(\alpha + l - 1) \cdots \alpha \Gamma(\alpha)}{(\alpha + k)(\alpha + k - 1) \cdots \alpha \Gamma(\alpha)(l-k)!}
\]

\[
= \frac{(\alpha + l)(\alpha + l - 1) \cdots (\alpha + k + 1)}{(\alpha - k)!} = \binom{\alpha + l}{l-k}. \tag{2.9}
\]

By (2.8) and (2.9), we get

\[
C_k = (-1)^k \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{n-2r} \left\{ \binom{n-m}{2r} m + \binom{m}{2r} (n-m) \right\} \times \binom{n-2r}{l} \binom{\alpha + l}{l-k} \frac{B_{2r} B_{n-2r-l}}{(n-2r)}. \tag{2.10}
\]

Therefore, by (2.7) and (2.10), we obtain the following theorem.
Theorem 2.4 For \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( m \in \mathbb{Z} \), with \( n - m \geq 0 \), we have

\[
B_{n-m}(x)B_m(x) = \sum_{k=0}^{n} (-1)^k \left\{ \sum_{r=0}^{\lfloor \frac{n-k}{2} \rfloor} \left( \begin{array}{c} n-m \\ 2r \end{array} \right) m + \left( \begin{array}{c} m \\ 2r \end{array} \right)(n-m) \right\} \times \binom{n-2r}{l} \binom{\alpha+l}{l-k} \frac{B_{2r}B_{n-2r-l}}{(n-2r)} \right\} L_{\alpha}^k(x).
\]

It is easy to show that

\[
t^2e^{t(xy)} \left( \frac{e^t}{e^t-1} \right)^2 = (x+y-1) t^2e^{t(xy-1)} - t^2 \frac{d}{dt} \left( \frac{e^{t(xy-1)}}{e^t-1} \right).
\]

From (2.11), we have

\[
\sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(y) = (1-n)B_n(x+y) + (x+y-1)nB_{n-1}(x+y) \quad \text{(see [11])}.
\]

Let \( x = y \). Then by (2.12), we get

\[
\sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(x) = (1-n)B_n(2x) + (2x-1)B_{n-1}(2x).
\]

Let us take \( p(x) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(x) \in \mathbb{P}_n \). Then \( p(x) \) can be generated by an orthogonal basis \( \{ L_{\alpha}^0(x), L_{\alpha}^1(x), \ldots, L_{\alpha}^n(x) \} \) in \( \mathbb{P}_n \) to be

\[
p(x) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(x) = \sum_{k=0}^{n} C_k L_{\alpha}^k(x).
\]

From (2.3), (2.13), and (2.14), we can determine the coefficients \( C_k \)'s to be

\[
C_k = \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha+k} e^{-x} \right) p(x) \, dx
\]

\[
= \frac{1}{\Gamma(\alpha + k + 1)} \left\{ (1-n) \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha+k} e^{-x} \right) B_n(2x) \, dx 
\right. 
\]

\[
+ n \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha+k} e^{-x} \right) (2x-1)B_{n-1}(2x) \, dx \right\}.
\]

By simple calculation, we get

\[
\frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} \left( \frac{d^k}{dx^k} x^{\alpha+k} e^{-x} \right) (2x-1)B_{n-1}(2x) \, dx
\]

\[
= 2(-1)^k \sum_{l=k}^{n-1} \binom{n-1}{l} 2^l B_{n-1-l} \left( \frac{\alpha+l+1}{l-k+1} \right) (l+1)!
\]

\[
+ (-1)^{k+1} \sum_{l=k}^{n-1} \binom{n-1}{l} 2^l B_{n-1-l} \left( \frac{\alpha+l}{l-k} \right) l!
\]

(2.16)
and
\[
\frac{1}{\Gamma(\alpha + k + 1)} \int_0^\infty \left( \frac{dk}{dx} x^{k+\alpha} e^{-x} \right) B_n(2x) \, dx \\
= (-1)^k \sum_{l=k}^n \binom{n}{l} 2^l B_{n-l} B_{l+k} (l + \alpha) (l - k).
\]

(2.17)

Therefore, by (2.13), (2.14), (2.15), (2.16), and (2.17), we obtain the following theorem.

**Theorem 2.5** For \( n \in \mathbb{Z}_+ \), we get
\[
\sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(x) = (1 - n) \sum_{k=0}^n \left( -1 \right)^k \sum_{l=k}^n \binom{n}{l} 2^l B_{n-l} B_{l+k} (l + \alpha) (l - k) \right) L_\alpha(x) \\
+ n \sum_{k=0}^n \left( -1 \right)^k \sum_{l=k}^{n-1} \binom{n-1}{l} 2^l B_{n-l} B_{l+k} (l + 1) (l + \alpha + 1) (l - k + 1) \\
- \sum_{l=k}^{n-1} \binom{n-1}{l} 2^l B_{n-l} B_{l+k} (l + \alpha + 1) (l - k). 
\]

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Acknowledgements
The authors would like to express their deep gratitude to the referees for their valuable suggestions and comments.

Received: 8 August 2012 Accepted: 6 November 2012 Published: 22 November 2012

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