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The generalized Fibonomial matrix

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ABSTRACT

The Fibonomial coefficients are known as interesting generalizations of binomial coefficients. In this paper, we derive a $(k+1)$th recurrence relation and generating matrix for the Fibonomial coefficients, which we call *generalized Fibonomial matrix*. We find a nice relationship between the eigenvalues of the Fibonomial matrix and the generalized right-adjusted Pascal matrix; that they have the same eigenvalues. We obtain generating functions, combinatorial representations, many new interesting identities and properties of the Fibonomial coefficients. Some applications are also given as examples.

1. Introduction

The well known Fibonacci numbers are defined by

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$, for $n > 1$.

The Fibonomial coefficient is defined by the relation for $n \geq m \geq 1$

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_{n-m})(F_1 F_2 \dots F_m)}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$ where F_n is the nth Fibonacci number. These coefficients satisfy the relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

Let p be a nonzero integer. Define the generalized Fibonacci and Lucas sequences by the recurrences:

$$u_n = pu_{n-1} + u_{n-2}$$
$$v_n = pv_{n-1} + v_{n-2}$$

where $u_0 = 0,\ u_1 = 1$ and $v_0 = 2$, $v_1 = p$, respectively, for all $n \geq 2$.

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When $p = 1$ and $p = 2$, $u_n = F_n$ (nth Fibonacci number) and $u_n = P_n$ (nth Pell number), respectively.

Jarden and Motzkin [13] were the first to study generalized Fibonomial coefficients formed by terms of sequence $\{u_n\}$ as follows: for $n \geq m \geq 1$

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{u_1 u_2 \ldots u_n}{(u_1 u_2 \ldots u_{n-m})(u_1 u_2 \ldots u_m)}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1$. When $p = 1$, the generalized Fibonomial coefficient $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ is reduced to the Fibonomial coefficient $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]_F$.

The $n \times n$ generalized Pascal matrix P_n whose (i, j) entry is given by

$$(P_n)_{ij} = \binom{j-1}{j+i-n-1} p^{i+j-n-1}.$$

For example,

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3p \\ 0 & 1 & 2p & 3p^2 \\ 1 & p & p^2 & p^3 \end{bmatrix}.$$

Recently there has been increasing interest in both the Fibonomial coefficients and certain generalized matrix of binomial coefficients, which we call generalized Pascal matrices. Regarding left or right adjustments, and certain coefficients generalizations, several authors give various names to Pascal matrices. For example Carlitz [1] considered the right adjusted Pascal matrix and he called it a "matrix of binomial coefficients". In [5], Edelman and Gilbert considered left adjusted matrix of binomial coefficients and he called it a "Pascal matrix". In [21], the author considered right adjusted and coefficient generalized matrix of binomial coefficient and he called it a "Netted Matrix".

Regarding generalization of binomial coefficients, several authors have studied the generalized Fibonomial coefficients and their properties (for more details see [7,9,13,19,23,24]). Meanwhile, some authors have considered the spectral properties of the generalized Pascal matrix [1,3,10,20]. Since some relationships between the generalized Pascal matrix and the Fibonomial coefficients have been constructed, the Fibonomial coefficients have been considered by some authors. In this paper, we give more powerful relationships between the Fibonomial coefficients and a right-adjusted generalized Pascal matrix.

Matrix methods and generating matrices are very useful for solving some problems stemming from number theory. In this paper, we define the generalized Fibonomial matrix and derive an $(k + 1)$th order linear recurrence relation for the generalized Fibonacci coefficients. Also, we show that the generalized Fibonomial and Pascal matrices have the same characteristic polynomials and therefore the same eigenvalues. We obtain some explicit and closed formulas for the coefficients and their sums by matrix methods. We give generating functions, properties and combinatorial representations for them. Further, we present some relationships between determinants of certain matrices and the generalized Fibonacci coefficients.

2. Generalized Fibonomial coefficients

This section is mainly devoted to deriving a recurrence relation and generating matrix for the generalized Fibonomial coefficients. For the sake of compactness, we shall use the following notations, for fixed k such that $1 \leq i \leq k + 1$:

$$a_{n,i} = (-1)^{(i-1)(i-2)/2} \left\{ \begin{matrix} n+k \\ k-i+1 \end{matrix} \right\} \left\{ \begin{matrix} n+i-2 \\ i-1 \end{matrix} \right\}$$

where $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ stands for the generalized Fibonomial coefficients and is defined by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \begin{cases} 0 & \text{if } m > n \text{ and } n \geq 0, \\ (-1)^{m(m-1)/2} & \text{if } m > n \text{ and } n < 0, \\ \dfrac{u_1 u_2 \ldots u_n}{(u_1 u_2 \ldots u_m)(u_1 u_2 \ldots u_{n-m})} & m \leq n. \end{cases}$$

For $k = 2$, the generalized Fibonomial coefficients and their properties were studied in [15].

For later use, we give the following useful result.

Lemma 1. *For $n > 0$ and $1 \le i \le k$*

$$a_{1,i}a_{n,1} + a_{n,i+1} = a_{n+1,i}$$

where $a_{n,i}$ be as before.

Proof. For case $i = 1$, the proof can be found in [9]. For the other cases, that is, $i > 1$, if we simplify the equality $a_{1,i}a_{n,1} + a_{n,i+1} = a_{n+1,i}$, it is reduced to the form:

$$u_{k+1}u_{n+i} + (-1)^{i-1}u_n u_{k-i+1} = u_i u_{n+k+1}.$$

The last equality can be easily obtained from the Binet formula of $\{u_n\}$. Thus the proof follows. □

For $k \ge 1$, define the $(k+1) \times (k+1)$ companion matrix G_k and the matrix $H_{n,k}$ as follows:

$$G_k = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\ 1 & & & \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \quad \text{and} \quad H_{n,k} = \begin{bmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,k+1} \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,k+1} \end{bmatrix}. \tag{1}$$

The matrix G_k is said to be *generalized Fibonomial matrix*.

Now we give our main result as follows;

Theorem 2. *For all $n > 0$,*

$$G_k^n = H_{n,k}.$$

Proof. By the definitions of matrix $H_{n,k}$ and Fibonomial coefficients, the proof is obvious for $n = 1$. Suppose that the equation holds for $n \ge 1$. Now we show that the equation holds for $n + 1$. Thus we write

$$G_k^{n+1} = G_k G_k^n = G_k H_{n,k}.$$

From Lemma 1 and the property of matrix multiplication, we get

$$G_k^{n+1} = G_k H_{n,k} = H_{n+1,k}.$$

Thus the proof of the theorem is complete. □

It is valuable to note that when $p = 1$ and $k = 1$, we obtain the well-known fact:

$$G_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H_{n,1} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Now we give a linear recurrence relation for the generalized Fibonomial coefficients.

