On the k-Lucas Numbers

Sergio Falcon

Department of Mathematics
University of Las Palmas de Gran Canaria, Spain
sfalcon@dma.ulpgc.es

Abstract

From a special sequence of squares of k-Fibonacci numbers, the k-Lucas sequences are obtained in a natural form. Then, we will study the properties of the k-Lucas numbers and will prove these properties will be related with the k-Fibonaci numbers. In this paper we examine some of the interesting properties of the k-Lucas numbers themselves as well as looking at its close relationship with the k-Fibonacci numbers. The k-Lucas numbers have lots of properties, similar to those of k-Fibonacci numbers and often occur in various formulae simultaneously with latter.

1 Introduction

In this section, we introduce the k-Fibonacci numbers and its generation is justified.

There is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [9, 10, 11]. The Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\phi = \dfrac{1 + \sqrt{5}}{2}$, which appears in modern research, particularly physics of the high energy particles [5] or theoretical physics [6, 7, 8].

In this section, we present a generalization of the classical Fibonacci numbers by mean of a recurrence equation with a parameter k. In the sequel, we

show some properties proven in papers [3, 4] which generalize the respective properties of the classical Fibonacci sequence.

Definition 1.1 *For any integer number $k \geq 1$, the k-th Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbf{N}}$ is defined recurrently by:*

$$F_{k,0} = 0, \ F_{k,1} = 1, \ and \ F_{k,n+1} = kF_{k,n} + F_{k,n-1} \ for \ n \geq 1.$$

As particular cases

- if $k = 1$, we obtain the classical Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$

- if $k = 2$, the Pell sequence appears $\{0, 1, 2, 5, 12, 29, 70, \ldots\}$

- if $k = 3$, we obtain the sequence $\{F_{3,n}\}_{n \in \mathbf{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}$

From definition of the k-Fibonacci numbers, the first of them are presented in Table 1, and from these expressions, one may deduce the value of any k-Fibonacci number by simple substitution. For example, the seventh element of the 4-Fibonacci sequence, $\{F_{4,n}\}_{n \in N}$, is $F_{4,7} = 4^6 + 5 \cdot 4^4 + 6 \cdot 4^2 + 1 = 5473$.

First k-Fibonacci numbers are showed in Table 1:

Table 1: First k-Fibonacci numbers

$$F_{k,1} = 1$$
$$F_{k,2} = k$$
$$F_{k,3} = k^2 + 1$$
$$F_{k,4} = k^3 + 2k$$
$$F_{k,5} = k^4 + 3k^2 + 1$$
$$F_{k,6} = k^5 + 4k^3 + 3k$$
$$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1$$
$$F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k$$

There are a large number of k-Fibonacci sequences indexed in *The Online Encyclopedia of Integer Sequences* [12], from now on OEIS, being the first

- $\{F_{1,n}\} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$: A000045

- $\{F_{2,n}\} = \{0, 1, 2, 5, 12, 29, \ldots\}$: A000129

- $\{F_{3,n}\} = \{0, 1, 3, 10, 33, 109, \ldots\}$: A006190

Some of the properties that the k-Fibonacci sequences verify are summarized bellow, (see [3, 4] for details of the proofs):

- **Binet formula:** $F_{k,n} = \dfrac{\sigma_k^n - (\sigma_k)^{-n}}{\sigma_k + \sigma_k^{-1}}$, where $\sigma_k = \dfrac{k + \sqrt{k^2 + 4}}{2}$ is the positive root of the characteristic equation $r^2 - k \cdot r - 1 = 0$ associated to the recurrence relation defining k-Fibonacci numbers.

- **First combinatorial formula:** $F_{k,n} = \dfrac{1}{2^{n-1}} \displaystyle\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i} \left(k^2 + 4\right)^i$

- **Second combinatorial formula:** $F_{k,n} = \displaystyle\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i}$

- **Catalan Identity:** $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r} F_{k,r}^2$

- **Simson Identity:** $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$

- **D'Ocagne Identity:** $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$

- **Odd k-Fibonacci:** $F_{k,2n+1} = F_{k,n}^2 + F_{k,n+1}^2$

- **Even k-Fibonacci:** $F_{k,2n} = \frac{1}{k}(F_{k,n+1}^2 - F_{k,n-1}^2)$

- **Sum of the first n terms:** $S_{k,n} = \displaystyle\sum_{i=1}^{n} F_{k,i} = \dfrac{1}{k}\left(F_{k,n+1} + F_{k,n} - 1\right)$

- **Sum of the first n even terms:** $\displaystyle\sum_{i=1}^{n} F_{k,2i} = \dfrac{1}{k}\left(F_{k,2n+1} - 1\right)$

- **Sum of the first n odd terms:** $\displaystyle\sum_{i=1}^{n} F_{k,2i+1} = \dfrac{1}{k}F_{k,2n+2}$

- **Generating function:** $f_k(x) = \dfrac{x}{1 - kx - x^2}$

2 Generation of the k-Lucas numbers from the k-Fibonacci numbers

In this section we introduce some sequences obtained from the k-Fibonacci sequences and then some properties of the k-Lucas numbers will be proved. Previously, we need the following theorem.

