

DERIVATIVE SEQUENCES OF JACOBSTHAL AND  
JACOBSTHAL-LUCAS POLYNOMIALS

Alwyn F. Horadam

University of New England, Armidale, NSW 2351, Australia

Piero Filipponi

Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy

e-mail: filippo@fub.it

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1. AIM OF THE PAPER

The *Jacobsthal polynomials*  $J_n(x)$  and the *Jacobsthal-Lucas polynomials*  $j_n(x)$ , whose properties have been investigated in [4], are a natural extension of the *Jacobsthal numbers*  $J_n$  and the *Jacobsthal-Lucas numbers*  $j_n$  which, in turn, have been investigated in [3]. These polynomials are defined by the second-order recurrence relations

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), [J_0(x) = 0, J_1(x) = 1] \quad (1.1)$$

and

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), [j_0(x) = 2, j_1(x) = 1], \quad (1.2)$$

respectively, where  $x$  is an indeterminate.

Since throughout this paper we shall make use of the notation and the formulas found in [3] and [4], the reader is assumed to be aware of the contents of these papers.

**Definitions:** Following the idea exploited in [1], let us define the polynomials  $J_n^{(1)}(x)$  and  $j_n^{(1)}(x)$  {see (3.9) and (3.10) of [4] for the combinatorial representations of  $J_n(x)$  and  $j_n(x)$ } as

$$J_n^{(1)}(x) = \frac{d}{dx} J_n(x) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} 2^r r \binom{n-1-r}{r} x^{r-1} \quad (n \geq 0), \quad (1.3)$$

$$j_n^{(1)}(x) = \frac{d}{dx} j_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{2^r n r}{n-r} \binom{n-r}{r} x^{r-1} \quad (n \geq 1), \quad (1.4)$$

and

$$J_0^{(1)}(x) = j_0^{(1)}(x) = 0, \quad (1.5)$$

where the symbol  $\lfloor \cdot \rfloor$  denotes the greatest integer function, and the bracketed superscript symbolizes the first derivative with respect to  $x$ .

The aim of this paper is to study some properties of the above sequences just as was done in [1] for the Fibonacci and Lucas polynomials. Here, we shall also confine ourselves to considering the case  $x = 1$ . Since letting  $x = 1$  in (1.1) and (1.2) will yield the Jacobsthal numbers and the Jacobsthal-Lucas numbers {cf. (2.3) and (2.4) of [3]}

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n, \quad (1.6)$$

the sequences  $\{J_n^{(1)}(1)\}$  and  $\{j_n^{(1)}(1)\}$  will be referred to as *Jacobsthal* and *Jacobsthal-Lucas derivative sequences*. For notational convenience, their terms  $J_n^{(1)}(1)$  and  $j_n^{(1)}(1)$  will be denoted

by  $H_n$  and  $K_n$ , respectively. From (1.3)-(1.5), the numbers  $H_n$  and  $K_n$  can be obtained readily for the first few values of  $n$ . They are shown in Table 1.

TABLE 1. The Numbers  $H_n$  and  $K_n$  for  $0 \leq n \leq 8$

$n$	0	1	2	3	4	5	6	7	8
$J_n^{(1)}(1) = H_n$	0	0	0	2	4	14	32	82	188
$j_n^{(1)}(1) = K_n$	0	0	4	6	24	50	132	294	688

2. CLOSED-FORM EXPRESSIONS FOR  $H_n$  AND  $K_n$

Closed-form expressions for  $H_n$  and  $K_n$  are, quite obviously, useful tools for discovering their properties. They are established in this section, where some equivalent expressions for these numbers are also found.

