UMBRAL CALCULUS ASSOCIATED WITH FROBENIUS-TYPE EULERIAN POLYNOMIALS

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ABSTRACT. In this paper, we study some properties of several polynomials arising from umbral calculus. In particular, we investigate the properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus. By using our properties, we can derive many interesting identities of special polynomials associated with Frobenius-type Eulerian polynomials. An application to normal ordering is presented.

1. INTRODUCTION

It is well known that the Euler numbers have a long history (see [7,8,13]). They are of fundamental importance in several parts of mathematics and mathematical physics (see [5–8]). In the last decades, several interesting extensions and modifications were considered along with related combinatorial, probabilistic, and statistical applications (see [9–12,16]). One of the well known extensions it is the Frobenius-Euler numbers and polynomials [5,6,13]). The aim of this paper is to study several properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus (see [22,23,30,31]). Note that umbral calculus has an application in the physics of gases (see [35]) and in the group theory and quantum mechanics (see [1,2]). Umbral calculus, in particular Sheffer sequences, has also been applied to the normal ordering of expressions involving bosonic creation and annihilation operators [3,4].

In this paper, umbral calculus is considered for some special Sheffer polynomials such as Frobenius-Euler polynomials, Changhee polynomials, Dahee polynomials and Bessel polynomials. Let $\Pi$ be the algebra structure of polynomials in a single variable $x$ over $\mathbb{C}$ and let $\Pi^\text{e}$ be the vector space of all linear functionals on $\Pi$. The action of a linear functional $L$ on a polynomial $p(x)$ is denoted by $(L|p(x))$. We note that $(L|p(x))$ satisfies $(cL + c'L|p(x)) = c(L|p(x)) + c'(L'|p(x))$, for any $c, c' \in \mathbb{C}$ and $L, L' \in \Pi^\text{e}$ (see [22,23,30,31]). Let

\begin{equation}
\mathcal{H} = \left\{ f(t) = \sum_{k \geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}.
\end{equation}

For $f(t) = \sum_{k \geq 0} a_k \frac{t^k}{k!} \in \mathcal{H}$, we define a linear functional on $\Pi$ by setting

\begin{equation}
(L|p(x)) = \sum_{k \geq 0} (L|\frac{t^k}{k!}) = a_k, \text{ for all } n \geq 0, \text{ (see } [22,23,30,31]).
\end{equation}

By (1.1) and (1.2), we have

\begin{equation}
\langle t^k | x^n \rangle = n! \delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see } [22,23,30,31]),
\end{equation}

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where $\delta_{n,k}$ is the Kronecker’s symbol. Let us assume that $f_L(t) = \sum_{k \geq 0} \langle L|x^k \rangle k!^k$. Then by (1.2), we easily obtain that $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$ and $f_L(t) = L$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphic from $\mathcal{H}$ onto $\mathcal{H}$. Henceforth, $\mathcal{H}$ is thought of as both set of formal power series and set of linear functionals. We call $\mathcal{H}$ the umbral algebra. The umbral calculus is the study of umbral algebra.

As is definition, the order $O(f(t))$ of a non-zero power series $f(t)$ is the smallest integer $k$ for which the coefficient $t^k$ does not vanish (see [22, 23, 30, 31]). If $O(f(t)) = 1$ (respectively, $O(f(t)) = 0$) then $f(t)$ is called a delta (respectively, an invertible) series. Let us assume that $f(t), g(t) \in \mathcal{H}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, so there exists a unique sequence $S_n(x)$ of polynomials with

$$\langle g(t)(f(t))^k|S_n(x) \rangle = n!\delta_{n,k}$$

for all $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [22, 23, 30, 31]). Let $f(t) \in \mathcal{H}$ and $p(x) \in \mathbb{P}$, then we note that

$$\langle e^{y|p(x)} \rangle = p(y), \quad \langle f(t)g(t)|p(x) \rangle = \langle f(t)g(t)p(x) \rangle,$$

and

$$f(t) = \sum_{k \geq 0} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k \geq 0} \langle t^k|p(x) \rangle \frac{x^k}{k!}.$$  

