Some Identities on the Generalized $q$-Bernoulli, $q$-Euler, and $q$-Genocchi Polynomials

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Mahmudov (2012, 2013) introduced and investigated some $q$-extensions of the $q$-Bernoulli polynomials $B_{n, q}^{(α)}(x, y)$ of order $α$, the $q$-Euler polynomials $E_{n, q}^{(α)}(x, y)$ of order $α$, and the $q$-Genocchi polynomials $G_{n, q}^{(α)}(x, y)$ of order $α$. In this paper, we give some identities for $B_{n, q}^{(α)}(x, y)$, $E_{n, q}^{(α)}(x, y)$, and $G_{n, q}^{(α)}(x, y)$ and the recurrence relations between these polynomials. This is an analogous result to the $q$-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The $q$-numbers and $q$-factorial are defined by

\begin{equation}
[a]_q = \frac{1 - q^a}{1 - q}, \quad q \neq 1, \tag{1}
\end{equation}

\begin{equation}
[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \tag{2}
\end{equation}

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The $q$-binomial coefficient is defined by

\begin{equation}
\binom{n}{k}_q = \frac{(q : q)_n}{(q : q)_{n-k}(q : q)_k}, \tag{2}
\end{equation}

where $(q : q)_n = (1 - q) \cdots (1 - q^n)$. The $q$-analogue of the function $(x + y)^n$ is defined by

\begin{equation}
(x + y)^n_q = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k(k-1))/2} x^{n-k} y^k. \tag{3}
\end{equation}

The $q$-binomial formula is known as

\begin{equation}
(n; q)_a = (1 - a q^n) \bar{q}^{n-1} \prod_{j=0}^{n-1} (1 - q_j a), \quad 0 < q < 1, |z| < 1.
\end{equation}

The $q$-exponential functions are given by

\begin{equation}
e_q(z) = \sum_{n=0}^\infty \frac{z^n}{[n]_q!}, \quad 0 < |q| < 1, |z| < \frac{1}{1 - |q|}. \tag{4}
\end{equation}
From these forms, we easily see that \( e_q(z)E_q(-z) = 1 \). Moreover, \( D_qe_q(z) = e_q(z) \) and \( D_qE_q(z) = E_q(qz) \), where \( D_q \) is defined by

\[
D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.
\]

The previous \( q \)-standard notation can be found in \([1, 2]\). Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials \([3, 4]\). There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. \([5, 6]\), Choi et al. \([1]\), Cheon \([7]\), Kim \([8]\), Kurt \([9]\), Kurt \([10]\), Luo and Srivastava \([11–13]\), Srivastava et al. \([14, 15]\), Natalini and Bernardini \([16]\), Tremblay et al. \([17, 18]\), Gaboury and Kurt \([19]\), Mahmudov \([20, 21]\), Araci et al. \([22]\), and Kupershmidt \([23]\).

Mahmudov defined and studied the properties of the following generalized \( q \)-Bernoulli numbers \( B_{n,q}(\alpha) \) and \( q \)-Euler polynomials \( E_{n,q}(\alpha, x, y) \) of order \( \alpha \) as follows \([2]\).

Let \( q \in \mathbb{C}, \alpha \in \mathbb{N}, \) and \( 0 < |q| < 1 \). The \( q \)-Bernoulli numbers \( B_{n,q}(\alpha) \) and polynomials \( B_{n,q}(\alpha, x, y) \) in \( x \) and \( y \) of order \( \alpha \) are defined by means of the generating functions:

\[
\sum_{n=0}^{\infty} B_{n,q}(\alpha) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^\alpha, \quad |t| < 2\pi, \quad \alpha \\
\sum_{n=0}^{\infty} B_{n,q}(\alpha, x, y) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(tx) E_q(ty), \quad |t| < 2\pi.
\]