Corollary 3. *For $n, k > 0$, the generalized Fibonomial coefficients satisfy the following order-$(k+1)$ linear recurrence relation*

$$a_{n+1,1} = \sum_{i=1}^{k+1} a_{1,i}a_{n-i+1,1}$$

or clearly

$$\begin{Bmatrix} n+k+1 \\ k \end{Bmatrix} = \begin{Bmatrix} k+1 \\ k \end{Bmatrix}\begin{Bmatrix} n+k \\ k \end{Bmatrix} + \begin{Bmatrix} k+1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + \cdots + (-1)^{(k-1)(k-2)/2}\begin{Bmatrix} k+1 \\ 1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix} + (-1)^{k(k-1)/2}\begin{Bmatrix} n \\ k \end{Bmatrix}.$$

Proof. Since $a_{n,1} = \left\{ {n+k \atop k} \right\}$ and from matrix multiplication, by equating $(1, 1)$ entries in the equation $H_{1,k}H_{n,k} = H_{n+1,k}$, the proof is easily seen. □

Considering the generalized Fibonomial matrix, we obtain following corollary.

Corollary 4. *For $n > 0$, the following identities hold;*

$$a_{n-1,1} = (-1)^{k(k-1)/2} a_{n,k+1},$$

$$a_{m+n+1-i,j} = \sum_{t=1}^{k+1} a_{n+1-i,t} a_{m+1-t,j} \quad \text{for all } m > 0,$$

$$a_{n+t+1-i,j} = \sum_{m=1}^{k+1} a_{n+r+1-i,m} a_{t-r+1-i,j} \quad \text{for } t > 0 \text{ and } t > r,$$

$$a_{n+1,1} = a_{1,1}a_{n,1} + a_{n,2},$$

$$a_{n+1,k+1} = a_{n,1}a_{1,k+1},$$

$$a_{n+1,i} = a_{1,i}a_{n,1} + a_{n,i+1} \quad \text{for } 2 \le i \le k.$$

Proof. Above identities can be proved by considering a property of matrix multiplication in $H_{n+1,k} = H_{n,k}H_{1,k}$, $H_{n+m,k} = H_{n,k}H_{m,k}$ and $H_{n+t,k} = H_{n+r,k}H_{t-r,k}$ for $n, m > 0$ and $t > r$. □

3. The eigenvalues of matrix G_k

In this section, we determine the eigenvalues of matrix G_k. Since the matrix G_k is a companion matrix, its characteristic polynomial can be easily derived.

Let $f_{n,k}(x)$ be a polynomial of degree $(k+1)$ related with the matrix $H_{n,k}$ whose coefficients are consist of the first row entries of $H_{n,k}$ as follows: for $n, k > 0$,

$$f_{n,k}(x) = \sum_{t=0}^{k+1} (-1)^{t(t+1)/2} \left\{ {n+k \atop k-t+1} \right\} \left\{ {n+t-2 \atop t-1} \right\} x^{k+1-t}.$$

Then we have the following Corollary.

Corollary 5. *For $k > 0$, the characteristic polynomial of G_k is given by*

$$f_{1,k}(x) = \sum_{i=0}^{k+1} (-1)^{i(i+1)/2} \left\{ {k+1 \atop i} \right\} x^{k-i+1}.$$

Here we should note that in [12,8,9,4], the authors gave the characteristic equation of the matrix for generalized Fibonomial coefficients as

$$C_n(x) = \sum_{h=0}^{n} (-1)^{h(h+1)/2} \left\{ {n \atop h} \right\} x^{n-h},$$

where $\left\{ {n \atop h} \right\}$ is defined as before.

Moreover in [4], the authors proved the conjecture of Horadam and Mahon, and they gave a very nice relationship between the characteristic polynomials of the matrix for generalized Fibonomial coefficients and the generalized Pascal matrix P_n. Let $R_n(x)$ be the characteristic polynomial of matrix P_n. From [4], we have that

$$C_n(x) = R_n(x).$$

Therefore we derive a nice relationship between the characteristic polynomials of matrix G_k and the polynomial of P_n as follows:

$$f_{1,n-1}(x) = C_n(x) = R_n(x).$$

We have the following result.

Corollary 6 ([4]). *Let* $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$. *The characteristic roots of* $C_{m+1}(x) = f_{1,m}(x)$ *are:*

$$\left\{(-1)^j \alpha^{m-2j}, (-1)^j \beta^{m-2j}\right\}_{j=0,1,\ldots k-1} \quad \text{if } m = 2k-1,$$
$$\left\{(-1)^k, (-1)^j \alpha^{m-2j}, (-1)^j \beta^{m-2j}\right\}_{j=0,1,\ldots k-1} \quad \text{if } m = 2k.$$

As an example, when $k = 5$, after some simplifications, we write

$$G_5 = \begin{bmatrix} a_1 & b_1 & -c_1 & -d_1 & e_1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and}$$

$$H_{n,5} = \begin{bmatrix} a_n & b_n & -c_n & -d_n & e_n & a_{n-1} \\ a_{n-1} & b_{n-1} & -c_{n-1} & -d_{n-1} & e_{n-1} & a_{n-2} \\ a_{n-2} & b_{n-2} & -c_{n-2} & -d_{n-2} & e_{n-2} & a_{n-3} \\ a_{n-3} & b_{n-3} & -c_{n-3} & -d_{n-3} & e_{n-3} & a_{n-4} \\ a_{n-4} & b_{n-4} & -c_{n-4} & -d_{n-4} & e_{n-4} & a_{n-5} \\ a_{n-5} & b_{n-5} & -c_{n-5} & -d_{n-5} & e_{n-5} & a_{n-6} \end{bmatrix}$$

where $a_n = \left\{ {n+5 \atop 5} \right\}$, $b_n = \left\{ {n+5 \atop 4} \right\} \left\{ {n \atop 1} \right\}$, $c_n = \left\{ {n+5 \atop 3} \right\} \left\{ {n+1 \atop 2} \right\}$, $d_n = \left\{ {n+5 \atop 2} \right\} \left\{ {n+2 \atop 3} \right\}$, $e_n = \left\{ {n+5 \atop 1} \right\} \left\{ {n+3 \atop 4} \right\}$.

The characteristic polynomial and its roots of G_5 are given by

$$f_{1,5}(x) = \sum_{i=0}^{6} (-1)^{i(i+1)/2} \left\{ {6 \atop i} \right\} x^{6-i}$$

and $\lambda_6 = \alpha^5$, $\lambda_5 = \beta^5$, $\lambda_4 = -\alpha^3$, $\lambda_3 = -\beta^3$, $\lambda_2 = \alpha$, $\lambda_1 = \beta$ where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

Thus we have the following result.

Corollary 7. *For* $k > 0$,

$$\prod_{i=1}^{k+1} (x - \lambda_i) = \sum_{i=0}^{k+1} (-1)^{i(i+1)/2} \left\{ {k+1 \atop i} \right\} x^{k+1-i}.$$

Considering the results of Corollary 6, we derive the following facts:

$$f_{1,4m+4}(x) = (x^4 - c_{4m}x^3 - d_{4m}x^2 - c_{4m}x + 1) f_{1,4m},$$
$$f_{1,4m+5}(x) = (x^4 - c_{4m+1}x^3 - d_{4m+1}x^2 + c_{4m+1}x + 1) f_{1,4m+1},$$
$$f_{1,4m+6}(x) = (x^4 - c_{4m+2}x^3 - d_{4m+2}x^2 - c_{4m+2}x + 1) f_{1,4m+2},$$
$$f_{1,4m+7}(x) = (x^4 - c_{4m+3}x^3 - d_{4m+3}x^2 + c_{4m+3}x + 1) f_{1,4m+3}.$$

In general, we obtain the following identity:

$$f_{1,t+4}(x) = (x^4 - c_t x^3 - d_t x^2 + (-1)^{t+1} c_t x + 1) f_{1,t} \tag{2}$$

where $c_t = v_{t+4} - v_{t+2}$ and $d_t = v_{t+4}v_{t+2} + (-1)^{t+1}2$.