(see [22, 23, 30, 31]). From (1.6), we see that

$$\langle t^k|p(x) \rangle = p^{(k)}(0) \quad \text{and} \quad \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0),$$

where $p^{(k)}(0)$ denotes the $k$-th derivative of $p(x)$ respect to $x$ at $x = 0$. From (1.7) we can derive the following equation $t^k p(x) = p^{(k)}(x)$ (see [22, 23, 30, 31]). For $S_n(x) \sim (g(t), f(t))$, we have

$$\frac{1}{g(f(t))} e^{y f(t)} = \sum_{k \geq 0} S_k(y) \frac{t^k}{k!},$$

for all $y \in \mathbb{C}$, where $\tilde{f}(t)$ is the compositional inverse of $f(t)$ (see [22, 23, 30, 31]). For $S_n(x) \sim (g(t), f(t))$ and $R_n(x) \sim (h(t), \ell(t))$, let us assume that $S_n(x) = \sum_{k=0}^{n} C_{n,k} R_k(x)$. Then we see that

$$C_{n,k} = \frac{1}{k!} \frac{\langle h(f(t)) \rangle}{g(f(t))} \frac{(\ell(f(t)))^k}{k!} \langle x^n \rangle,$$

(1.9)

For all $n, k \geq 0$ (see [22, 23, 30, 31]).

Throughout this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. The Frobenius-type Eulerian polynomials of order $r$ are also given by

$$\left(\frac{1 - \lambda}{e^{t \lambda} - \lambda}\right)^r e^{xt} = \sum_{n \geq 0} A_r^{(n)}(x) \frac{t^n}{n!} \quad \text{see} \quad [7, 8, 14, 15, 19, 20],$$

(1.10)

where $r$ is a positive integer. In particular case, $x = 0$, $A_r^{(0)}(0) = A_r^{(n)}(\lambda)$ are called the $n$-th Frobenius-Euler numbers of order $r$. As is well known, the Frobenius-Euler polynomials of order $r$ are defined by the generating function to be

$$\left(\frac{1 - \lambda}{e^{t \lambda} - \lambda}\right)^r e^{xt} = \sum_{n \geq 0} F_r^{(r)}(x) \frac{t^n}{n!} \quad \text{see} \quad [23–25].$$

(1.11)
In the special case, \( x = 0 \), \( F^{(r)}_n(0, \lambda) = F^{(r)}_n(\lambda) \) are called the \( n \)-th Frobenius-Euler numbers of order \( r \). The Hermite polynomials are defined by the generating function to be

\[
(1.12) \quad e^{2xt - t^2} = \sum_{n \geq 0} H_n(x) \frac{t^n}{n!} \quad \text{(see \[21,22,30,31\]).}
\]

In the special case, \( x = 0 \), \( H_n(0) = H_n \) are called the \( n \)-th Hermite numbers. From (1.12), we note that \( H_n(x) = \sum_{j=0}^{n} 2^j \binom{n}{j} H_{n-j} x^j \). It is well known that the Poisson-Charlier sequence is given by

\[
C_n(x; a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim \left( e^{a(e^t-1)}, a(e^t - 1) \right) \quad \text{and} \quad \sum_{k \geq 0} C_n(k; a) \frac{t^k}{k!} = \left( \frac{t-a}{a} \right)^n e^t,
\]

(see \[22,23,30,31\]), where \( a \neq 0 \), \( n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \) and \((x)_k = x(x-1) \cdots (x-k+1)\). The solution of the Bessel differential equation \( x^2 y'' + 2(x+1)y' + n(n+1)y = 0 \) is given by

\[
(1.13) \quad y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)! k!} \left( \frac{x}{2} \right)^k.
\]

The Stirling numbers of the second kind is defined by the generating function to be

\[
(1.14) \quad (e^t - 1)^n = n! \sum_{j \geq n} S_n(j, n) \frac{t^j}{j!} \quad \text{(see \[22,30\]).}
\]

In this paper, we present some properties of several polynomials arising from umbral calculus. In particular, we investigate the properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus. Finally, in the last section, we establish a connection between our results and the problem of normal ordering.