Theorem 2.1 *or any integer n, number $(k^2 + 4)F_{k,n}^2 + 4(-1)^n$ is a perfect square.*

Proof. From the Binet formula for the k-Fibonacci numbers, we obtain
$$F_{k,n}^2 = \frac{\sigma_k^{2n} - 2(-1)^n + \sigma_k^{-2n}}{k^2 + 4}.$$
If n is even, then $(k^2 + 4)F_{k,n}^2 + 4 = \sigma_k^{2n} - 2 + \sigma_k^{-2n} + 4 = (\sigma_k^n + \sigma_k^{-n})^2$
If n is odd, then it is $(k^2 + 4)F_{k,n}^2 - 4 = \sigma_k^{2n} + 2 + \sigma_k^{-2n} - 4 = (\sigma_k^n - \sigma_k^{-n})^2$

Now we show the first sequences obtained after finding the square root of the numbers of the preceding form and we show also the reference code of these sequences in OEIS:

- $L_1 = \{L_{1,n}\} = \{2, 1, 3, 4, 7, 11, 18, 29, \ldots\}$: A000032

- $L_2 = \{L_{2,n}\} = \{2, 2, 6, 14, 34, 82, 198, 478, \ldots\}$: A002203

- $L_3 = \{L_{3,n}\} = \{2, 3, 11, 36, 119, 393, 1298, 4287, \ldots\}$: A006497

First one is the well known Lucas sequence and second one is the Pell-Lucas sequence. For this reason, we have decided to call them *The k-Lucas Sequences.*

Elements of these sequences, say $L_k = \{L_{k,n}\}$, verify the following recurrence law:
$$L_{k,n+1} = k\, L_{k,n} + L_{k,n-1} \quad \text{for } n \geq 1 \tag{1}$$
with initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$

Classical Lucas numbers $\{L_{1,n}\}$ are related with the Artin's Constant [1].

Sequences $\{F_{k,n}\}$ and $\{L_{k,n}\}$ are called conjugate sequences in a k-Fibonacci-Lucas sense [14].

Expressions of first k-Lucas numbers are presented in Table 2, and from these expressions, anyone may deduce the value of any k-Lucas number by simple substitution on the corresponding $L_{k,n}$ as we have done for $F_{k,n}$. First k-Lucas numbers are showed in Table 2:

Table 2: First k-Lucas numbers

$$L_{k,0} = 2$$
$$L_{k,1} = k$$
$$L_{k,2} = k^2 + 2$$
$$L_{k,3} = k^3 + 3k$$
$$L_{k,4} = k^4 + 4k^2 + 2$$
$$L_{k,5} = k^5 + 5k^3 + 5k$$
$$L_{k,6} = k^6 + 6k^4 + 9k^2 + 2$$
$$L_{k,7} = k^7 + 7k^5 + 14k^3 + 7k$$

Particular cases:

- For $k = 1$, the classical Lucas sequence appears: $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$

- For $k = 2$, we obtain the Pell-Lucas sequence: $\{2, 2, 6, 14, 34, 82, 198, \ldots\}$

Theorem 2.2 (Binet formula) *k-Lucas numbers are given by the formula* $L_{k,n} = \sigma_k^n + (-\sigma_k)^{-n}$ *with* $\sigma_k = \dfrac{k + \sqrt{k^2 + 4}}{2}$.

Proof. Characteristic equation of the recurrence (1) is $r^2 - kr - 1 = 0$, which solutions are $\sigma_k = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $\sigma_k' = \dfrac{k - \sqrt{k^2 + 4}}{2}$. So, solution of equation (1) is $L_{k,n} = C_1\sigma_k^n + C_2\sigma_k'^n$. By doing $n = 0 \rightarrow L_{k,0} = 2$ and $n = 1 \rightarrow L_{k,1} = k$, we obtain the values $C_1 = C_2 = 1$. Finally, taking into account $\sigma_k \cdot \sigma_k' = -1 \rightarrow \sigma_k' = -\frac{1}{\sigma_k}$, and then $L_{k,n}$. ∎

As particular cases of this formula, and following to [11]:

- If $k = 1$ we obtain the classical Lucas numbers, and then $\sigma_1 = \frac{1+\sqrt{5}}{2}$ is well-known as *the golden ratio*, ϕ, while σ_1' is usually written as φ. In this notation the general term of the classical Lucas sequence is given by

$$L_n = \phi^n + \varphi^n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

- If $k = 2$, then $\sigma_2 = 1 + \sqrt{2}$ and it is known as *the silver ratio* and the correspondent sequence is the Pell-Lucas sequence in wich $LP_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$