Rearranging the right-hand-side of (2) and equating the corresponding coefficients of x^n, gives the following new result:

Corollary 8. *For all* $t \geq j$,

$$\left\{ \begin{matrix} t+5 \\ i \end{matrix} \right\} = \left\{ \begin{matrix} t+1 \\ i \end{matrix} \right\} + (-1)^{i+1} c_t \left\{ \begin{matrix} t+1 \\ i-1 \end{matrix} \right\} + d_t \left\{ \begin{matrix} t+1 \\ i-2 \end{matrix} \right\} + (-1)^{i+t} c_t \left\{ \begin{matrix} t+1 \\ i-3 \end{matrix} \right\} + \left\{ \begin{matrix} t+1 \\ i-4 \end{matrix} \right\}$$

where c_t *and* d_t *are defined as before.*

Proof. From (2), we write

$$\begin{aligned} f_{1,t+4}(x) &= (x^4 - c_t x^3 - d_t x^2 + (-1)^{t+4} c_t x + 1) f_{1,t} \\ &= x^{t+5} - x^{t+4} \left(\left\{ \begin{matrix} t+1 \\ 1 \end{matrix} \right\} + c_t \left\{ \begin{matrix} t+1 \\ 0 \end{matrix} \right\} \right) \\ &\quad - x^{t+3} \left(\left\{ \begin{matrix} t+1 \\ 2 \end{matrix} \right\} - c_t \left\{ \begin{matrix} t+1 \\ 1 \end{matrix} \right\} + d_t \left\{ \begin{matrix} t+1 \\ 0 \end{matrix} \right\} \right) \\ &\quad + x^{t+2} \left(\left\{ \begin{matrix} t+1 \\ 3 \end{matrix} \right\} + c_t \left\{ \begin{matrix} t+1 \\ 2 \end{matrix} \right\} + d_t \left\{ \begin{matrix} t+1 \\ 1 \end{matrix} \right\} - c_t \left\{ \begin{matrix} t+1 \\ 0 \end{matrix} \right\} \right) \\ &\quad + x^{t+1} \left(\left\{ \begin{matrix} t+1 \\ 4 \end{matrix} \right\} - c_t \left\{ \begin{matrix} t+1 \\ 3 \end{matrix} \right\} + d_t \left\{ \begin{matrix} t+1 \\ 2 \end{matrix} \right\} + c_t \left\{ \begin{matrix} t+1 \\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} t+1 \\ 0 \end{matrix} \right\} \right) \\ &\quad - x^{t} \left(\left\{ \begin{matrix} t+1 \\ 5 \end{matrix} \right\} + c_t \left\{ \begin{matrix} t+1 \\ 4 \end{matrix} \right\} + d_t \left\{ \begin{matrix} t+1 \\ 3 \end{matrix} \right\} - c_t \left\{ \begin{matrix} t+1 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} t+1 \\ 1 \end{matrix} \right\} \right) \cdots \\ &\quad + (-1)^{i(i+1)/2} x^{t+5-i} \left(\left\{ \begin{matrix} t+1 \\ i \end{matrix} \right\} + (-1)^{i+1} c_t \left\{ \begin{matrix} t+1 \\ i-1 \end{matrix} \right\} \right. \\ &\quad \left. + \ d_t \left\{ \begin{matrix} t+1 \\ i-2 \end{matrix} \right\} + (-1)^{i+t} c_t \left\{ \begin{matrix} t+1 \\ i-3 \end{matrix} \right\} + \left\{ \begin{matrix} t+1 \\ i-4 \end{matrix} \right\} \right) + \cdots + 1. \end{aligned}$$

Comparing the coefficients of x^i for $1 \leq i \leq n$ above and the polynomial $f_{1,t+4}$, the proof is complete. □

In [3], the authors show that

$$\text{tr}\,(P_n) = \frac{u_{(k+1)n}}{u_n},$$

where P_n is the generalized Pascal matrix.

Since the matrices $H_{n,k}$ and P_n have the same eigenvalues, alternatively we also have that

$$\text{tr}(H_{n,k}) = \frac{u_{(k+1)n}}{u_n}.$$

By Corollary 6, we can give the following result for both the generalized Fibonomial and Pascal matrices.

Theorem 9. *For* $n > 0$,

$$\text{tr}(H_{n,k}) = \sum_{i=0}^{\lfloor k-1/2 \rfloor} (-1)^{in} v_{(k-2i)n} + \frac{1}{2} \left(1 + (-1)^k \right).$$

4. Diagonalization of G_k and the generalized Binet formula

In this section, we consider diagonalization of the matrix G_k and then give the generalized Binet formula for the generalized Fibonomial coefficients. From Corollary 6, we know that if $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are the eigenvalues of matrix G_k, then they are all distinct. Thus we can diagonalize the matrix G_k.

Define the $(k+1)\times(k+1)$ Vandermonde matrix V and diagonal matrix $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ as shown:

$$V = \begin{bmatrix} \lambda_1^k & \lambda_2^k & \ldots & \lambda_{k+1}^k \\ \vdots & \vdots & & \vdots \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_{k+1}^2 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_{k+1} \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{k+1} \end{bmatrix}.$$

Since $\lambda_i \neq \lambda_j$ for $1 \leq i, j \leq k+1$, $\det V \neq 0$.

Let $V_j^{(i)}$ is the $(k+1) \times (k+1)$ matrix obtained from V^{T} by replacing the jth column of V by w_i where

$$w_i = \begin{bmatrix} \lambda_1^{n-i+k+1} & \lambda_2^{n-i+k+1} & \ldots & \lambda_{k+1}^{n-i+k+1} \end{bmatrix}^{\mathrm{T}}.$$

Recalling $a_{n,i} = (-1)^{(i-1)(i-2)/2} \left\{ {n+k \atop k-i+1} \right\} \left\{ {n+i-2 \atop i-1} \right\}$, we give the generalized Binet formulas for the generalized Fibonomial coefficients by the following theorem.

Theorem 10. *For $n, k > 0$,*

$$a_{n-i+1,j} = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

Proof. One can check that $G_k V = VD$. Since V is the invertible matrix and $G_k^n = H_{n,k}$, we write $G_k^n V = H_{n,k} V = VD^n$. Clearly we get the following linear equation system:

$$\begin{aligned} &h_{i1}\lambda_1^k + h_{i2}\lambda_1^{k-1} + \cdots + h_{i,k-1}\lambda_1^2 + h_{i,k}\lambda_1 + h_{i,k+1} = \lambda_1^{n-i+k+1} \\ &h_{i1}\lambda_2^k + h_{i2}\lambda_2^{k-1} + \cdots + h_{i,k-1}\lambda_2^2 + h_{i,k}\lambda_2 + h_{i,k+1} = \lambda_2^{n-i+k+1} \\ &\vdots \\ &h_{i1}\lambda_{k+1}^k + h_{i2}\lambda_{k+1}^{k-1} + \cdots + h_{i,k-1}\lambda_{k+1}^2 + h_{i,k}\lambda_{k+1} + h_{i,k+1} = \lambda_{k+1}^{n-i+k+1}. \end{aligned}$$

Thus by Cramer's Rule, we have the conclusion. $\square$

After some calculations, we present some identities as examples of Theorem 10.