2. Umbral calculus associated with Frobenius-type Eulerian polynomials

From (1.8), (1.12) and (1.10) we note that

\[
(2.1) \quad A^{(r)}_n(x|\lambda) \sim \left( \frac{e^{t(1-\lambda)} - \lambda}{1 - \lambda} \right)^r \quad \text{and} \quad H_n(x) \sim (e^{r^2/4}, t/2).
\]

Let us assume that

\[
(2.2) \quad A^{(r)}_n(x|\lambda) = \sum_{k=0}^{n} C_{n,k} H_k(x).
\]
By (1.9) and (2.2), we get that

$$C_{n,k} = \frac{1}{k!} \left( \frac{e^{t^2/4}}{(e^{(t^2/4)\lambda} - \lambda)^t} \right) \left( \frac{1}{(e^{(t^2/4)\lambda} - \lambda)^t} \right)^r |x^n|^{2^k} = \frac{1}{2^k} \binom{n}{k} \left( \frac{1 - \lambda}{(1 - \lambda)^t} \right)^r |x^{n-k}|^{2^k}$$

$$= \frac{1}{2^k} \binom{n}{k} \sum_{j=0}^{(n-k)/2} \frac{(n-k)2j}{4j!} \left( \frac{1}{(e^{(t^2/4)\lambda} - \lambda)^t} \right)^r |x^{n-k-2j}|$$

$$= \frac{1}{2^k} \binom{n}{k} \sum_{j=0}^{(n-k)/2} \frac{(2j)!}{2^{2j}j!} \binom{n-k}{2j} \left( \frac{1}{(e^{(t^2/4)\lambda} - \lambda)^t} \right)^r A_{n-k-2j}^{(r)}(x|\lambda)$$

$$= \frac{1}{2^k} \binom{n}{k} \sum_{j=0, j \text{ even}}^{n-k} \frac{(2j)!}{2^{2j}j!} \binom{n-k}{2j} A_{n-k-2j}^{(r)}(\lambda)$$

$$= \sum_{j=0, j \text{ even}}^{n-k} \frac{A_{n-k-2j}^{(r)}(\lambda)}{2^{2j}j!} |x^{n-k}|^{2^j}$$

Therefore, by (2.2), we obtain the following result.

**Theorem 2.1.** Let \( r \in \mathbb{Z}_+ \). For \( n \geq 0 \),

$$A_n^{(r)}(x|\lambda) = n! \sum_{k=0}^{n} \binom{n}{k} \frac{A_{n-k-2j}^{(r)}(\lambda)}{2^{2j}j!} |x^{n-k}|^{2^j}$$

Now, let us assume that

(2.3) \[
H_n(x) = \sum_{k=0}^{n} C_{n,k} A_k^{(r)}(x|\lambda).
\]

Then, by (1.9) and (2.3), we get

$$C_{n,k} = \frac{1}{k!} \left( \frac{e^{2t(\lambda-1) - \lambda}}{1 - \lambda} \right)^t |x^n|^{2^k} e^{-t^2}$$

$$= 2^k \binom{n}{k} \sum_{j=0}^{(n-k)/2} \frac{(-1)^j(n-k)2j}{j!} \left( \frac{e^{2t(\lambda-1) - \lambda}}{1 - \lambda} \right)^t |x^{n-k-2j}|$$

$$= \frac{2^k}{(1 - \lambda)^t} \binom{n}{k} \sum_{j=0}^{(n-k)/2} \frac{(-1)^j(n-k)2j}{j!} \left( \frac{e^{2t(\lambda-1) - \lambda}}{1 - \lambda} \right)^t |x^{n-k-2j}|$$