The \( q \)-Euler numbers \( \mathcal{E}_{n,q}(\alpha) \) and polynomials \( \mathcal{E}_{n,q}(\alpha, x, y) \) in \( x \) and \( y \) of order \( \alpha \) are defined by means of the generating functions:

\[
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\alpha) \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha, \quad |t| < \pi, \quad \alpha \\
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\alpha, x, y) \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx) E_q(ty), \quad |t| < \pi.
\]

The familiar \( q \)-Stirling numbers \( S_{2,q}(n, k) \) of the second kind are defined by

\[
\left( \frac{e_q(t) - 1}{k} \right)^n = \sum_{n=0}^{\infty} S_{2,q}(n, k) \frac{t^n}{[n]_q!}.
\]

It is obvious that

\[
\mathcal{B}_{n,q}(1)(x, y) := \mathcal{B}_{n,q}(x, y), \quad \mathcal{E}_{n,q}(1)(x, y) := \mathcal{E}_{n,q}(x, y), \\quad \alpha \\
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\mathcal{B}_{n,q}(0, 0) := \mathcal{B}_{n,q}, \quad \mathcal{E}_{n,q}(0, 0) := \mathcal{E}_{n,q}, \quad \alpha \\
\mathcal{B}_{n,q}(\alpha) := \mathcal{B}_{n,q}(0, 0), \quad \mathcal{E}_{n,q}(\alpha) := \mathcal{E}_{n,q}(0, 0), \quad \alpha \\
\lim_{q \to 1^-} \mathcal{B}_{n,q}(\alpha, x, y) = \mathcal{B}_{n}(\alpha, x + y), \quad \lim_{q \to 1^-} \mathcal{E}_{n,q}(\alpha, x, y) = \mathcal{E}_{n}(\alpha, x + y), \quad \alpha \\
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\lim_{q \to 1^-} \mathcal{B}_{n,q}(\alpha) = \mathcal{B}_{n}(\alpha), \quad \lim_{q \to 1^-} \mathcal{E}_{n,q}(\alpha) = \mathcal{E}_{n}(\alpha), \quad \alpha.
\]

From (8) and (10), it is easy to check that

\[
\mathcal{B}_{n,q}(\alpha, x, y) = \sum_{k=0}^{n} \binom{n}{k} q_{n-k,q}(x,0) \mathcal{B}_{k,q}((a-1), 0, y), \quad \alpha \\
\mathcal{E}_{n,q}(\alpha, x, y) = \sum_{k=0}^{n} \binom{n}{k} q_{n-k,q}(x,0) \mathcal{E}_{k,q}((a-1), 0, y).
\]

In this work, we give some identities for the \( q \)-Bernoulli polynomials. Also, we give some relations between the \( q \)-Bernoulli polynomials and \( q \)-Euler polynomials and the \( q \)-Genocchi polynomials and \( q \)-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.
Theorem 1. There are the following relations between the $q$-Bernoulli polynomials and $q$-Stirling numbers of the second kind:

$$\mathcal{S}_{n,q}^{(\alpha)}(x, y) = \left[\frac{1}{n+1} \right]_q \sum_{k=0}^{n} \binom{n+k}{l}_q \mathcal{S}_{k,q}^{(\alpha)} (x, y) \times S_{2,q}(n + k - l, k),$$

where $q \in \mathbb{C}, \alpha, n \in \mathbb{N}$, and $0 < |q| < 1$.

Theorem 2. The $q$-Stirling numbers of the second kind satisfy the following relations:

$$\mathcal{S}_{n,q}^{(\alpha)}(x, y) = \sum_{j=0}^{\infty} \left( -\alpha \right)_j^q \frac{1}{j!} [j]_q !$$

$$\times \sum_{p=0}^{n} \binom{n}{p}_q S_{2,q}(n - p, j)$$

$$\times \sum_{l=0}^{p} \binom{p}{l}_q x^{p-l} y^l \left( \frac{1}{2} \right)_q$$

$$\mathcal{B}_{n,q}^{(\alpha)} = \left[\alpha \right]_q \sum_{j=0}^{\infty} \left( -\alpha \right)_j^q$$

$$\times \sum_{k=0}^{j} \binom{j}{k}_q S_{2,q}(n + k, k) [n]_q ! [-k]_q q^{-k}$$

$$\mathcal{B}_{n,q}^{(-\alpha)}(x, y) = \left[\alpha \right]_q \sum_{m=0}^{n} \binom{n+\alpha}{m}_q S_{2,q}(m, \alpha)$$

$$\times (x + y)^{n-m} [n]_q !$$

where $q \in \mathbb{C}, \alpha, n \in \mathbb{N}$, and $0 < |q| < 1$.

