The Lehmer matrix and its recursive analogue

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Abstract

This paper considers the Lehmer matrix and its recursive analogue. The determinant of Lehmer matrix is derived explicitly by both its LU and Cholesky factorizations. We further define a generalized Lehmer matrix with $(i, j)$ entries $g_{ij} = \min\{u_{i+1}, u_{j+1}\}$ where $u_n$ is the $n$th term of a binary sequence $\{u_n\}$. We derive both the LU and Cholesky factorizations of this analogous matrix and we precisely compute the determinant.

1 Introduction

D.H. Lehmer (see [2]) constructed an $n \times n$ symmetric matrix $A = (a_{ij})_{i,j}$ whose $(i, j)$ entry is

$$a_{ij} = \frac{\min\{i, j\}}{\max\{i, j\}} = \left\{ \begin{array}{ll} i/j & j \geq i, \\ j/i & i > j. \end{array} \right.$$ 

Define the second order recurrence $\{U_n (p, q)\}$ as follows:

$$U_n (p, q) = pU_{n-1} (p, q) - qU_{n-2} (p, q),$$
This paper considers the Lehmer matrix and its recursive analogue. The determinant of Lehmer matrix is derived explicitly by both its LU and Cholesky factorizations. We further define a generalized Lehmer matrix with \((i; j)\) entries \(g_{ij} = \min f_{i+1}; u_{j+1}\) \(\max f_{i+1}; u_{j+1}\) where \(u_n\) is the \(n\)th term of a binary sequencefung. We derive both the LU and Cholesky factorizations of this analogous matrix and we precisely compute the determinant.
where \( U_0 (p, q) = 0 \) and \( U_1 (p, q) = 1 \) for \( n > 1 \).

As an interesting example, we mention that the set of natural numbers can be obtained from the sequence \( \{ U_n (p, q) \} \) by taking \( p = 2, q = 1 \). Throughout this paper, we consider the case \( q = -1 \) and we denote \( u_n = U_n (p, -1) \).

We now define an \( n \times n \) generalized Lehmer matrix, namely \( F_n = (g_{ij})_{1 \leq i, j \leq n} \) defined below:

\[
g_{ij} = \frac{\min \{ u_{i+1}, j+1 \}}{\max \{ u_{i+1}, u_{j+1} \}} = \begin{cases} 
\frac{u_{i+1}}{u_{j+1}} & \text{if } j \geq i, \\
\frac{u_{j+1}}{u_{i+1}} & \text{if } i > j.
\end{cases}
\]

where \( u_n \) is the \( n \)th term of the sequence \( \{ u_n \} \). In this paper, we obtain the general LU factorization and other explicit formulas for both the Lehmer matrix and its recursive analogue.

The Lehmer matrix is part of a family of matrices known as test matrices, which are used to evaluate the accuracy of matrix inversion programs since the exact inverses are known (see [1, 2]). It is hoped that our generalized Lehmer matrix will add to the literature of special matrices with known inverse.

## 2 The Lehmer Matrix

We start by obtaining the LU factorization of the Lehmer matrix \( A \). Using the inverses of \( L \) and \( U \), we obtain the explicit form for the inverse of \( A \), whose inverse is well-known, thus obtaining another proof of this result.

We define the \( n \times n \) invertible lower triangular matrix \( L = (\ell_{ij}) \) where \( \ell_{ij} = j/i \) for \( i \geq j \) and 0 otherwise. Next, we define the \( n \times n \) invertible upper triangular matrix \( U = (u_{ij}) \) with \( u_{ij} = \frac{2^{i-1}}{ij} \) for
\(i \leq j\) and 0 otherwise. For example, when \(n = 5\), we get

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 1 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & 1 \\
\end{bmatrix}
\quad \text{and} \quad
U = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
0 & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \frac{3}{10} \\
0 & 0 & \frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\
0 & 0 & 0 & \frac{7}{16} & \frac{7}{20} \\
0 & 0 & 0 & 0 & \frac{9}{25} \\
\end{bmatrix}.
\]

The following result holds.

