An Explicit Formula for Bernoulli Numbers in Terms of Stirling Numbers of the Second Kind

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Abstract: In the paper, the authors recover an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind. The main results may be summarized as the following theorem.

**Theorem 1** For \( n \geq k \geq 0 \), we have

\[
B_{n,k}(0, 1, \ldots, 1) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} S(n - i, k - i) \tag{4}
\]

and

\[
B_{n,k} \left( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2} \right) = \frac{n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+k}{k-i} S(n+i, i). \tag{5}
\]

For \( n \geq 0 \), we have

\[
B_n = \sum_{i=0}^{n} (-1)^{i+1} \binom{n+1}{i} S(n+i, i). \tag{6}
\]

Keywords: explicit formula; Bernoulli number; Stirling number of the second kind; Bell polynomial of the second kind

1 Introduction

It is well known that Bernoulli numbers \( B_k \) for \( k \geq 0 \) may be generated by

\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k} \tag{1}
\]

for \( |x| < 2\pi \). See [1, p. 48]. In combinatorics, Stirling numbers of the second kind \( S(n, k) \) for \( n \geq k \geq 0 \) may be computed by

\[
S(n, k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{(k-\ell)} \binom{k}{\ell} \ell^n \tag{2}
\]

and may be generated by

\[
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \tag{3}
\]

See [1, p. 206]. Bell polynomials of the second kind \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) are defined by

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_k \leq n} \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{i=1}^{k-1} \frac{(x_i)}{i!} \prod_{i=1}^{k} (x_i)^{\ell_i} \tag{4}
\]

for \( n \geq k \geq 1 \), See [1, p. 134, Theorem A].

The aim of this paper is to recover an explicit formula for computing Bernoulli numbers \( B_n \) in terms of Stirling numbers of the second kind \( S(n, k) \).

The main results may be summarized as the following theorem.

2 Proof of Theorem 1

In combinatorics, Faà di Bruno formula may be described in terms of Bell polynomials of the second kind

\[
\sum_{\ell_1 + \cdots + \ell_k = n} \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{i=1}^{k-1} \frac{(x_i)}{i!} \prod_{i=1}^{k} (x_i)^{\ell_i} = \sum_{k=0}^{\infty} B_{n,k}(x_1, \ldots, x_{n-k+1}) \frac{x^n}{n!} \tag{5}
\]
For $B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$ by

$$
\frac{d^n}{dx^n}f \circ g(x) = \sum_{k=1}^{n} f^{(k)}(g(x))B_{n,k}(g''(x),g'''(x),\ldots,g^{(n-k+1)}(x)). \tag{7}
$$

See [1, p. 139, Theorem C]. It is easy to see that

$$
\frac{x}{e^x-1} = \frac{1}{\int_0^1 e^x \, dt}
$$

Applying in (7) the functions $f(y) = \frac{1}{y}$ and $y = g(x) = \int_0^1 e^x \, dt$ results in

$$
\frac{d^n}{dx^n} \left( \frac{x}{e^x-1} \right) = \frac{d^n}{dx^n} \left( \frac{1}{\int_0^1 e^x \, dt} \right)
$$

$$
= \sum_{k=1}^{n} (-1)^k \frac{k!}{(\int_0^1 e^x \, dt)^{k+1}} B_{n,k} \left( \int_0^1 t e^x \, dt, \int_0^1 t^2 e^x \, dt, \ldots, \int_0^1 t^{n-k+1} e^x \, dt \right)
$$

$$
= \sum_{k=1}^{n} (-1)^k k! B_{n,k} \left( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2} \right)
$$

as $x \to 0$. On the other hand, differentiating $n$ times on both sides of (1) leads to

$$
\frac{d^n}{dx^n} \left( \frac{x}{e^x-1} \right) = \sum_{k=n}^{\infty} B_k \frac{k^n}{(k-n)!} \to B_n, \quad x \to 0.
$$

As a result, we obtain

$$
B_n = \sum_{k=1}^{n} (-1)^k k! B_{n,k} \left( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2} \right). \tag{8}
$$

In [1, p. 133], it was listed that

$$
\frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m t^m m! \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) t^n n! \tag{9}
$$

for $n \geq k \geq 0$. Letting $x_1 = 0$ and $x_m = 1$ for $m \geq 2$ in (9) and employing (3) give

$$
\sum_{n=k}^{\infty} B_{n,k}(0,1,\ldots,1) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{m=2}^{\infty} \frac{t^m}{m!} \right)^k = \frac{1}{k!} (e^t - 1 - i)^k
$$

$$
= \frac{1}{k!} \sum_{i=0}^{k} (-1)^k (i) (k+i) \frac{(e^t - 1 - i)^i}{i!}
$$

$$
= \sum_{i=0}^{k} (-1)^{i-k} \frac{(k+i)!}{(k-i)!} \frac{S(j,i)}{j!}.
$$

This implies that

$$
B_{n,k}(0,1,\ldots,1) = n! \sum_{i=0}^{k} \frac{(-1)^{k-i} S(n-k+i,i)}{(k-i)! (n-k+i)!}
$$

$$
= \frac{k}{(n-k)!} S(n-k+i,i)
$$

$$
= \sum_{i=0}^{k} (-1)^{i-j} \frac{n}{i} S(n-i,k-i).
$$

The formula (4) follows.