Case I When $k = 3$, $\det(V) = -u_2^2 u_3 \Delta^3$ and for $n > 0$ we have that

$$\begin{aligned} \left\{ {n+2 \atop 3} \right\} &= \left(u_{3n+3} + (-1)^{n+1} u_3 u_{n+1}\right)/u_2 u_3 \Delta, \\ \left\{ {n+3 \atop 2} \right\} \left\{ {n \atop 1} \right\} &= \left(u_{3n+5} + (-1)^{n+1}\left(u_2 u_n + u_{n+5}\right)\right)/u_2 \Delta, \\ \left\{ {n+3 \atop 1} \right\} \left\{ {n+1 \atop 2} \right\} &= \left(u_3 u_{3n+4} + (-1)^{n+1}\left[u_2^2\left(u_{n+4} + u_n\right) - u_{n-4}\right]\right)/u_2 u_3 \Delta \end{aligned}$$

where $\Delta = p^2 + 4$.

Especially when $p = 1$, $u_n = F_n$ (nth Fibonacci number) and so

$$\begin{aligned} F_n F_{n+1} F_{n+2} &= \left(F_{3n+3} + 2(-1)^{n+1} F_{n+1}\right)/5, \\ F_n F_{n+2} F_{n+3} &= \left[F_{3n+5} + (-1)^{n+1}\left(F_{n+5} + F_n\right)\right]/5, \\ F_n F_{n+1} F_{n+3} &= \left[2F_{3n+4} + (-1)^{n+1}\left(F_{n+4} + L_{n-2}\right)\right]/10. \end{aligned}$$

Case II When $k=4$, $\det(V)=u_2^4 v_2 u_3^2 \Delta^5$ and for $n \geq 0$

$$\begin{Bmatrix} n+3 \\ 4 \end{Bmatrix} = \frac{v_{4n+6}-v_4+v_1^2+(-1)^{n+1} v_1 v_2 v_{2n+3}}{u_2^2 v_2 u_3 \Delta^2},$$

$$\begin{Bmatrix} n+4 \\ 3 \end{Bmatrix} \begin{Bmatrix} n \\ 1 \end{Bmatrix} = \frac{v_{4n+9}+v_5-v_3+v_1+(-1)^n (v_1 v_{2n+2}-v_{2n+5}-v_{2n+9})}{u_1 u_2 u_3 \Delta^2},$$

$$\begin{Bmatrix} n+4 \\ 2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ 2 \end{Bmatrix} = \frac{v_{4n+8}-v_6-v_2+v_0+(-1)^n (v_1 v_{2n+1}+v_{2n+6}-v_{2n+8})}{u_2^2 \Delta^2},$$

$$\begin{Bmatrix} n+4 \\ 1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ 3 \end{Bmatrix} = \frac{v_{4n+7}-v_5+v_3-v_1+(-1)^n (v_1 v_{2n+4}-v_{2n+7}-v_{2n-1})}{u_2 u_3 \Delta^2}.$$

Case III When $k=5$, $\det(V)=u_2^4 u_3^3 u_4^2 u_5 \Delta^{\frac{15}{2}}$ and for $n \geq 0$

$$\begin{Bmatrix} n+4 \\ 5 \end{Bmatrix} = \frac{u_{5n+10}+(-1)^{n+1} (u_2 u_{3n+9}+u_3 u_{3n+4})-\left(u_3 (u_{n+6}+u_{n-2})+u_2^2 u_{n+2}\right)}{u_2 u_3 u_4 u_5 \Delta^2},$$

$$\begin{Bmatrix} n+5 \\ 4 \end{Bmatrix} \begin{Bmatrix} n \\ 1 \end{Bmatrix}$$
$$= \frac{u_{5n+14}+(-1)^{n+1} (u_{3n+14}+u_{3n+10}-u_3 u_{3n+6})+(u_4 u_{n+7}-u_3(u_{n+2}+u_{n-2}))}{u_2 u_3 u_4 \Delta^2},$$

$$\begin{Bmatrix} n+5 \\ 3 \end{Bmatrix} \begin{Bmatrix} n+1 \\ 2 \end{Bmatrix}$$
$$= \frac{u_{5n+13}+(-1)^{n+1} (u_2 u_{3n+12}-u_3 u_{3n+5})-u_{n+11}-u_2 u_3 u_{n+4}-u_3 u_{n-3}}{u_2^2 u_3 \Delta^2},$$

$$\begin{Bmatrix} n+5 \\ 2 \end{Bmatrix} \begin{Bmatrix} n+2 \\ 3 \end{Bmatrix}$$
$$= \frac{u_{5n+12}+(-1)^{n+1} (u_3 u_{3n+10}+u_2 u_{3n+3})-u_4 u_{n+7}-2u_2 u_{n+1}-u_{n-2}-u_{n-6}}{u_2^2 u_3 \Delta^2},$$

$$\begin{Bmatrix} n+5 \\ 1 \end{Bmatrix} \begin{Bmatrix} n+3 \\ 4 \end{Bmatrix}$$
$$= \frac{u_{5n+11}+(-1)^{n+1} (u_2 (u_{3n+10}+u_{3n+6})-u_{3n+1})-u_3 u_{n+7}-u_3 u_{n+3}-u_4 u_{n-2}}{u_2 u_3 u_4 \Delta^2}.$$

Let $V_j^{(e_i)}$ be a $(k+1)\times(k+1)$ matrix obtained from the Vandermonde matrix V by replacing the jth column of V by e_i where V is defined as before and e_i is the ith element of the natural basis for $\mathbb{R}^n$ and

$$V_j^{(e_i)} = \begin{bmatrix} \lambda_1^k & \cdots & \lambda_{j-1}^k & 0 & \lambda_{j+1}^k & \cdots & \lambda_{k+1}^k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \cdots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \cdots & \lambda_{k+1}^{k-i+1} \\ \lambda_1^{k-i} & \cdots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \cdots & \lambda_{k+1}^{k-i} \\ \lambda_1^{k-i-1} & \cdots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \cdots & \lambda_{k+1}^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \cdots & \lambda_{j-1} & 0 & \lambda_{j+1} & \cdots & \lambda_{k+1} \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \end{bmatrix}.$$
$$\qquad\qquad\qquad\qquad \underset{e_i}{\downarrow}$$

Let $q_j^{(i)} = \frac{\left|V_j^{(e_i)}\right|}{|V|}$ where the $(k+1) \times (k+1)$ matrices $V_j^{(e_i)}$ and V are defined as before.

Then we give the following theorem.