**Theorem 1.** For \(n > 0\), the LU factorization of Lehmer matrix is given by

\[A = LU\]

where \(L\) and \(U\) were defined previously.

**Proof.** We split the proof into three cases.

**Case 1:** \(i = j\). By \(\sum_{k=1}^{i} (2k - 1) = i^2\), then

\[
a_{ii} = \sum_{k=1}^{n} \ell_{ik}u_{ki} = \sum_{k=1}^{i} \ell_{ik}u_{ki} = \sum_{k=1}^{i} \frac{k}{i} (2k - 1) = \sum_{k=1}^{i} \frac{2k - 1}{i^2} = 1.
\]

**Case 2:** \(i > j\). Thus

\[
a_{ij} = \sum_{k=1}^{n} \ell_{ik}u_{kj} = \sum_{k=1}^{j} \ell_{ik}u_{kj} = \sum_{k=1}^{j} \frac{k}{ij} (2k - 1) = \frac{1}{ij} \sum_{k=1}^{j} 2k - 1 = \frac{j}{i}.
\]

**Case 3:** \(j > i\). Then

\[
a_{ij} = \sum_{k=1}^{n} \ell_{ik}u_{kj} = \sum_{k=1}^{i} \ell_{ik}u_{kj} = \sum_{k=1}^{i} \frac{k}{ij} (2k - 1)
\]

\[= \frac{1}{ij} \sum_{k=1}^{i} 2k - 1 = \frac{i}{j}.
\]
which completes the proof. \qed

We display an example below:

\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\
\frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{3}{4} & \frac{3}{6} & \frac{3}{8} & \frac{3}{10} \\
\frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\
\frac{7}{15} & \frac{7}{20} \\
\frac{9}{25}
\end{bmatrix}.
\]

As a consequence of Theorem 1, we obtain an explicit value of the determinant of the Lehmer matrix in the following corollary.

**Corollary 1.** For \( n > 0 \),

\[
\det A = \frac{(2n)!}{2^n (n!)^3}
\]

**Proof.** The proof follows from the LU factorization of matrix \( A \) by considering \( \det A = \det U = \prod_{i=1}^{n} \frac{2i-1}{i^2} \).

The \( n \)th Catalan number is given in terms of binomial coefficients by

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}
\]

Thus we may note that

\[
\det A = \frac{(n+1)}{2^n n!} C_n.
\]

We continue our analysis by determining the \( L_1L_1^T \) (named after Cholesky) factorization of the Lehmer matrix, where \( L_1 \) is a lower triangular matrix. The Cholesky factorization was obtained for a different kind of matrix defined using binary sequences by the second author in [3].
Theorem 2. The Cholesky factorization of the Lehmer matrix is given by
\[ A = L_1 L_1^T \]
where \( L_1 = \{f_{ij}\} \) is a lower triangular matrix with \( f_{ij} = \frac{\sqrt{2i-1}}{i} \) for all \( i \geq j \).

Proof. If \( i > j \), then
\[
a_{ij} = \sum_{r=1}^{n} f_{ir} f_{jr} = \sum_{r=1}^{j} f_{ir} f_{jr} = \sum_{r=1}^{j} \frac{\sqrt{2r-1}}{i} \frac{\sqrt{2r-1}}{j} = \frac{ij}{ij} \sum_{r=1}^{j} (2r - 1) = \frac{j}{i}.
\]

If \( i = j \), then
\[
a_{ii} = \sum_{r=1}^{n} f_{ir}^2 = \sum_{r=1}^{i} f_{ir}^2 = \sum_{r=1}^{i} \left( \frac{\sqrt{2r-1}}{i} \right)^2 = \frac{i^2}{i^2} = 1.
\]

Finally, if \( i < j \), then
\[
a_{ij} = \sum_{r=1}^{n} f_{ir} f_{jr} = \sum_{r=1}^{i} f_{ir} f_{jr} = \frac{1}{ij} \sum_{r=1}^{i} (2r - 1) = \frac{i}{j},
\]
which proves the theorem.

