A new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers

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Abstract

In the paper, the authors concisely review some explicit formulas and establish a new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers of the second kind.

Keywords: explicit formula; Bernoulli number; Genocchi number; Stirling number of the second kind

MSC: Primary 11B68; Secondary 11B73

1. Introduction and main results

It is well known that the Bernoulli numbers $B_n$ for $n \geq 0$ may be defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi,$$

(1)

that Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

(2)

that the Genocchi numbers $G_n$ for $n \in \mathbb{N}$ are given by the generating function

$$\frac{2t}{e^t + 1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}.$$  

(3)
and that the Stirling numbers of the second kind which may be generated by
\[
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} \frac{S(n, k)}{n!} x^n, \quad k \in \mathbb{N}
\] (4)
and may be computed by
\[
S(k, m) = \frac{1}{m!} \sum_{\ell=1}^{m} (-1)^{m-\ell} \binom{m}{\ell} \ell^k, \quad 1 \leq m \leq k.
\] (5)

By the way, the Stirling number of the second kind \(S(n, k)\) may be interpreted combinatorially as the number of ways of partitioning a set of \(n\) elements into \(k\) nonempty subsets.

The Bernoulli numbers \(B_n\) for \(n \in \{0\} \cup \mathbb{N}\) satisfy
\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_{2n+2} \neq 0, \quad B_{2n+3} = 0.
\] (6)

For \(n \in \mathbb{N}\), the Genocchi numbers meet \(G_{2n+1} = 0\). The first few Genocchi numbers \(G_n\) are listed in Table 1.1. The

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_n)</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-3</td>
<td>17</td>
<td>-155</td>
<td>2073</td>
<td>-38227</td>
<td>929569</td>
<td>-28820618</td>
</tr>
</tbody>
</table>

Genocchi numbers \(G_{2n}\) may be represented in terms of the Bernoulli numbers \(B_{2n}\) and Euler polynomials \(E_{2n-1}(0)\) as
\[
G_{2n} = 2(1 - 2^n)B_{2n} = 2nE_{2n-1}(0), \quad n \in \mathbb{N}.
\] (7)

See [1, p. 49]. As a result, we have
\[
G_n = 2(1 - 2^n)B_n, \quad n \in \mathbb{N}.
\] (8)

The first formula for the Bernoulli numbers \(B_n\) listed in [2] is
\[
B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n, \quad n \geq 0,
\] (9)
which is a special case of the general formula [13, (2.5)]. The formula (9) is equivalent to
\[
B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} S(n, k), \quad n \in \{0\} \cup \mathbb{N},
\] (10)
which was listed in [3, p. 536] and [4, p. 560]. Recently, four alternative proofs of the formula (10) were provided in [7, 16]. A generalization of the formula (10) was supplied in [6]. In all, we may collect at least seven alternative proofs for the formula (9) or (10) in [2, 4, 7, 13, 14, 16] and closely related references therein.

In [2, p. 48, (11)], it was deduced that
\[
B_n = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \frac{n!}{(n+j)!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^{n+j}, \quad n \geq 0,
\] (11)
which may be rearranged as
\[
B_n = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i+1} \frac{1}{(n+i)!} S(n + i, i), \quad n \geq 0.
\] (12)
The formula (12) was rediscovered in the paper [8]. On 21 January 2014, the authors searched out that the formula (12) was also derived in [12, p. 59] and [17, p. 140].
In [11, p. 1128, Corollary], among other things, it was found that

\[
B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1} \tag{13}
\]

for \( k \in \mathbb{N} \), where \( A_m \) is defined by

\[
\sum_{m=1}^{\infty} m^k = \sum_{m=0}^{k+1} A_m n^m.
\]

In [15, Theorem 1.4], among other things, it was presented that

\[
B_{2k} = \frac{(-1)^{k-1}k}{2^{2(k-1)}(2^k-1)} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-i-1} (-1)^{i+\ell} \binom{2k}{\ell} (k-i-\ell)^{2k-1}, \quad k \in \mathbb{N}. \tag{14}
\]

In [10, Theorem 3.1], it was obtained that

\[
B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1, m+1)S(2k, 2k-m)}{(2k)\binom{2k}{m}} - \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k, m)S(2k+1, 2k-m+1)}{(\binom{2k}{m})}, \quad k \in \mathbb{N}. \tag{15}
\]

The aim of this paper is to find the following new explicit formula for the Bernoulli numbers \( B_k \), or say, the Genocchi numbers \( G_k \), in terms of the Stirling numbers of the second kind \( S(k, m) \).

**Theorem 1.1** For all \( k \in \mathbb{N} \), the Genocchi numbers \( G_k \) may be computed by

\[
G_k = 2(1 - 2^k)B_k = (-1)^k k \sum_{m=1}^{k} (-1)^m \frac{(m-1)!}{2^m-1} S(k, m). \tag{16}
\]

## 2. Proof of Theorem 1.1

Differentiating on both sides of the equation (3) and employing Leibniz identity for differentiation give

\[
\left( \frac{2t}{e^t+1} \right)^{(k)} = 2 \left[ t \left( \frac{1}{e^t+1} \right)^{(k)} + k \left( \frac{1}{e^t+1} \right)^{(k-1)} \right] = \sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!}.
\]

In [9, Theorem 2.1] and [18, Theorem 3.1], it was obtained that, when \( \lambda > 0 \) and \( t \neq -\frac{\ln \lambda}{\alpha} \) or when \( \lambda < 0 \) and \( t \in \mathbb{R} \),

\[
\left( \frac{1}{\lambda e^{\alpha t} - 1} \right)^{(k)} = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left( \frac{1}{\lambda e^{\alpha t} - 1} \right)^m. \tag{17}
\]

Specially, when \( \lambda = -1 \) and \( \alpha = 1 \), the identity (17) becomes

\[
\left( \frac{1}{e^t+1} \right)^{(k)} = (-1)^{k+1} \sum_{m=1}^{k+1} (-1)^m (m-1)! S(k+1, m) \left( \frac{1}{e^t+1} \right)^m. \tag{18}
\]

Consequently, it follows that

\[
G_k = \lim_{t \to 0} \sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!} = 2k \lim_{t \to 0} \left( \frac{1}{e^t+1} \right)^{(k-1)} = 2k(-1)^k \sum_{m=1}^{k} (-1)^m (m-1)! S(k, m) \lim_{t \to 0} \left( \frac{1}{e^t+1} \right)^m = (-1)^k \sum_{m=1}^{k} (-1)^m \frac{(m-1)!}{2^m-1} S(k, m).
\]

The proof of Theorem 1.1 is complete.

**Remark 2.1** This paper is a slightly modified version of the preprint [5].
References