Theorem 11. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be the distinct roots of $x^{k+1} - a_{1,1}x^k - a_{1,2}x^{k-1} - \cdots - a_{1,k}x - a_{1,k+1} = 0$. For any integer n and $1 \le i \le k+1$,*

$$a_{n,i} = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k}.$$

Proof. We consider the following system of k linear equations with k unknowns $x_1, x_2, \ldots, x_k$: for $1 \le i \le k$

$$\begin{aligned}
&\lambda_1^k x_1 + \lambda_2^k x_2 + \cdots + \lambda_{k+1}^k x_{k+1} = 0 \\
&\vdots \\
&\lambda_1^{k-i+1} x_1 + \lambda_2^{k-i+1} x_2 + \cdots + \lambda_{k+1}^{k-i+1} x_{k+1} = 0 \\
&\lambda_1^{k-i} x_1 + \lambda_2^{k-i} x_2 + \cdots + \lambda_{k+1}^{k-i} x_{k+1} = 1 \\
&\lambda_1^{k-i-1} x_1 + \lambda_2^{k-i-1} x_2 + \cdots + \lambda_{k+1}^{k-i-1} x_{k+1} = 0 \\
&\vdots \\
&\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_{k+1} x_{k+1} = 0 \\
&x_1 + x_2 + \cdots + x_{k+1} = 0.
\end{aligned}$$

By the solution of Vandermonde's determinants and Cramer rule, we get

$$q_j^{(i)} = \frac{\left|V_j^{(e_i)}\right|}{|V|} \quad (i = 1, 2, \ldots, k+1).$$

Thus for $n, k > 0$ and $1 \le i \le k+1$,

$$a_{n,i} = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k},$$

which completes the proof. □

For example, if we take $k = 2$, then $\gamma_1 = \alpha^2, \gamma_2 = \beta^2, \gamma_3 = -1$ are the roots of $x^3 - a_{1,1}x^2 - a_{1,2}x - a_{1,3} = 0$. After some computations, we get

$$q_1^{(1)} = \frac{1}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \qquad q_2^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)}, \qquad q_3^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)},$$

$$q_1^{(2)} = -\frac{\gamma_2 + \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \qquad q_2^{(2)} = \frac{\gamma_1 + \gamma_3}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)},$$

$$q_3^{(2)} = -\frac{\gamma_1 + \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)},$$

$$q_1^{(3)} = \frac{\gamma_2 \gamma_3}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \qquad q_2^{(3)} = -\frac{\gamma_1 \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)},$$

$$q_3^{(3)} = \frac{\gamma_1 \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}.$$

Therefore, by Theorem 11, we get

$$a_{n,1} = \left\{ \begin{matrix} n+2 \\ 2 \end{matrix} \right\} = \frac{\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)} + \frac{\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$

$$a_{n,2} = \left\{ \begin{matrix} n+2 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = -\frac{(\gamma_2+\gamma_3)\gamma_1^{n+2}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{(\gamma_1+\gamma_3)\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)} - \frac{(\gamma_1+\gamma_2)\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}$$

and since $\gamma_1\gamma_2\gamma_3 = -1$,

$$a_{n,3} = -\left\{ \begin{matrix} n+1 \\ 2 \end{matrix} \right\} = \frac{\gamma_2\gamma_3\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_1\gamma_3\gamma_2^{n+2}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_1\gamma_2\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$

$$= -\left(\frac{\gamma_1^{n+1}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+1}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)} \right) = -a_{n-1,1}.$$

Note that also by using the definition of $a_{n,i}$ for $k = 2$, the equality $a_{n,3} = -a_{n-1,1}$ can be obtained.

5. On sums of the generalized Fibonomial coefficients

In this section, we consider the sum of the generalized Fibonomial coefficients. In order to compute this sum, we shall define a new generating matrix by extending G_k which is given in (1).

Define the $(k+2) \times (k+2)$ matrices T_k and $W_{n,k}$ as follows:

$$T_k = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & G_k & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad W_{n,k} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n & & & \\ \vdots & & H_{n,k} & \\ S_{n-k} & & & \end{bmatrix}$$

where the matrices G_k and $H_{n,k}$ are as before and also S_n is given by

$$S_n = \sum_{i=0}^{n-1} a_{i,1} = \sum_{i=0}^{n-1} \left\{ \begin{matrix} k+i \\ k \end{matrix} \right\}.$$

Then we have the following result.

Theorem 12. *For $n, k > 0$,*

$$T_k^n = W_{n,k}.$$

Proof. Since $S_{n+1} = a_{n,1} + S_n$ and by Theorem 2, we write the matrix recurrence relation $W_{n,k} = W_{n-1,k}T_k$. By the induction method, we write $W_{n,k} = W_{1,k}T_k^{n-1}$. From the definition of $W_{n,k}$, we obtain $W_{1,k} = T_k^1$ and so $W_{n,k} = T_k^n$. Thus we have the conclusion. □

Here we should note that from Corollary 6, we know that the polynomial $f_{1,k}$ has the root 1 for $k \equiv 0 \pmod 4$. Expanding the $\det(\lambda I_{k+2} - T_k)$ with respect to the first row, it is easily seen that the matrix T_k also has the eigenvalue 1. Thus we see that the matrix T_k has a double eigenvalue for $k \equiv 0 \pmod 4$. For $k \not\equiv 0 \pmod 4$, we can diagonalize the matrix T_k and so we derive an explicit formula for this sum.

Define the $(k+2) \times (k+2)$ matrix M as shown:

$$M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \delta & & & \\ \vdots & & V & \\ \delta & & & \end{bmatrix}$$

where $\delta = \left(1 - \sum_{i=1}^{k+1} a_{1,i}\right)^{-1}$ and the Vandermonde matrix V is defined as before.

We can check that $T_kM = MD_1$ where T_k is as before and D_1 is a diagonal matrix such that $D_1 = \mathrm{diag}(1, \lambda_1, \lambda_2, \ldots, \lambda_{k+1})$. Considering the matrix V, we compute $\det M$ with respect to the first row and then we find $\det M = \det V$.

Then we give the following theorem.

Theorem 13. *For $n, k > 0$ and $k \not\equiv 0 \pmod 4$,*

$$S_n = \frac{a_{n,1} + a_{n,2} + \cdots + a_{n,k+1} - 1}{\sum_{i=1}^{k+1} a_{1,i} - 1}.$$

Proof. Since the matrix M is invertible, we write $M^{-1}T_kM = D_1$. Thus the matrix T_k is similar to the matrix D_1. Then we write $T_k^nM = MD_1^n$. By Theorem 12, $W_{n,k}M = MD_1^n$. Equating the (2, 1) th elements of $W_{n,k}M = MD_1^n$ and from a matrix multiplication, we obtain

$$S_n + \delta\left(a_{n,1} + a_{n,2} + \cdots + a_{n,k+1}\right) = \delta.$$

Thus the proof is complete. □

As an application of Theorem 13, we give the following case without computations. When $k = 3$, we obtain that for $n \geq 0$

$$\sum_{i=0}^{n} \left\{ \begin{matrix} 3+i \\ 3 \end{matrix} \right\} = u_2u_3\left(u_nu_{n+2}u_{n+4} + u_{n+1}u_{n+3}u_{n+5} - u_3\right)/v_3.$$

6. Generating functions

In this section, we give generating functions of the generalized Fibonomial coefficients.

We define k sequences $\{f_n^i\}$ of kth order linear recurrence relation, for $n > 0$ and $1 \leq i \leq k$, as

$$f_n^i = c_1f_{n-1}^i + c_2f_{n-2}^i + \cdots + c_kf_{n-k}^i \tag{3}$$

with initial conditions

$$f_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \leq n \leq 0$$

where c_j, $1 \leq j \leq k$, are constant coefficients, and f_n^i is the nth term of the ith sequence.