As an example, for \( n = 5 \) and \( p = 1 \) (the Fibonacci sequence case), we have
\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\
\frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\
\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
\frac{\sqrt{4}}{4} & \frac{\sqrt{4}}{4} & \frac{\sqrt{4}}{4} & 0 & 0 \\
\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\
\frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{4}}{4} & \frac{\sqrt{5}}{5} \\
0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{4}}{4} & \frac{\sqrt{5}}{5} & \frac{\sqrt{6}}{6} \\
0 & 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{6}}{6} & \frac{\sqrt{7}}{7} \\
0 & 0 & 0 & \frac{\sqrt{7}}{7} & \frac{\sqrt{8}}{8} \\
0 & 0 & 0 & 0 & \frac{\sqrt{8}}{8}
\end{bmatrix}.
\]
By Theorem 2, we find that, since $A = L_1 L_1^T$, we have that \( \det(A) = \prod_{i=1}^{n} f_{ii}^2 = \prod_{i=1}^{n} (\frac{2i-1}{i^2}) = \frac{(2n)!}{2^n (n!)^2} \), that is, Corollary 1.

3 The Inverse of the Lehmer Matrix

Now we find an explicit formula for the inverse of the Lehmer matrix. For this purpose, we use its LU factorization as $A^{-1} = U^{-1} L^{-1}$. We first derive the inverses of the matrices $L$ and $U$.

**Lemma 1.** Let $L^{-1} = (t_{ij})$ denote the inverse of $L$. Then

\[
t_{ij} = \begin{cases} 
  1 & \text{if } i = j, \\
  -\frac{i}{2} & \text{if } i = j + 1, \\
  0 & \text{otherwise},
\end{cases}
\]

**Proof.** The proof can be easily checked from the product $L^{-1} L$. \( \square \)

**Lemma 2.** Let $U^{-1} = (w_{ij})$ denote the inverse of $U$. Then

\[
w_{ij} = \begin{cases} 
  \frac{i^2}{2i-1} & \text{if } i = j \\
  -\frac{i(i+1)}{2i+1} & \text{if } i + 1 = j, \\
  0 & \text{otherwise},
\end{cases}
\]

**Proof.** The proof follows from the product $U^{-1} U$. \( \square \)

The inverse of the Lehmer matrix is found in the following theorem.

**Theorem 3.** For $n > 0$, let $A^{-1} = (b_{ij})$, then

\[
b_{ij} = \begin{cases} 
  \frac{4i^3}{4n^2 - 1} & \text{if } i = j < n \\
  \frac{2i - 1}{2n - 1} & \text{if } i = j = n, \\
  -\frac{i(i+1)}{2i+1} & \text{if } |i - j| = 1, \\
  0 & \text{otherwise},
\end{cases}
\]
Proof. Since $A^{-1} = U^{-1}L^{-1}$, using the previous two lemmas, we obtain for $1 \leq i \leq n-1$,

$$b_{ii} = \sum_{k=1}^{n} w_{ik} t_{ki} = w_{ii} + w_{i,i+1} t_{i,i+1},$$

$$= \frac{i^2}{2i-1} + \frac{i(i+1)}{2i+1} \frac{i}{(i+1)} = \frac{i^2}{2i-1} + \frac{i^2}{2i+1} = \frac{4i^3}{4i^2 - 1}.$$

When $i = j = n$, it is easy to see that $b_{nn} = w_{nn} = \frac{n^2}{2n-1}$. If $i = j + 1$, then

$$b_{i+1,i} = \sum_{k=1}^{n} w_{i+1,k} t_{ki} = w_{i+1,i+1} t_{i+1,i},$$

$$= \frac{(i+1)^2}{2i+1} \left( \frac{-i}{i+1} \right) = -\frac{i(i+1)}{2i+1}.$$

The last case $j = i + 1$ can be similarly done, and the proof is complete.

Therefore we recover the known fact that the inverse of the Lehmer matrix is a symmetric tridiagonal matrix.