By using formulas (1.4), (1.5), (3.3), and (3.4) of [4], we easily see that

$$\Delta^{(1)}(x) = \frac{d}{dx} \Delta(x) = 4 / \Delta(x),$$

$$\alpha^{(1)}(x) = \frac{d}{dx} \alpha(x) = \Delta^{(1)}(x) / 2 = 2 / \Delta(x),$$

$$\beta^{(1)}(x) = \frac{d}{dx} \beta(x) = -\Delta^{(1)}(x) / 2 = -2 / \Delta(x),$$

$$[\alpha^n(x)]^{(1)} = \frac{d}{dx} \alpha^n(x) = n\alpha^{n-1}(x)\alpha^{(1)}(x) = 2n\alpha^{n-1}(x) / \Delta(x),$$

and

$$[\beta^n(x)]^{(1)} = \frac{d}{dx} \beta^n(x) = n\beta^{n-1}(x)\beta^{(1)}(x) = -2n\beta^{n-1}(x) / \Delta(x).$$

Hence, we have

$$J_n^{(1)}(x) = \frac{d}{dx} \left[ \frac{\alpha^n(x) - \beta^n(x)}{\Delta(x)} \right] = 2 \frac{nj_{n-1}(x) - 2J_n(x)}{\Delta^2(x)}, \tag{2.1}$$

and

$$j_n^{(1)}(x) = 2nJ_{n-1}(x). \tag{2.2}$$

Letting  $x = 1$  in (2.1) and (2.2) leads to the relations

$$H_n = J_n^{(1)}(1) = \frac{2(nj_{n-1} - 2J_n)}{9} \tag{2.3}$$

and

$$K_n = j_n^{(1)}(1) = 2nJ_{n-1} \tag{2.4}$$

which express  $H_n$  and  $K_n$  in terms of  $J_n$  and  $j_n$ .

By (2.3) and (2.4) above, and (1.6), the following relations can be obtained readily:

$$H_n = \frac{2^n(3n-4) - (6n-4)(-1)^n}{27} \tag{2.5}$$

and

$$K_n = \frac{n[2^n + 2(-1)^n]}{3}, \tag{2.6}$$

which express  $H_n$  and  $K_n$  in terms of their subscripts.

Observe that using (2.5) and (2.6) above, along with (1.6), we obtain the relations

$$H_n = \frac{(3n-4)J_n - n(-1)^n}{9} \tag{2.7}$$

and

$$K_n = \frac{n[j_n + (-1)^n]}{3}, \tag{2.8}$$

which express  $H_n$  in terms of  $J_n$  and  $K_n$  in terms of  $j_n$ , respectively.

### 3. BASIC PROPERTIES OF $H_n$ AND $K_n$

Some relations involving  $H_n$  and  $K_n$  are established in this section, most of which are the analogs of those found by Horadam in [3] for  $J_n$  and  $j_n$ . Some simple but sometimes tedious manipulations involving the use of (2.3)-(2.8) provide the required proofs. To save space, only the proofs of Theorems 1-3 will be given in detail in Subsection 3.2.

#### 3.1. Results

##### *Generating functions*

$$\sum_{n=0}^{\infty} H_n y^n = \frac{2y^3}{(2y^2 + y - 1)^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} K_n y^n = \frac{2y^2(2-y)}{(2y^2 + y - 1)^2}. \tag{3.2}$$

These functions can be obtained readily from (3.1) and (3.2) of [4].

##### *Recurrence relations*

$$H_{n+2} = H_{n+1} + 2H_n + 2J_n, \tag{3.3}$$

$$K_{n+2} = K_{n+1} + 2K_n + 2j_n. \tag{3.4}$$

These relations can be obtained readily by calculating at  $x = 1$  the first derivative with respect to  $x$  of both sides of (1.1) and (1.2).

##### *Some identities*

$$H_n K_n = \frac{n}{9} [K_{2n-1} - 2J_{n-1}(4J_n - j_{n-1})] \tag{3.5}$$

$$= \frac{n}{3} H_{2n} - \frac{n}{81} [(-2)^{n+2} + 3n4^n - 4] \tag{3.5}$$

$$H_{n+1} + 2H_{n-1} = K_n - 2J_{n-1} \tag{3.6}$$

$$= 2(n-1)J_{n-1} \quad [\text{by (2.4)}], \tag{3.6}$$

$$K_{n+1} + 2K_{n-1} = 9H_n + 2J_n + 2^n, \tag{3.7}$$

$$H_n + K_n = 2H_{n+1} \quad [\text{from (2.5) and (2.6)}], \tag{3.8}$$

$$K_n - H_n = 4(H_{n-1} + J_{n-1}), \quad (3.9)$$

$$K_n = 3H_n + \frac{1}{9}[4(-1)^n(3n-1) + 2^{n+2}]. \quad (3.10)$$

Observe that identity (3.8) is an important feature of  $H_n$  and  $K_n$ , being analogous to  $J_n + j_n = 2J_{n+1}$  for Jacobsthal and Jacobsthal-Lucas numbers.