Theorem 3. The $q$-Euler polynomials satisfy the following relation:

$$\sum_{k=0}^{n} \binom{n}{k}_q \mathcal{E}_{k,q}^{(\alpha)}(x, y) = 2(x + y)_q^n - \mathcal{E}_{n,q}(x, y),$$

where $q \in \mathbb{C}, \alpha, n \in \mathbb{N}$, and $0 < |q| < 1$.

Theorem 4. The polynomials $B_{n,q}(x, y)$ and $\mathcal{G}_{n,q}(x, y)$ satisfy the following difference relationships:

$$\mathcal{B}_{n,q}(x, y) = \sum_{l=0}^{n+1} \binom{n+1}{l}_q \frac{1}{[n+1]_q} \mathcal{G}_{l+1,q}^{(\alpha)}(x, y) \mathcal{B}_{n+1-l,q}^{(\alpha)}$$

$$\mathcal{G}_{n,q}(x, y) = -2 \sum_{l=0}^{n} \binom{n}{l}_q \frac{1}{[l+1]_q} \mathcal{G}_{l+1,q}^{(\alpha)}(x, y) \mathcal{B}_{n-l,q}^{(\alpha)}$$

where $q \in \mathbb{C}, \alpha, n \in \mathbb{N}$, and $0 < |q| < 1$.

Theorem 5. There is the following relation between the generalized $q$-Euler polynomials and generalized $q$-Bernoulli polynomials:

$$\mathcal{G}_{n,q}^{(\alpha)}(x, y) = \left[\sum_{l=0}^{n+1} \binom{n+1}{l}_q \frac{1}{[n+1]_q} \mathcal{B}_{l,q}^{(\alpha)}(mx, 0) \right]$$

$$\mathcal{B}_{n+1-l,q}^{(\alpha)}(mx, 0)$$

$$\times \sum_{m=0}^{n} \binom{n+1}{l}_q \mathcal{G}_{l,q}^{(\alpha)}(0, y) m^l n^l$$

where $q \in \mathbb{C}, \alpha, n \in \mathbb{N}$, and $0 < |q| < 1$.

2. Proof of the Theorems

Lemma 6. The generalized $q$-Bernoulli polynomials, $q$-Euler polynomials, and $q$-Genocchi polynomials satisfy the following relations:

$$\sum_{k=0}^{n} \binom{n}{k}_q \mathcal{B}_{k,q}(x, y) \mathcal{B}_{n-k,q}^{(-\alpha)} = (x + y)_q^n$$

$$\sum_{k=0}^{n} \binom{n}{k}_q \mathcal{B}_{k,q}^{(\alpha)}(0, y) \mathcal{B}_{n-k,q}^{(-\alpha)} = q^{(n(n-1))/2} y^n$$

$$\mathcal{B}_{n,q}^{(\alpha)}(x, y) = \sum_{l=0}^{n} \binom{n}{l}_q \mathcal{B}_{l,q}^{(\alpha)}(0, y) \mathcal{B}_{n-l,q}^{(-\alpha)}$$

$$\times \sum_{k=0}^{l} \binom{l}{k}_q \mathcal{B}_{k,q}(x, 0) \mathcal{B}_{l-k,q}^{(-\alpha)}$$
\[ \mathcal{G}_{n,k}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k,q}^{(\alpha)}(x,0) B_{-k,q}^{(-\alpha)}(0,y) \]
\[ = 2[n]_q(x+y)^{n-1}, \]
\[ \mathcal{G}_{n,q}^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k,q}^{(\alpha)}(x,0) B_{-k,q}^{(-\beta)}(x,0). \]