(2.4)

From (1.14), we note that

$$\left( e^{2t(\lambda-1) - \lambda} \right)^t = \left( e^{2t(\lambda-1) - 1} - 1 - \lambda \right)^t = \sum_{d=0}^{r} \binom{r}{d} (1 - \lambda)^{r-d} \left( e^{2t(\lambda-1) - 1} \right)^d$$

$$= \sum_{d=0}^{r} \sum_{m=0}^{d} (-1)^{r-d-j} m^{2+m-d} \binom{r}{d} (\lambda - 1)^{r+m} (m + d)! S_2(m + d, d) t^{m+d},$$
which implies
\[
(21) \quad (e^{2t(1-\lambda)} - \lambda)^r x^{n-k-2j} \\
= \sum_{d=0}^{r} \sum_{m=0}^{d} \frac{(-1)^{r-d}d!2^{m+d}d^r(2j)!\lambda^{r+m}}{(m+d)!} S_2(m+d, d)(n-k-2j)(n-k-2j-d)_m x^{n-k-2j-d}.
\]

By (24) and (25), we have
\[
C_{n,k} = r! \binom{n}{k} \sum_{j=0}^{(n-k)/2} \sum_{d=0}^{r} \frac{(-1)^{j+d}(2j)!d!}{j!} \binom{n-k}{2j} \binom{r}{d} \lambda^{n-k-2j-d}S_2(n-k-2j, d).
\]

Therefore, by (23), we can state the following result.

**Theorem 2.2.** Let \( r \in \mathbb{Z}_+. \) For all \( n \geq 0, \)
\[
H_n(x) = \sum_{k=0}^{n-1} \frac{(n-k)!}{(n-1-k)!2k!} x^{n-k} \sim (1, t - t^2/2).
\]

By (2.6), we can derive the generating function of \( J_n(x) \) as follows: \( \sum_{n \geq 0} J_n(x) t^n = e^{(1+\sqrt{1-2t})}. \) By (1.13), we easily obtain
\[
J_n(x) = \sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!2k!2^k} x^{n-k} \sim (1, t - t^2/2).
\]

Let us assume that
\[
(2.8) \quad A^{(r)}_n(x|\lambda) = \sum_{k=0}^{n} C_{n,k} J_k(x).
\]

Then, by (1.9) and (2.8), we have
\[
C_{n,k} = \binom{n}{k} \left( \frac{1-\lambda}{e^{t(\lambda-1)} - \lambda} \right)^r \left( \frac{2-t}{t} \right)^k x^{n-k} \\
= (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} \frac{C_k(j; 2)}{j!} \left( \frac{1-\lambda}{e^{t(\lambda-1)} - \lambda} \right)^r e^{-tj} x^{n-k} \\
= (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} C_k(j; 2) \binom{n-k}{j} \left( \frac{1-\lambda}{e^{t(\lambda-1)} - \lambda} \right)^r (x-1)^{n-k-j} \\
= (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} C_k(j; 2) \binom{n-k}{j} A^{(r)}_{n-k-j}(-1|\lambda).
\]

Therefore, by (2.8) we have the following result.
Theorem 2.3. Let \( r \in \mathbb{Z}_+ \). For \( n \geq 0 \), we have
\[
A_{n}^{(r)}(x|\lambda) = \sum_{k=0}^{n} \left( \sum_{j=0}^{n-k} \binom{n-k}{j} \binom{n-1}{k-j} \right) C_{k}(j;2)A_{n-k-j}^{(r)}(-1|\lambda) J_{k}(x).
\]

It is well known that Eulerian-type Chaughee polynomials are defined by the generating function to be
\[
(2.9) \quad \sum_{n=0}^{\infty} Ch_{n}(x|\lambda) = \frac{(1+t)^{\lambda-1}-\lambda}{1-\lambda} (1+t)^{x} \quad \text{(see [19,22,23,30,31]).}
\]