By virtue of

$$
B_{n,k} \left( \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_{n-k+2}}{n-k+2} \right)
$$

$$
= \frac{n!}{(n+k)!} B_{n+k,k}(0,x_2,\ldots,x_{n+1}),
$$

see [1, p. 136], and the formula (4), we obtain

$$
B_{n,k} \left( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2} \right)
$$

$$
= \frac{n!}{(n+k)!} B_{n+k,k}(0,1,\ldots,1)
$$

$$
= \frac{n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{i} \left( \frac{n+k}{i} \right) S(n+k-i,k-i),
$$

from which, the formula (5) follows.

Substituting (5) into (8) leads to

$$
B_n = \sum_{k=0}^{n} \frac{k! n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{i} \left( \frac{n+k}{i} \right) S(n+i,i)
$$

$$
= \sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^{i} \left( \frac{k}{i} \right) S(n+i,i)
$$

$$
= \sum_{i=0}^{n} \sum_{k=i}^{n} (-1)^{i-j} \left( \frac{n+1}{i+j} \right) S(n+i,i),
$$

which may be rewritten as the formula (6). The proof of Theorem 1 is complete.

### 3 Remarks

Finally we list several remarks on something to do with our main results.

**Remark 1** The formula (5) may be alternatively proved as follows.
Taking $x_m = \frac{1}{m+1}$ for all $m \in \mathbb{N}$ in (9) and utilizing (3) yield

$$
\sum_{n=k}^{\infty} B_{n,k} \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n-k+2}\right) \frac{t^n}{n!} = \frac{1}{k!} \left[ \sum_{m=1}^{\infty} \frac{t^m}{(m+1)!} \right]^k
$$

$$
= \frac{1}{k!} \left( \frac{e^t - 1 - t}{t} \right)^k
$$

$$
= \frac{1}{k!} \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{\ell!} \left( \frac{e^t - 1}{t} \right)^{\ell}
$$

$$
= \frac{1}{k!} \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{\ell!} \frac{1}{t^\ell} \sum_{i=\ell}^{\infty} S(i,\ell) t^i / i!
$$

$$
= \frac{1}{k!} \frac{(-1)^{k-\ell}}{\ell!} \sum_{i=\ell}^{\infty} S(i,\ell) t^{i-\ell} / i!.
$$

This implies that

$$
B_{n,k} \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n-k+2}\right) = n! \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{\ell!} (k-\ell)! (n+\ell)! S(n+\ell,\ell).
$$

The formula (5) follows.

**Remark 2** In [1, p. 220] and [3, pp. 559–560], the following explicit formula for computing Bernoulli numbers $B_n$ in terms of Stirling numbers of the second kind $S(n,k)$ was presented: For $n \geq 0$, we have

$$
B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} S(n,k).
$$

Recently, four alternative proofs for the formula (10) were supplied in [4] and [11].

The first formula for Bernoulli numbers $B_n$ listed in [2] is

$$
B_n = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f^j, \quad n \geq 0, \quad (11)
$$

which is a special case of the general formula [8, (2.5)].

We observe that the formula (11) is equivalent to the one (10). In all, we may collect at least seven alternative proofs for the formula (10) or (11) in the references [2], [3], [4], [8], and [11].

**Remark 3** In [6, p. 1128, Corollary], among other things, it was found that, for $k \geq 1$,

$$
B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k} \frac{A_{2(k-i)}}{2(k-i)+1},
$$

where $A_m$ is defined by

$$
\sum_{m=1}^{n} m^k = \sum_{m=0}^{k+1} A_m t^m.
$$

In [5, Theorem 3.1], it was presented that Bernoulli numbers $B_{2k}$ may be computed by

$$
B_{2k} = 1 + \sum_{m=0}^{2k-1} \frac{S(2k+1,m+1)S(2k,2k-m)}{2^k (m-1)}
$$

$$
- \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m}}
$$

for $k \in \mathbb{N}$. In [10, Theorem 1.4], among other things, it was discovered that

$$
B_{2k} = \frac{(-1)^{k-1} k}{2^{2k-2}(2k-1)} \sum_{i=0}^{k-1} (-1)^{i+\ell} \frac{2^k}{\ell} (k-i)^{2k-1}.
$$

for $n \in \mathbb{N}$.

**Remark 4** The object of the paper [2], motivated by the paper [8], is to set matters straight by presenting a bibliography, including 33 references, on explicit formulas for Bernoulli numbers and to show how one can easily manufacture expressions for Bernoulli numbers.

In [2, p. 48, (11)], it was deduced that

$$
B_n = \sum_{j=0}^{n} (-1)^j \frac{n+1}{n+j} \frac{n!}{(n+j)!} \sum_{k=0}^{j} (-1)^j k^k k^n.
$$

for $n \geq 0$. This may be rearranged as the form of the formula (6).

On 21 January 2014, the authors searched out that the formula (6) was ever derived in [7, p. 59] and [13, p. 140] by different tools from Faà di Bruno formula (7).

For more information on the history and literature of explicit formulas for computing Bernoulli numbers, please refer to [2], [7], [8], [12], and [13] and plenty references therein.

**Remark 5** This paper is a revised version of the preprint [9].

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