Using the approach of Kalman in [14], Er showed in [6] that

$$M_n = A^n$$

where the matrices A and Q_n are

$$A = \begin{bmatrix} c_1 & c_2 & \ldots & c_{k-1} & c_k \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{k\times k} \quad \text{and} \quad Q_n = \begin{bmatrix} f_n^1 & f_n^2 & \ldots & f_n^k \\ f_{n-1}^1 & f_{n-1}^2 & \ldots & f_{n-1}^k \\ \vdots & \vdots & & \vdots \\ f_{n-k+1}^1 & f_{n-k+1}^2 & \ldots & f_{n-k+1}^k \end{bmatrix}_{k\times k}. \tag{4}$$

By defining $G(i, x) = f_0^ix^0 + f_1^ix^1 + f_2^ix^2 + \cdots + f_n^ix^n + \cdots$, we give the following theorem.

For the reader's convenience, we give the following result with proof.

Theorem 14. *For $k > 0$ and $1 \leq i \leq k$,*

$$G(i, x) = \frac{f_0^i + \sum_{m=1}^{k} \sum_{v=m+1}^{k} c_vf_{m-v}^ix^m}{1 - c_1x - c_2x^2 - \cdots - c_kx^k}.$$

Proof. Using definition of $G(i,x)$,

$$\left(1-c_1x-c_2x^2-\cdots-c_kx^k\right)G(i,x)=f_0^i+f_1^ix+f_2^ix^2+\cdots+f_k^ix^k+\cdots+f_n^ix^n+\cdots$$
$$-c_1f_0^ix-c_1f_1^ix^2-c_1f_2^ix^3-\cdots-c_1f_{k-1}^ix^k-\cdots-c_1f_{n-1}^ix^n-\cdots$$
$$-c_kf_0^ix^k-c_kf_1^ix^{k+1}-c_kf_2^ix^{k+2}-\cdots-c_kf_{n-k}^ix^n-\cdots.$$

After some arrangements, we write

$$\left(1-c_1x-c_2x^2-\cdots-c_kx^k\right)G(i,x)=f_0^i+\left(f_1^i-c_1f_0^i\right)x+\left(f_2^i-c_1f_1^i-c_2f_0^i\right)x^2+\cdots$$
$$+\left(f_{k-1}^i-c_1f_{k-2}^i-c_2f_{k-3}^i-\cdots-c_{k-1}f_0^i\right)x^{k-1}$$
$$+\left(f_k^i-c_1f_{k-1}^i-c_2f_{k-2}^i-\cdots-c_{k-1}f_0^i-c_kf_1^i\right)x^k+\cdots$$
$$+\left(f_n^i-c_1f_{n-1}^i-c_2f_{n-2}^i-\cdots-c_kf_{n-k}^i\right)x^n+\cdots.$$

Now we compute the coefficients of x^n of the equation above. From the definition of $\left\{f_n^i\right\}$,

$$f_1^i-c_1f_0^i=c_2f_{-1}^i+\cdots+c_kf_{1-k}^i$$
$$f_2^i-c_1f_1^i-c_2f_0^i=c_3f_{-1}^i+\cdots+c_kf_{2-k}^i$$
$$\vdots$$
$$f_{k-1}^i-c_1f_{k-2}^i-c_2f_{k-3}^i-\cdots-c_{k-1}f_0^i=c_kf_{-1}^i.$$

For $n\geq k$ and from the definition of $\left\{f_n^i\right\}$, the coefficients of x^n's are all zero. Thus the proof is complete. □

Now letting $g(i,x)=a_{0,i}+a_{1,i}x+a_{2,i}x^2+a_{3,i}x^3+\cdots+a_{n,i}x^n+$, we have the following corollary. Then we have the following Corollary.

Corollary 15. *For* $1\leq i\leq k+1$,

$$g(i,x)=\frac{a_{0,i}-\sum\limits_{m=1}^{k}\sum\limits_{v=m+1}^{k+1}a_{1,v}a_{m-v,i}x^m}{1-a_{1,1}x-a_{1,2}x^2-\cdots-a_{1,k+1}x^{k+1}}$$

where $a_{n,i}$ *is defined as before.*

For example, when $i=1$ in Corollary 15, we get

$$\sum_{n=0}^{\infty}\left\{\begin{matrix}n+k\\k\end{matrix}\right\}x^n=\frac{1}{1-\left\{\begin{smallmatrix}k+1\\k\end{smallmatrix}\right\}x-\left\{\begin{smallmatrix}k+1\\k-1\end{smallmatrix}\right\}x^2+\left\{\begin{smallmatrix}k+1\\k-2\end{smallmatrix}\right\}x^3+\cdots-(-1)^{k(k-1)/2}x^{k+1}}.$$

For $k=3$ ($i=1,2,3$ subsequently) we obtain

$$\sum_{n=0}^{\infty}\left\{\begin{matrix}n+3\\3\end{matrix}\right\}x^n=\frac{1}{1-\left\{\begin{smallmatrix}4\\3\end{smallmatrix}\right\}x-\left\{\begin{smallmatrix}4\\2\end{smallmatrix}\right\}x^2+\left\{\begin{smallmatrix}4\\1\end{smallmatrix}\right\}x^3+x^4},$$

$$\sum_{n=0}^{\infty}\left\{\begin{matrix}n\\1\end{matrix}\right\}\left\{\begin{matrix}n+3\\2\end{matrix}\right\}x^n=\frac{u_3u_4/u_2x-v_1v_2x^2-x^3}{1-\left\{\begin{smallmatrix}4\\3\end{smallmatrix}\right\}x-\left\{\begin{smallmatrix}4\\2\end{smallmatrix}\right\}x^2+\left\{\begin{smallmatrix}4\\1\end{smallmatrix}\right\}x^3+x^4}$$

and

$$\sum_{n=0}^{\infty}\left\{\begin{matrix}n+1\\2\end{matrix}\right\}\left\{\begin{matrix}n+3\\1\end{matrix}\right\}x^n=\frac{v_1v_2x+x^2}{1-\left\{\begin{smallmatrix}4\\3\end{smallmatrix}\right\}x-\left\{\begin{smallmatrix}4\\2\end{smallmatrix}\right\}x^2+\left\{\begin{smallmatrix}4\\1\end{smallmatrix}\right\}x^3+x^4}.$$

For example, the following generating function for the triple product of consecutive Fibonacci numbers can be found in [16]:

$$\sum_{n=0}^{\infty} F_n F_{n+1} F_{n+2} x^n = \frac{2x}{1-3x-6x^2+3x^3+x^4}.$$

Indeed the above generating function can also be rewritten via the Fibonomial coefficients $\left\{ {n \atop 3} \right\}_F$ as;

$$\sum_{n=0}^{\infty} \left\{ {n+2 \atop 3} \right\}_F x^n = \frac{x}{1-3x-6x^2+3x^3+x^4}.$$

For the generating function of the powers of Fibonacci numbers, we can refer to [24,11].