From (1.8) and (2.9), we can derive \( Ch_{n}(x|\lambda) \sim \frac{\Gamma(\lambda-1)}{e(\lambda-1)-\lambda}, e^{t}-1 \). Let us assume that
\[
(2.10) \quad A_{n}^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} Ch_{k}(x|\lambda).
\]

Then, by (1.9), we obtain
\[
C_{n,k} = \frac{1}{k!} \left( \frac{1-\lambda}{e(\lambda-1)-\lambda} \right)^{r+1} \left( e^{t}-1 \right)^{k} \left| x^{n} \right|
\]
\[
= \sum_{j=0}^{n-k} S_{2}(j+k,k) \left( \frac{1-\lambda}{e(\lambda-1)-\lambda} \right)^{r+1} \left| t^{k+j} x^{n} \right|
\]
\[
= \sum_{j=0}^{n-k} S_{2}(j+k,k) \binom{n}{j+k} \left( \frac{1}{e(\lambda-1)-\lambda} \right)^{r+1} x^{n-k-j}
\]
\[
= \sum_{j=0}^{n-k} S_{2}(j+k,k) \binom{n}{j+k} A_{n-k-j}^{(r+1)}(\lambda).
\]

Hence, by (2.10) we can state the following theorem.

Theorem 2.4. Let \( r \in \mathbb{Z}_+ \). For \( n \geq 0 \), we have
\[
A_{n}^{(r)}(x|\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} S_{2}(j+k,k) \binom{n}{j+k} A_{n-k-j}^{(r+1)}(\lambda) Ch_{k}(x|\lambda).
\]

Let us consider the Eulerian-type Dahee polynomials of the second kind as follows:
\[
(2.11) \quad D_{n}^{t}(x|\lambda) \sim \frac{1-\lambda}{e(\lambda-1)-\lambda} \frac{e^{t}-1}{e^{t}-\lambda}.
\]

From (1.8) and (2.11), we can derive the generating function of (2.11) as follows:
\[
\sum_{n>0} D_{n}^{t}(x|\lambda) \frac{x^{n}}{n!} = \frac{1-\lambda t^{x+\lambda-1} - \lambda(1-t)^{\lambda-1}(1-\lambda t^{x})}{(1-\lambda)(1-t)^{x+\lambda-1}},
\]
where \( t \neq 1 \). Let us assume that
\[
(2.12) \quad A_{n}^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} D_{k}^{t}(x|\lambda).
\]
Then, by (1.9) and (2.12), we have
\[
C_{n,k} = \frac{1}{k!} \left( \frac{1 - \lambda}{e^{\lambda(\lambda-1)} - \lambda} \right)^{r+1} \left( \frac{e^t - 1}{e^t - \lambda} \right)^k |x^n|
\]
\[
= \frac{1}{(1 - \lambda)^k} \sum_{j=0}^{n-k} \binom{n}{j+k} S_2(j+k,k) \left( \frac{1 - \lambda}{e^{\lambda(\lambda-1)} - \lambda} \right)^{r+1} F_{n-k-j}^{(r+1)}(x|\lambda)
\]
\[
= \frac{1}{(1 - \lambda)^k} \sum_{j=0}^{n-k} \sum_{m=0}^{n-k-j} \binom{n}{j+k} S_2(j+k,k) F_{n-k-j-m}^{(r+1)}(x|\lambda) A_m^{(r+1)}(\lambda).
\]
Therefore, by (2.12), we obtain the following result.