(24)

**Proof.** The proof of this lemma can be found from (7)–(12).

**Proof of Theorem 1.** By (8) and (13) we have
\[ \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right) e_q(tx) E_q(ty) \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k,q}^{(\alpha+k)}(x,0) S_{2,q}(n-k) \frac{t^n}{[n]_q!} \]
\[ \times (x,y) S_{2,q}(n-l,k) \frac{t^n}{[n]_q!}. \]

(25)

Comparing the coefficients of \((t^n/[n]_q!)\), we find (18). Similarly, we have (19).

**Proof of Theorem 2.** Combining (10) and (13), we obtain
\[ \left( \frac{2}{e_q(t) + 1} \right)^\alpha = \left( 1 + \frac{e_q(t) - 1}{2} \right)^{(-\alpha)} \]
\[ = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left( \frac{e_q(t) - 1}{2} \right)^j, \]
\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left( \frac{e_q(t) - 1}{2} \right)^j e_q(tx) E_q(ty) \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k,q}^{(\alpha)}(x,0) S_{2,q}(n-k) \frac{t^n}{[n]_q!} \]
\[ \times \sum_{l=0}^{p} \binom{p}{l} \frac{t^n}{[n]_q!} \times \sum_{k=0}^{\infty} \sum_{l=0}^{n-k} \binom{n+k}{l} \mathcal{G}_{l,q}^{(\alpha+k)}(x,0) S_{2,q}(n+l-k) \frac{t^n}{[n]_q!}. \]

(26)

Equating the coefficients of \((t^n/[n]_q!)\), we obtain (16). Similarly, we have (17).
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\[ - \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} \]

\[ \times \sum_{n=0}^{\infty} (x + y)^{\frac{n}{q}} \frac{t^n}{[n]_q!}. \]  

(28)

Using the Cauchy product and comparing the coefficients of \( t^n/[n]_q! \), we have

\[ \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \mathcal{G}_{k,q}(x, y) = 2(x + y)^{\frac{n}{q}} - \mathcal{G}_{k,q}(x, y). \]  

(29)

Finally, we consider the interesting relationships between the \( q \)-Bernoulli polynomials and \( q \)-Genocchi polynomials and the \( q \)-Euler polynomials and \( q \)-Bernoulli polynomials. These relations are \( q \)-analogues to the Srivastava-Pintér addition theorems.

**Proof of Theorem 4.** It follows immediately that

\[ \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \frac{1}{2} \frac{2te_q(tx)E_q(ty)}{e_q(t) + 1} \]

\[ + \frac{1}{f} \frac{2t}{e_q(t) - 1} \frac{e_q(tx)E_q(ty)}{e_q(t) + 1} \]

\[ \times \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x, y) \frac{t^n}{[n]_q!}. \]

(30)

Equating the coefficients of \( t^n/[n]_q! \), we have (21).

In a similar fashion, (12) yields

\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \frac{1}{f} \left( \frac{2t}{e_q(t) - 1} \right) \left( \frac{te_q(tx)E_q(ty)}{e_q(t) + 1} \right) \]

\[ = \frac{1}{f} \left( 2t - 2 \frac{2t}{e_q(t) + 1} \right) \left( \frac{t}{e_q(t) - 1} e_q(tx)E_q(ty) \right) \]

(31)

Comparing the coefficients of \( t^n/[n]_q! \), we have (22).

**Proof of Theorem 5.** By (10), we write

\[ \sum_{n=0}^{\infty} \mathcal{G}^{(n)}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \left( \frac{2}{e_q(t) + 1} \right)^{\alpha} \]

\[ \times E_q(ty) \frac{e_q(t/m) - 1}{e_q(t/m)} \]

(32)
\[
= \sum_{n=0}^{\infty} \frac{g_{n,q}^{(\alpha)}(x,y)}{[n]_q^!} t^n
\]

By equating the coefficients of \((t^n/[n]_q^!))\), we get the theorem.

**Remark 7.** There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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