For $k=4$, we get

$$\sum_{n=0}^{\infty} \left\{ {n+4 \atop 4} \right\} x^n = \frac{1}{1-\left\{ {5 \atop 4} \right\} x - \left\{ {5 \atop 3} \right\} x^2 + \left\{ {5 \atop 2} \right\} x^3 + \left\{ {5 \atop 1} \right\} x^4 - x^5},$$

$$\sum_{n=0}^{\infty} \left\{ {n \atop 1} \right\} \left\{ {n+4 \atop 3} \right\} x^n = \frac{v_2u_5x - v_2u_5x^2 - u_5x^3 + x^4}{1-\left\{ {5 \atop 4} \right\} x - \left\{ {5 \atop 3} \right\} x^2 + \left\{ {5 \atop 2} \right\} x^3 + \left\{ {5 \atop 1} \right\} x^4 - x^5},$$

$$\sum_{n=0}^{\infty} \left\{ {n+1 \atop 2} \right\} \left\{ {n+4 \atop 2} \right\} x^n = \frac{v_2u_5x + u_5x^2 - x^3}{1-\left\{ {5 \atop 4} \right\} x - \left\{ {5 \atop 3} \right\} x^2 + \left\{ {5 \atop 2} \right\} x^3 + \left\{ {5 \atop 1} \right\} x^4 - x^5},$$

$$\sum_{n=0}^{\infty} \left\{ {n+2 \atop 3} \right\} \left\{ {n+4 \atop 1} \right\} x^n = \frac{u_5x - x^2}{1-\left\{ {5 \atop 4} \right\} x - \left\{ {5 \atop 3} \right\} x^2 + \left\{ {5 \atop 2} \right\} x^3 + \left\{ {5 \atop 1} \right\} x^4 - x^5}.$$

For $k=5$, we get

$$\sum_{n=0}^{\infty} \left\{ {n+5 \atop 5} \right\} x^n = \frac{1}{1-\left\{ {6 \atop 5} \right\} x - \left\{ {6 \atop 4} \right\} x^2 + \left\{ {6 \atop 3} \right\} x^3 + \left\{ {6 \atop 2} \right\} x^4 - \left\{ {6 \atop 1} \right\} x^5 - x^6},$$

$$\sum_{n=0}^{\infty} \left\{ {n \atop 1} \right\} \left\{ {n+5 \atop 4} \right\} x^n = \frac{(u_5u_6/u_2)\,x - v_2v_3u_5x^2 - (u_5u_6/u_2)\,x^3 + u_6x^4 + x^5}{1-\left\{ {6 \atop 5} \right\} x - \left\{ {6 \atop 4} \right\} x^2 + \left\{ {6 \atop 3} \right\} x^3 + \left\{ {6 \atop 2} \right\} x^4 - \left\{ {6 \atop 1} \right\} x^5 - x^6},$$

$$\sum_{n=0}^{\infty} \left\{ {n+1 \atop 2} \right\} \left\{ {n+5 \atop 3} \right\} x^n = \frac{v_2v_3u_5x + (u_5u_6/u_2)\,x^2 - u_6x^3 - x^4}{1-\left\{ {6 \atop 5} \right\} x - \left\{ {6 \atop 4} \right\} x^2 + \left\{ {6 \atop 3} \right\} x^3 + \left\{ {6 \atop 2} \right\} x^4 - \left\{ {6 \atop 1} \right\} x^5 - x^6},$$

$$\sum_{n=0}^{\infty} \left\{ {n+2 \atop 3} \right\} \left\{ {n+5 \atop 2} \right\} x^n = \frac{(u_5u_6/u_2)\,x - u_6x^2 - x^3}{1-\left\{ {6 \atop 5} \right\} x - \left\{ {6 \atop 4} \right\} x^2 + \left\{ {6 \atop 3} \right\} x^3 + \left\{ {6 \atop 2} \right\} x^4 - \left\{ {6 \atop 1} \right\} x^5 - x^6},$$

$$\sum_{n=0}^{\infty} \left\{ {n+3 \atop 4} \right\} \left\{ {n+5 \atop 1} \right\} x^n = \frac{(u_5u_6/u_4)\,x + x^2}{1-\left\{ {6 \atop 5} \right\} x - \left\{ {6 \atop 4} \right\} x^2 + \left\{ {6 \atop 3} \right\} x^3 + \left\{ {6 \atop 2} \right\} x^4 - \left\{ {6 \atop 1} \right\} x^5 - x^6}.$$

7. Combinatorial representations

In this section, we give combinatorial representations for the generalized Fibonomial coefficients. In [2], the authors considered the $k \times k$ companion matrix A that we give in (4) and its nth power to derive an explicit formula for the elements in the nth power of the matrix A. Let us recall this result, as follows.

Theorem 16 *([2]). Let the matrix $A = (a_{ij})$ be as in (4). The (i, j) entry $a_{ij}^{(n)}$ in the matrix A_k^n is given by the following formula:*

$$a_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1,t_2,\ldots t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} \ldots c_k^{t_k} \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + kt_k = n - i + j$, and the coefficients are defined as 1 for $n = i - j$.

Thus we give the following results.

Corollary 17. *Let $a_{n,i}$ denote the generalized Fibonomial coefficients. Then*

$$a_{n-i+1,j} = \sum_{(t_1,t_2,\ldots,t_{k+1})} \frac{t_j + t_{j+1} + \cdots + t_{k+1}}{t_1 + t_2 + \cdots + t_{k+1}} \times \binom{t_1 + t_2 + \cdots + t_{k+1}}{t_1, t_2, \ldots, t_{k+1}} a_{1,1}^{t_1} \ldots a_{1,k+1}^{t_{k+1}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k+1)t_{k+1} = n - i + j$.

Proof. Considering the matrices G_k and A, the proof is obvious from the result of Theorem 16. □

Corollary 18. *For $n \geq 0$,*

$$\left\{ {n+k \atop k} \right\} = \sum_{(t_1,t_2,\ldots,t_{k+1})} \binom{t_1 + t_2 + \cdots + t_{k+1}}{t_1, t_2, \ldots, t_{k+1}} a_{1,1}^{t_1} \ldots a_{1,k+1}^{t_{k+1}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k+1)t_{k+1} = n$.

Proof. In Corollary 17, if we take $i = j = 1$, then $a_{n,1} = \left\{ {n+k \atop k} \right\}$, so the proof of Corollary 17 follows. □

For example, one can obtain

$$\left\{ {n \atop 1} \right\} \left\{ {n+3 \atop 2} \right\} = \sum_{(r_1,r_2,r_3,r_4)} \frac{r_2 + r_3 + r_4}{r_1 + r_2 + r_3 + r_4} \binom{r_1 + r_2 + r_3 + r_4}{r_1, r_2, r_3, r_4} (-1)^{r_3+r_4} \left\{ {4 \atop 3} \right\}^{r_1+r_3} \left\{ {4 \atop 2} \right\}^{r_2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n + 1$ and

$$\left\{ {n+1 \atop 2} \right\} \left\{ {n+3 \atop 1} \right\} = \sum_{(r_1,r_2,r_3,r_4)} \frac{r_3 + r_4}{r_1 + r_2 + r_3 + r_4} \times \binom{r_1 + r_2 + r_3 + r_4}{r_1, r_2, r_3, r_4} (-1)^{r_3+r_4+1} \left\{ {4 \atop 3} \right\}^{r_1+r_3} \left\{ {4 \atop 2} \right\}^{r_2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n + 2$.