**Theorem 2.5.** Let \( r \in \mathbb{Z}_+ \). For \( n \geq 0 \), we have
\[
A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \left( \sum_{j=0}^{n-k} \sum_{m=0}^{n-k-j} \frac{n-k-j}{j+m} S_2(j+k,k) F_{n-k-j-m}^{(r+1)}(x|\lambda) A_m^{(r+1)}(\lambda) \right) C_h(x|\lambda).
\]

Let \( p_n(x) \sim (1,f(t)) \) and \( q_n(x) \sim (1,g(t)) \) \((n \geq 0)\). Then we note that
\[
q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x).
\]
Let us consider the following Sheffer sequences
\[
S_n(x|\lambda) \sim \left( 1, \frac{1}{1-\lambda} \lambda t \right) \quad \text{and} \quad x^n \sim (1,t).
\]
From (2.13) and (2.14), we can derive
\[
S_n(x|\lambda) = x \left( \frac{1 - \lambda}{e^{\lambda(\lambda-1)} - \lambda} \right)^n x^{n-1}
\]
and
\[
S_n(x|\lambda) = (1-\lambda)^n x(e^{\lambda(\lambda-1)} - 1 + \lambda)^{-n} x^{n-1}
\]
\[
= x \sum_{j=0}^{n} \binom{n}{j} \left( \frac{e^{\lambda(\lambda-1)}}{1-\lambda} - 1 \right)^j x^{n-1}
\]
\[
= x \sum_{j=0}^{n} \binom{n+1}{j} (1-\lambda)^{-j} (e^{\lambda(\lambda-1)} - 1)^j x^{n-1}.
\]
By (1.14), we easily get
\[
(e^{\lambda(\lambda-1)} - 1)^j = \sum_{m=0}^{j} \frac{j!}{(m+j)!} S_2(m+j,j)(\lambda-1)^{j+m} t^{j+m}.
\]
Thus, from (2.16) and (2.17), we have
\[
S_n(x|\lambda) = x \sum_{j=0}^{n} \sum_{m=0}^{n-j} \frac{j!}{(m+j)!} \binom{n}{j+m} (\lambda-1)^{j+m} t^{j+m} x^{n-1}.
\]
\[
S_n(x|\lambda) = x \sum_{j=0}^{n-j} \sum_{m=0}^{n-j} \frac{j!}{(m+j)!} \binom{n+1}{j+m} (\lambda-1)^{j+m} x^{n-1}.
\]
Hence, by (2.15) and (2.18), we obtain the following result.
Theorem 2.6. Let $r \in \mathbb{Z}_+$. For $n \geq 1$, we have

$$x A_{n-1}^{(r)}(x|\lambda) = \sum_{j=0}^{n-1} \sum_{m=0}^{n-1-j} j! \binom{n+1-j}{j} \binom{n-1}{m+j} (\lambda - 1)^m S_2(m+j,j)x^{n-1-m-j}. $$

If we consider the following Sheffer sequences

\begin{equation}
 p_n(x) \sim \left(1, \left(\frac{e^{t(\lambda - 1)}}{1 - \lambda} \right)^t \right) \text{ and } x^n \sim (1, t),
\end{equation}

then, by (2.13) and (2.19), we get

\begin{equation}
 p_n(x) = x \left(\frac{1 - \lambda}{e^{t(\lambda - 1)} - \lambda} \right)^{rn} x^{n-1} = x A_{n-1}^{(rn)}(x|\lambda).
\end{equation}

Therefore, we can state the following result.

Theorem 2.7. Let $r \in \mathbb{Z}_+$. For $n \geq 1$, we have

$$x A_{n-1}^{(rn)}(x|\lambda) \sim \left(1, \left(\frac{1 - \lambda}{e^{t(\lambda - 1)} - \lambda} \right)^r t \right).$$

3. Orthogonality-type

Let $\Pi_n = \{ p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n \}$. Then we note that $\Pi_n$ is the $(n + 1)$-dimensional vector space over $\mathbb{C}$. It is not difficult to see that $\{ A_0^{(r)}(x|\lambda), A_1^{(r)}(x|\lambda), \ldots, A_n^{(r)}(x|\lambda) \}$ is a basis for $\Pi_n$. For $p(x) \in \Pi_n$, let us assume that

\begin{equation}
 p(x) = \sum_{k=0}^{n} a_k A_k^{(r)}(x|\lambda), \ (n \geq 0).
\end{equation}