#### *Simson formula analogs*

$$H_{n+1}H_{n-1} - H_n^2 = \frac{1}{81}[(-2)^n(9n^2 - 18n + 5) - 4^n - 4], \quad (3.11)$$

$$K_{n+1}K_{n-1} - K_n^2 = -\frac{1}{9}[(-2)^n(9n^2 - 5) + 4^n + 4]. \quad (3.12)$$

#### *Limits*

$$\lim_{n \rightarrow \infty} H_{n+1} / H_n = \lim_{n \rightarrow \infty} K_{n+1} / K_n = 2, \quad (3.13)$$

$$\lim_{n \rightarrow \infty} K_n / H_n = 3. \quad (3.14)$$

#### *Evaluation of some finite sums*

$$S_n \stackrel{\text{def}}{=} \sum_{k=0}^n H_k = 2H_n - \frac{1}{18}[2^{n+2} - (-1)^n(6n-5) - 9], \quad (3.15)$$

$$T_n \stackrel{\text{def}}{=} \sum_{k=0}^n K_k = 2K_n - \frac{1}{6}[(-1)^n(6n-1) + 2^{n+2} - 3]. \quad (3.16)$$

Alternative, but perhaps less elegant, expressions for  $S_n$  and  $T_n$  can be obtained after several tedious manipulations involving the use of (2.4) and (2.7). They are

$$S_n = \frac{1}{108}[2^n(27n-56) - 9K_n - (-1)^n(12\lceil n/2 \rceil - 5) + 51], \quad (3.15')$$

where the symbol  $\lceil x \rceil$  denotes the least integer not less than  $x$ , and

$$T_n = \frac{1}{2}[K_n + J_{n+1} + 2^n(n-2) + 1]. \quad (3.16')$$

$$\sum_{k=0}^n \binom{n}{k} H_k = 2(n-2)3^{n-3} + \frac{1}{9}\left[2\delta_{1,n} + \frac{4}{3}\delta_{0,n}\right], \quad (3.17)$$

where  $\delta_{a,b} = 1$  (0) if  $a = (\neq)b$  is the Kronecker symbol,

$$\sum_{k=0}^n \binom{n}{k} K_k = 2n3^{n-2} - \frac{2}{3}\delta_{1,n}. \quad (3.18)$$

#### *Convolution properties*

$$H_n = \sum_{k=0}^n J_k J_{n-k} - \frac{1}{9}[2^n + (-1)^n(3n-1)], \quad (3.19)$$

$$K_n = \frac{1}{3} \sum_{k=0}^n J_k J_{n-k} - \frac{1}{9}[2^{n+2} - (-1)^n(3n-5)], \quad (3.20)$$

$$\sum_{k=0}^n H_k H_{n-k} = 3^{-7} [2^{n-1} (9n^3 - 72n^2 + 159n - 80) + (-1)^n (18n^3 - 72n^2 + 30n + 40)], \quad (3.21)$$

$$\sum_{k=0}^n K_k K_{n-k} = 3^{-5} [2^{n-1} (9n^3 - 57n + 16) + (-1)^n (18n^3 - 42n - 8)]. \quad (3.22)$$

**Remarks:**

- (i) The geometric series formula has to be used along with (2.3)-(2.8) to prove (3.15)-(3.22).  
 (ii) The identities (3.19) and (3.20) can be checked easily by using (2.5), (2.6), and the identities

$$\sum_{k=0}^n J_k J_{n-k} = \frac{1}{9} [(n+1)j_n - 2J_{n+1}] \quad (3.23)$$

and

$$\sum_{k=0}^n j_k j_{n-k} = (n+1)j_n + 2J_{n+1}, \quad (3.24)$$

which are obtainable by using (1.6) and the geometric series formula.