8. Determinantal representations

In this section, we determine some relationships between determinants of certain matrices and the generalized Fibonomial coefficients. Similar relationships have been derived by some authors (see for more detail [17,18,22]). In particular, Lind [17] gave the first result for the relationship between the determinant of certain Hessenberg matrices and the generalized Fibonomial coefficients. For convenience, we give the result of Lind [17].

Let $D_{n,k}$ denote the recurrent $n \times n$ determinant $|a_{rs}|$, where

$$a_{rs} = -(-1)^{(s+r+1)(s-r+2)/2} \left\{ {k+1 \atop s-r+1} \right\}_F$$

for $r, s = 1, 2, \ldots, n$.

Then the author showed that $D_{n,k} = \left\{ {n+k \atop k} \right\}_F$ where $\left\{ {n \atop i} \right\}_F$ is the Fibonomial coefficient. The analogous result holds when the Fibonacci sequence is replaced by an ordinary second-order recurring sequence.

Now, by constructing superdiagonal matrices, we give some new results that are given by Lind in [17].

Definition 19. For $n > k > 0$, let $M_n = \left[m_{ij}\right]$ denote the k-superdiagonal matrix of order n with $m_{ii} = a_{1,1}$ for $1 \le i \le n$, $m_{i,i+1} = a_{1,2}$ for $1 \le i \le n-1, \ldots, m_{i,i+k} = a_{1,k+1}$ for $1 \le i \le n-k$.

Clearly the matrix M_n is in the form

$$M_n = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} & & & 0 \\ -1 & a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} & & \\ & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\ & & & -1 & a_{1,1} & \cdots & \vdots \\ & & & & -1 & a_{1,1} & a_{1,2} \\ 0 & & & & & -1 & a_{1,1} \end{bmatrix}. \quad (6)$$

Theorem 20. *For $n > 0$,*

$$|M_n| = a_{n,1}.$$

Proof (*Induction on n*). If $n = 2$, then we have

$$|M_2| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ -1 & a_{1,1} \end{vmatrix} = a_{1,1}a_{1,1} + a_{1,2} = a_{2,1}.$$

Suppose this equation holds for n. Then we show that the equality is true for $n+1$. Expanding $|M_{n+1}|$ by the Laplace expansion of determinant according to the last column and by the definition of M_n, we get

$$|M_{n+1}| = a_{1,1}\left|M_n\right| + a_{1,2}\left|M_{n-1}\right| + a_{1,3}\left|M_{n-2}\right| + \cdots + a_{1,k+1}\left|M_{n-k}\right|.$$

By our assumption and the recurrence relation of $\left\{a_{n,1}\right\}$, we write

$$\begin{aligned} |M_{n+1}| &= a_{1,1}a_{n,1} + a_{1,2}a_{n-1,1} + a_{1,3}a_{n-2,1} + \cdots + a_{1,k+1}a_{n-k,1} \\ &= a_{n+1,1}. \end{aligned}$$

Thus the theorem is proven. □

For example, if we take $p = 1$, then $u_n = F_n$ (nth Fibonacci number) and by Theorem 20, we have

$$\begin{vmatrix} 3 & 6 & -3 & -1 & & 0 \\ -1 & 3 & 6 & -3 & \ddots & \\ & -1 & 3 & 6 & \ddots & -1 \\ & & \ddots & \ddots & \ddots & -3 \\ & & & -1 & 3 & 6 \\ 0 & & & & -1 & 3 \end{vmatrix}_{n\times n} = \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_F.$$

Let $M_n(k)$ denote the matrix obtained by the matrix $M_n = \left[m_{ij}\right]$ taking $m_{1,j} = 0$ for $1 \le j \le k$. For example

$$M_4(2) = \begin{bmatrix} 0 & 0 & a_{1,3} & a_{1,4} \\ -1 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & -1 & a_{1,1} & a_{1,2} \\ 0 & 0 & -1 & a_{1,1} \end{bmatrix}.$$

Now we determine some relationships between the sequences $\{a_{n,i}\}$ for $1 < i \leq k$ and the consecutive determinants of matrix $M_n(k)$ for some specific number k.

Theorem 21. *For $n > k \geq i \geq 1$,*

$$|M_n(i)| = a_{n-i,i+1}.$$

Proof. Expanding $|M_n(i)|$ with respect to the first row and using the definitions of $M_n(i)$ and M_n, we get

$$|M_n(i)| = a_{1,i+1}|M_{n-i-1}| + a_{1,i+2}|M_{n-i-2}| + \cdots + a_{1,k+1}|M_{n-k-1}|.$$

Similarly expanding the $|M_n|$ with respect to the first row and using the definitions of M_n and $|M_n(i)|$, we may write, after some simplifications,

$$|M_n(i)| = |M_n| - a_{1,1}|M_{n-1}| - a_{1,2}|M_{n-2}| - a_{1,3}|M_{n-3}| - \cdots - a_{1,i}|M_{n-i}|$$

which, by our assumption and Lemma 1, satisfies

$$\begin{aligned}|M_n(i)| &= a_{n,1} - a_{1,1}a_{n-1,1} - a_{1,2}a_{n-2,1} - a_{1,3}a_{n-3,1} - \cdots - a_{1,i}a_{n-i,1}\\ &= a_{n-1,2} - a_{1,2}a_{n-2,1} - a_{1,3}a_{n-3,1} - \cdots - a_{1,i}a_{n-i,1}\\ &\ \ \vdots\\ &= a_{n-i+1,i} - a_{1,i}a_{n-i,1}\\ &= a_{n-i,i+1}.\end{aligned}$$

Thus the proof is complete.

We now define an $n \times n$ upper Hessenberg matrix D_n as in the following compact form:

$$D_n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & & & \\ 0 & & M_{n-1} & \\ & & & \\ 0 & & & \end{bmatrix} \tag{7}$$

where M_n is defined as before. □

Theorem 22. *For $n > 1$,*

$$|D_n| = S_n$$

where S_n is defined as before.

Proof. By Theorem 20, the proof follows from by induction. □

To derive other similar relationships between determinants of certain matrices and the sums of the other products, we define $(n + 1) \times (n + 1)$ matrix $T_{n,i}$ for $1 \leq i \leq k$ as follows:

$$T_{n,i} = \begin{bmatrix} & & & & 0 \\ & & & & \vdots \\ & & M_n(i) & & 0 \\ & & & & 1 \\ & & & & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix} \quad i\text{th row}$$

where the matrix $M_n(i)$ is defined as before.

Expanding $\left|T_{n,i}\right|$ according to the last row, we have the following Theorem.

Theorem 23. *For* $n \geq k \geq i \geq 1$,

$$\left|T_{n,i}\right| = \sum_{i=0}^{n-i-1} a_{i,1}.$$

9. Conclusion

In this paper, we consider the recurrence $\{u_n\}$ and its generalized Fibonomial coefficients. Using results in this paper, one can obtain many applications to the recurrence $\{u_n\}$ or its special cases, that is, Fibonacci or Pell sequences. Moreover, one can obtain many analogues for the recurrence $\{U_n\}$ defined by $U_n = pU_{n-1} - qU_{n-2}$ with $U_0 = 0$ and $U_1 = 1$. However one should be aware that, in case of recurrence $\{U_n\}$, we cannot obtain a generator matrix by using just the matrix itself.

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