From (1.4), (2.1) and (3.1), we can derive

$$\left\langle \left(\frac{e^{t(\lambda - 1)}}{1 - \lambda} \right)^r t^k | p(x) \right\rangle = \sum_{j=0}^{n} a_j \left\langle \left(\frac{e^{t(\lambda - 1)}}{1 - \lambda} \right)^r t^k | A_j^{(r)}(x|\lambda) \right\rangle = \sum_{j=0}^{n} j! a_j \delta_{j,k} = k!a_k. $$

So,

$$a_k = \frac{1}{k!} \left\langle \left(\frac{e^{t(\lambda - 1)}}{1 - \lambda} \right)^r t^k | p(x) \right\rangle = \frac{1}{k!} \left\langle \left(\frac{e^{t(\lambda - 1)}}{1 - \lambda} \right)^r | D^k p(x) \right\rangle = \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \langle 0 | D^k p(x + j(\lambda - 1)) \rangle. $$

Therefore, by (3.1), we obtain the following theorem.

Theorem 3.1. For $r \in \mathbb{Z}_+$ and $p(x) \in \Pi_n$, let $p(x) = \sum_{k=0}^{n} a_k A_k^{(r)}(x|\lambda)$. Then

$$a_k = \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j(\lambda - 1)). $$
Now, we present several applications for the above theorem. At first, let us take

\begin{equation}
(3.2) \quad p(x) = L_n(-x) = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x^n \sim (1, t/(1-t)),
\end{equation}

where \( L_n(x) \) is the \( n \)-th Laguerre polynomial. By Theorem 3.1, we obtain

\[
a_k = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j-1) = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \sum_{m=1}^{n} \binom{r}{j} (-\lambda)^{r-j} \binom{n-1}{m-1} \binom{m}{k} \frac{n!k!}{m!} (j-1)^{m-k-1} j^{m-k}
\]

Hence, by Theorem 3.1 we can state the following theorem.

**Theorem 3.2.** For \( r, n \in \mathbb{Z}_+ \), we have

\[
L_n(-x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{m=1}^{n} \binom{r}{j} (-\lambda)^{r-j} \binom{n-1}{m-1} \binom{m}{k} \frac{n!k!}{m!} (j-1)^{m-k-1} j^{m-k} \right\} A_k^{(r)}(x|\lambda).
\]

Let us take \( p(x) = J_n(x) = \sum_{m=0}^{\infty} \frac{(n-1+m)!}{m!(n-1-m)!2^m} x^{n-m} \sim (1, t - t^2/2) \), where \( J_n(x) \) is the \( n \)-th Bessel function. By Theorem 3.1, we obtain

\[
a_k = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j-1) = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \sum_{m=0}^{n-1} \binom{r}{j} (-\lambda)^{r-j} \frac{(n-1+m)!}{m!(n-1-m)!2^m} \binom{n-m}{k} \frac{n!k!}{m!} (j-1)^{m-k-1} j^{m-k-1}
\]

Hence, by Theorem 3.1 we can state the following theorem.

**Theorem 3.3.** For \( r, n \in \mathbb{Z}_+ \), we have

\[
J_n(x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{m=0}^{n-1} \binom{r}{j} \binom{n-m}{k} \frac{(j-1)^{m-k-1} j^{m-k-1}}{m!(n-1-m)!2^m} \right\} A_k^{(r)}(x|\lambda).
\]