We give the following example as a consequence of the above theorem: for $n = 4$,

$$A^{-1} = \begin{bmatrix}
\frac{4}{3} & -\frac{2}{3} & 0 & 0 \\
-\frac{2}{3} & \frac{32}{15} & -\frac{6}{5} & 0 \\
0 & -\frac{6}{5} & \frac{108}{35} & -\frac{12}{7} \\
0 & 0 & -\frac{12}{7} & \frac{16}{7}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{4}{3} & -\frac{6}{5} & 0 \\
0 & 0 & \frac{9}{5} & -\frac{12}{7} \\
0 & 0 & 0 & \frac{16}{7}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & 0 \\
0 & -\frac{2}{3} & 1 & 0 \\
0 & 0 & -\frac{3}{4} & 1
\end{bmatrix}.$$
We also give a relation between the terms of inverse of the Lehmer matrix and triangular numbers. Recall that the $n$th triangular number $T_n$ is defined as the sum of the first $n$ natural numbers, that is, $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We can re-write $A^{-1} = (b_{ij})$ as $b_{ij} = -\frac{2T_i}{2i+1}$ for $|i - j| = 1$, and $b_{ii} = \frac{4i^3}{4i^2-1}$.

4 Recursive Analogue of the Lehmer Matrix

In this section we investigate the same questions for our generalized recursive analogue of the Lehmer matrix $F_n$ defined in the first section, namely, $F_n = (g_{ij})$:

$$g_{ij} = \frac{\min \{u_{i+1}, u_{j+1}\}}{\max \{u_{i+1}, u_{j+1}\}} = \begin{cases} \frac{u_{i+1}}{u_{j+1}} & \text{if } j \geq i, \\ \frac{u_{j+1}}{u_{i+1}} & \text{if } i > j. \end{cases}$$

where $u_n$ is the $n$th term of the sequence $\{u_n\}$.

For example, when $n = 5$ and $p = 1$, the matrix $F_5$ takes the following form:

$$F_5 = \begin{pmatrix}
1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\
\frac{u_2}{u_3} & 1 & \frac{u_3}{u_4} & \frac{u_3}{u_5} & \frac{u_3}{u_6} \\
\frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & \frac{u_4}{u_5} & \frac{u_4}{u_6} \\
\frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & \frac{u_5}{u_6} \\
\frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1
\end{pmatrix}.$$

In order to give the LU factorization of the matrix $F_n$, we define two triangular matrices.

Define the $n \times n$ unit lower triangular matrix $L_2 = (c_{ij})$ with $c_{ij} = \frac{u_{i+1}}{u_{i+1}}$ for all $i \geq j$ and $u_{ij} = 0$ for all $i < j$. 

For example, when \( n = 5 \), the matrix takes the form:

\[
L_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{u_2}{u_3} & 1 & 0 & 0 & 0 \\
\frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & 0 & 0 \\
\frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & 0 \\
\frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1
\end{bmatrix}.
\]

Before defining an upper triangular matrix for the LU factorization of the matrix \( F_n \), we need to introduce a new sequence \( \{t_n\} \) by the following relation:

\[
t_n = (p - 1) u_n + u_{n-1}, \text{ that is, } t_n = u_{n+1} - u_n, \ n > 1,
\]

where \( u_n \) is defined as before.

Define the \( n \times n \) upper triangular matrix \( U_2 = (d_{ij}) \) with \( d_{1j} = \frac{u_2}{u_{j+1}} \) for \( 1 \leq j \leq n \), \( d_{ij} = \frac{(u_i + u_{i+1})}{u_{i+1}u_{j+1}} t_i \) for \( 1 < i \leq j \leq n \).

From the definition of the sequence \( \{t_n\} \), we rewrite the matrix \( U_2 \) with \( d_{1j} = \frac{u_2}{u_{j+1}} \) for \( 1 \leq j \leq n \), \( d_{ij} = \frac{u_{i+1} - u_i}{u_{i+1}u_{j+1}} \) for \( 1 < i \leq j \leq n \).