**Congruence properties**

Congruence properties of  $H_n$  and  $K_n$  deserve a thorough investigation. Nevertheless, in this paper we shall confine ourselves to considering the residue of these numbers modulo their subscripts. That  $K_n$  is divisible by  $n$  for all  $n > 0$  is patent by (2.4). A brief computer experiment showed that the values of  $n \leq 1000$  for which  $H_n$  is divisible by  $n$  are 1, 2, 4, 20, 100, 220, 500, 620, and 820.

**Theorem 1:** There exist infinitely many values of  $n$  for which  $H_n \equiv 0 \pmod{n}$ .

**Theorem 2:** If  $p \neq 3$  is a prime, then

$$H_p \equiv -12 \left[ \frac{1 + p(-1)^{p \pmod{3}}}{3} \right]^3 \pmod{p}.$$

**Theorem 3:** If  $p \neq 3$  is a prime, then  $K_p \equiv 0 \pmod{p^2}$ .

**3.2. Proofs of Special Results**

**Proof of Theorem 1:** We shall prove that, if  $n = n(k) = 5^k 4$  ( $k = 0, 1, 2, \dots$ ), then  $H_n \equiv 0 \pmod{n}$ . Let  $B_n$  denote the numerator of the fraction on the right-hand side of (2.5). Since  $n(k)$  and 27 are coprime, it suffices to prove that  $B_{n(k)} \equiv 0 \pmod{n(k)}$ . After some simple manipulation, it is apparent that this is equivalent to proving that  $4[2^{n(k)} - (-1)^{n(k)}] \equiv 0 \pmod{n(k)}$ , that is, to proving the validity of the congruence

$$2^{n(k)} \equiv 1 \pmod{5^k}. \quad (3.25)$$

By Euler's theorem, it is known that  $2^{n(k)} = 2^{5^k 4} = 2^{\phi(5^{k+1})} \equiv 1 \pmod{5^{k+1}}$ , whence (3.25) is satisfied *a fortiori*.

By Table 1, it is immediately seen that the congruence  $H_n \equiv 0 \pmod{n}$  holds for  $n = 1, 2$ , and 4. We now state a proposition that gives the general solution to the problem of finding all  $n > 4$

for which this congruence is satisfied. Of course, this general solution encompasses the case  $n = 5^k 4$  considered in the proof of Theorem 1.

**Proposition 1:** For  $n > 4$ ,  $H_n \equiv 0 \pmod{n}$  if and only if

$$n = 4p_2^{a_2} p_3^{a_3} \cdots (p_2 = 5, p_3 = 7, p_4 = 11, \dots; a_2 \geq 1, a_i \geq 0 \text{ for } i > 3),$$

and  $\text{ord}(2, p_i^{a_i})$  divides  $n$  for all  $i$  such that  $a_i \geq 1$ , where (see [2], p. 71) the symbol  $\text{ord}(a, b)$  [defined for  $\text{g.c.d.}(a, b) = 1$ ] denotes the least exponent  $x$  for which  $a^x \equiv 1 \pmod{b}$ .

The proof of Proposition 1 is extremely long and cumbersome; it is omitted for the sake of brevity, but it is available on request.

**Proof of Theorem 2:** By (2.5), we get the congruence

$$H_p \equiv \frac{-2^{p+2} - 4}{27} \equiv -\frac{12}{27} \pmod{p} \text{ (by Fermat's Little Theorem).}$$

The desired result is obtained readily by observing that the multiplicative inverse of 27 modulo a prime  $p \neq 3$  is  $\{[1 + p(-1)^{p \pmod{3}}] / 3\}^3$ .

**Proof of Theorem 3:** First, by Table 1, we observe that  $K_2 \equiv 0 \pmod{4}$  and  $K_3 \equiv 6 \pmod{9}$ . Then, for  $p > 5$ , let us define  $M_p = K_p / p$  and prove that  $M_p \equiv 0 \pmod{p}$ . By (2.4) and (1.6), we can write

$$M_p = 2J_{p-1} = \frac{2^p - 2}{3} \equiv \frac{2 - 2}{3} \equiv 0 \pmod{p} \text{ (by Fermat's Little Theorem).}$$

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