4. Application to normal ordering

Since the seminal work of Katriel [17] the combinatorial aspects of normal ordering arbitrary words in the creation and annihilation operators \( a^\dagger \) and \( a \) of a single-mode boson having the usual commutation relations \([a, a^\dagger] = a a^\dagger - a^\dagger a = 1, [a, a] = 0 \) and \([a^\dagger, a^\dagger] = 0 \) have been studied intensively since the seventies, see [3, 4, 17, 18, 26–29, 32–34] and references therein. From a more mathematical point of view the consequences of the noncommutative calculus of operators has been considered, in particular by Maslov [29]. Recall that normal ordering \( \mathcal{N}(F(a, a^\dagger)) \) is a functional representation of
an operator function $F(a, a^\dagger)$ in which all the creation operators stand to the left of the annihilation operators. For example, Katriel [17] showed that

$$\mathcal{N}[a^\dagger a]^n = \sum_{k=0}^{n} S_2(n, k)(a^\dagger)^k a^k.$$ 

By the properties of coherent states (for instance, see [3]), the last identity can be written as

$$\langle z | e^{a^\dagger a} | z \rangle = \sum_{n\geq 0} \langle z | (a^\dagger a)^n | z \rangle \frac{t^n}{n!} = e^{z^2(t^2 - 1)}.$$ 

Now, we can state the relation between normal ordering and Sheffer sequences as follows. Let $S_n(x) \sim (g(t), f(t))$ and $R_n(x) \sim (h(t), \ell(t))$ be any two Sheffer sequences. Then one has

$$\langle z | [M_{g,f}(a, a^\dagger)]^n | z \rangle = \sum_{k=0}^{n} \frac{1}{k!} \left\langle \begin{array}{c} h(\tilde{f}(t)) \\ g(\tilde{f}(t)) \end{array} \right\rangle^{(k)} | a^n \rangle \langle z | [M_{h,\ell}(a, a^\dagger)]^k | z \rangle,'$$

where $f$ denotes the compositional inverse of $f$ and $M_{g,f}(x, y) = \left( y - \frac{g'(x)}{g(x)} \right) \frac{f'(x)}{f(x)}$. By (4.1) and the results in the previous sections, we can obtain several nice normal ordering identities. In the following, we present several examples.

**Example 4.1.** Let $(g, f) = (1, t - t^2/2)$ and $(h, \ell) = \left( \left( \frac{e^{(1-\lambda)x} - 1}{1-\lambda} \right)^r, t \right)$, so $M_{g,f}(a, a^\dagger) = a^\dagger - \frac{(1-\lambda)e^{a(1-\lambda)}}{e^{(1-\lambda)x} - \lambda}$ and $M_{h,\ell}(a, a^\dagger) = a^\dagger - \frac{(1-\lambda)e^{a(1-\lambda)}}{e^{(1-\lambda)x} - \lambda}$. Then, by the proof of Theorem 2.1 and (4.1), we obtain

$$n! \sum_{k=0}^{n} \frac{1}{k!} \frac{A_{n-k,j}(\lambda)}{2^{r+k}j!(j/2)!} \langle z | (2a^\dagger - a)^k | 0 \rangle = \langle z | (a^\dagger - \frac{(1-\lambda)e^{a(1-\lambda)}}{e^{(1-\lambda)x} - \lambda})^n | 0 \rangle.$$

**Example 4.2.** Let $(g, f) = \left( \left( \frac{e^{(1-\lambda)x} - 1}{1-\lambda} \right)^r, t \right)$ and $(h, \ell) = (1, t - t^2/2)$, so $M_{g,f}(a, a^\dagger) = a^\dagger - \frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{(1-\lambda)x} - \lambda}$ and $M_{h,\ell}(a, a^\dagger) = a^\dagger(1-a)^{-1} = a^\dagger \sum_{j\geq 0} a^j$. Then, by the proof of Theorem 2.3 and (4.1), we obtain

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n-k}{k} C_k(j; 2) A_{n-k-j}(\lambda)}{(n-k-j)!} \langle z | (a^\dagger(1-a)^{-1})^k | 0 \rangle = \langle z | (a^\dagger - \frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{(1-\lambda)x} - \lambda})^n | 0 \rangle.$$

**REFERENCES**


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