For example, when \( n = 4 \), the matrix takes the form:

\[
U_2 = \begin{bmatrix}
1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\
0 & \frac{u_2^2-u_3^2}{u_3^2} & \frac{u_2^2-u_3^2}{u_3u_4} & \frac{u_2^2-u_3^2}{u_3u_5} & \frac{u_2^2-u_3^2}{u_3u_6} \\
0 & 0 & \frac{u_2^2-u_3^2}{u_4^2} & \frac{u_2^2-u_3^2}{u_4u_5} & \frac{u_2^2-u_3^2}{u_4u_6} \\
0 & 0 & 0 & \frac{u_2^2-u_3^2}{u_5^2} & \frac{u_2^2-u_3^2}{u_5u_6} \\
0 & 0 & 0 & 0 & \frac{u_2^2-u_3^2}{u_6^2}
\end{bmatrix}.
\]

**Theorem 4.** For \( n > 0 \), the factorization of matrix \( F_n = (g_{ij}) \) is given by

\[
F_n = L_2 U_2.
\]
where $U_2$ and $L_2$ were defined previously.

Proof. Let $L_2U_2 = (h_{ij})$. We consider two cases, $i > j$ and $i \leq j$. For the first case, we write

$$h_{ij} = \sum_{m=1}^{n} c_{im}d_{mj} = \sum_{m=1}^{j} c_{im}d_{mj}$$

$$= c_{i1}d_{1j} + \sum_{m=2}^{j} \left( \frac{u_{m+1} \left( u_{m+1}^2 - u_m^2 \right)}{u_{i+1}u_{m+1}u_{j+1}} \right)$$

$$= \frac{u_2^2}{u_{i+1}u_{j+1}} + \frac{1}{u_{i+1}u_{j+1}} \sum_{m=2}^{j} \left( u_{m+1}^2 - u_m^2 \right)$$

$$= \frac{u_2^2}{u_{i+1}u_{j+1}} + \frac{1}{u_{i+1}u_{j+1}} \left( u_{j+1}^2 - u_i^2 \right) = \frac{u_{j+1}}{u_{i+1}} = g_{ij}.$$

If $i \leq j$, then similarly

$$h_{ij} = \sum_{m=1}^{n} c_{im}d_{mj} = \sum_{m=1}^{i} c_{im}d_{mj}$$

$$= c_{i1}d_{1j} + \sum_{m=2}^{i} \left( \frac{u_{m+1} \left( u_{m+1}^2 - u_m^2 \right)}{u_{i+1}u_{m+1}u_{j+1}} \right)$$

$$= \frac{u_2^2}{u_{i+1}u_{j+1}} + \frac{1}{u_{i+1}u_{j+1}} \sum_{m=2}^{i} \left( u_{m+1}^2 - u_m^2 \right)$$

$$= \frac{u_{i+1}}{u_{j+1}} = g_{ij},$$

and the claim is shown. \qed

Now we can find the value of $\det(F_n)$ by considering its $LU$ factorization.

**Corollary 2.** For $n > 0$,

$$\det(F_n) = \prod_{i=2}^{n} \left( \frac{u_{i+1}^2 - u_i^2}{u_i^2} \right).$$
As a special cases of the matrix $F_n$, we take the matrix $F^0_n$ obtained using the Fibonacci sequence, that is, $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$. The determinant of this matrix becomes

$$
\det (F^0_n) = \frac{F_{n-1}!F_{n+2}!}{2(F_{n+1}!)^2},
$$

where $F_n!$ is the Fibonomial factorial, that is, $F_n! = F_1F_2\cdots F_n$.

Next we give the Cholesky factorization of the generalized Lehmer matrix $F_n$. For this purpose we define a lower triangular matrix $L_3$ with $m_{i,1} = \frac{u_2}{u_{i+1}}$ for $1 \leq i \leq n$, $m_{ij} = \frac{1}{u_{i+1}}\sqrt{u_{j+1}^2 - u_j^2}$ for $1 < j \leq i \leq n$ and 0 otherwise.

When $n = 4$, the matrix $L_3$ takes the form:

$$
L_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{u_2}{u_3} & \frac{1}{u_4}\sqrt{u_3^2 - u_2^2} & 0 & 0 \\
\frac{u_2}{u_4} & \frac{1}{u_5}\sqrt{u_4^2 - u_3^2} & \frac{1}{u_6}\sqrt{u_5^2 - u_4^2} & 0 \\
\frac{u_2}{u_5} & \frac{1}{u_6}\sqrt{u_5^2 - u_4^2} & \frac{1}{u_7}\sqrt{u_6^2 - u_5^2} & \frac{1}{u_8}\sqrt{u_7^2 - u_6^2}
\end{bmatrix}.
$$

The proof of the next theorem is analogous to the proof of Theorem 4, so it will be omitted.

**Theorem 5.** The Cholesky factorization of the recursive analogue of the Lehmer matrix is given by

$$
F_n = L_3L_3^T
$$

where $L_3$ is the lower triangular matrix defined previously.

## 5 The Inverse of the Generalized Lehmer Matrix

Here we give the inverse of the recursive analogue of the Lehmer matrix $F^{-1}_n$ by considering its LU factorization. Before this, we give the inverses of the matrices $L_2$ and $U_2$ in the following lemmas, stated without proofs, as they are immediate.
Lemma 3. Let $U_2^{-1} = (\hat{w}_{ij})$ denote the inverse of $U_2$. Then

$$
\hat{w}_{ij} = \begin{cases} 
1 & \text{if } i = j = 1 \\
-\frac{u_i^2}{u_i^2 - u_{i+1}^2} & \text{if } 1 < i = j, \\
\frac{u_{i+1}^2}{u_{i+1}^2 - u_{i+2}^2} & \text{if } i + 1 = j, \\
0 & \text{otherwise,}
\end{cases}
$$

Lemma 4. Let $L_2^{-1} = (\hat{t}_{ij})$ denote the inverse of $L$. Then

$$
\hat{t}_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
-\frac{u_i}{u_{i+1}} & \text{if } i = j + 1, \\
0 & \text{otherwise,}
\end{cases}
$$

Thus the inverse of the matrix $F_n$ is found in the following theorem.

Theorem 6. For $n > 0$, let $F_n^{-1} = (q_{ij})$, then $q_{11} = \frac{u_2}{u_3 - u_2}$, $q_{nn} = \frac{u_{n+1}^2}{u_{n+1}^2 - u_n^2}$, $q_{i,i+1} = q_{i+1,i} = \frac{u_{i+1}^2}{u_{i+1}^2 - u_{i+2}^2}$ for $1 \leq i \leq n - 1$, $q_{ii} = \frac{u_{i+1}^2}{(u_{i+1}^2 - u_i^2)(u_{i+2}^2 - u_{i+1}^2)}$ for $2 \leq i \leq n - 1$ and $0$ otherwise.

Proof. Since $F_n^{-1} = U_2^{-1}L_2^{-1}$, the proof follows from the previous two lemmas and from matrix multiplication. \(\square\)

For example, for $n = 4$,

$$
F_5^{-1} = \begin{pmatrix}
\frac{u_1^2}{u_2^2 - u_1^2} & \frac{u_2u_3}{u_2^2 - u_1^2} & 0 & 0 \\
\frac{u_2u_3}{u_1^2 - u_4^2} & \frac{u_3^2}{u_1^2 - u_4^2} & \frac{u_1u_4}{u_2^2 - u_4^2} & 0 \\
0 & \frac{u_1u_4}{u_3^2 - u_4^2} & \frac{u_3u_4}{u_3^2 - u_4^2} & \frac{u_2u_4}{u_3^2 - u_4^2} \\
0 & 0 & \frac{u_2u_4}{u_3^2 - u_4^2} & \frac{u_3u_4}{u_3^2 - u_4^2}
\end{pmatrix}.
$$

6 Further comment

With a bit more care, one can certainly remove the constraint $q = -1$ on the sequence $U_n$, and prove similar results like in the present paper for the corresponding generalized Lehmer matrix.
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References