- Finally, if $k = 3$, then $\sigma_3 = \frac{3+\sqrt{13}}{2}$ is known as *the bronze ratio*

Theorem 2.3 (First relation) *A first kind of consequence of the generation of the k-Lucas numbers, is the formula*

$$L_{k,n}^2 = (k^2 + 4)F_{k,n}^2 + 4(-1)^n \tag{2}$$

Proof. It is enough to apply the Binet formula to this expression. ∎

Theorem 2.4 (Second relation) *Between the k-Lucas numbers and the k-Fibonacci numbers it is verified*

$$L_{k,n} = F_{k,n-1} + F_{k,n+1} \quad \text{for } n \geq 1 \tag{3}$$

Proof. By induction. If $n = 1$, then $F_{k,0} + F_{k-2} = k = L_{k,1}$. Let us suppose formula is true until $n - 1$. then $L_{k,n-2} = F_{k,n-3} + F_{k,n-1}$ and $L_{k,n-1} = F_{k,n-2} + F_{k,n}$.

So,

$$\begin{aligned} L_{k,n} &= k\,L_{k,n-1} + L_{k,n-2} = k(F_{k,n-2} + F_{k,n}) + F_{k,n-3} + F_{k,n-1} \\ &= (k\,F_{k,n-2} + F_{k,n-3}) + (k\,F_{k,n} + F_{k,n-1}) = F_{k,n-1} + F_{k,n+1} \end{aligned}$$

If $k = 1$, then it is $L_n = F_{n-1} + F_{n+1}$ [2].

Theorem 2.5 (Third relation)

$$L_{k,n}^2 + L_{k,n+1}^2 = (k^2 + 4)F_{k,2n+1} \tag{4}$$

Proof. It is anought to take into account the Binnet Identity and formula (2).

Theorem 2.6 (Asymptotic behaviour) $\displaystyle\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-r}} = \sigma_k^r$

Proof. It is enough to take into account $\displaystyle\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-r}} = \sigma_k^r$ [4] and equation (2).

Theorem 2.7 (Combinatorial formula for k-Lucas number) *Taking into account $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$, and expanding Binnet Identity, it is easy to find out the combinatorial formula for the k-Lucas number:*

$$L_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} k^{n-2i}(k^2+4)^i \tag{5}$$

In a particular case when $k = 1$, for the classical Lucas numbers formula $L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 5^i$ is obtained.

If $k = 2$, for the Lucas-Pell numbers it is $LP_n = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 2^i$

Theorem 2.8 (Catalan Identity) *For $r > n$, relation $L_{k,n-r}L_{k,n+r} - L_{k,n}^2 = (-1)^{n+r}L_{k,2r} + 2(-1)^{n+1}$ is verified.*

Proof. By replacing Binet Identity in expression $L_{k,n-r}L_{k,n+r} - L_{k,n}^2$, and taking into account $\sigma_k \cdot \sigma_k' = -1 \to \sigma_k' = -\sigma_k$, we find out

$$L_{k,n-r}L_{k,n+r} - L_{k,n}^2 =$$

$$= \left(\sigma_k^{n-r} + (-\sigma_k)^{-n+r}\right)\left(\sigma_k^{n+r} + (-\sigma_k)^{-n-r}\right) - \left(\sigma_k^n + (-\sigma_k)^{-n}\right)^2$$

$$= (-1)^{n+r}(\sigma_k^{-2r} + \sigma_k^{2r}) - 2(-1)^n = (-1)^{n+r}L_{k,2r} + 2(-1)^{n+1}$$

In the particular case when $r = 1$, the Simson (or Cassini) formula for the k-Lucas numbers is obtained: $L_{k,n-1}L_{k,n+1} - L_{k,n}^2 = (-1)^{n+1}(k^2 + 4)$ only taking into account that $L_{k,2} = k^2 + 2$.

If $r = 2$, then it is $L_{k,n}^2 - L_{k,n-2}L_{k,n+2} = (-1)^{n-1}L_{k,4} + 2(-1)^n = (-1)^{n-1}k^2(k^2 + 4)$

Finally, if $k = 1$, then $L_n^2 - L_{n-1}L_{n+1} = (-1)^n 5$

Corollary 2.9 (Gelin-Cesàro's Identity) *As an application of Catalan Identity, it is easy to find out the following formula that generalizes the Gelin-Cesàro's Identity for the Lucas numbers:*

$$L_{k,n-2}L_{k,n-1}L_{k,n+1}L_{k,n+2} - L_{k,n}^4 + k^2(k^2 + 4)^2 = (-1)^n(k^4 + 3k^2 - 4)L_{k,n}^2$$

If $k = 1$, for the classical Lucas numbers, we obtain
$$L_{n-2}L_{n-1}L_{n+1}L_{n+2} + 25 = L_n^4$$

There are a large set of formulas that relate the k-Lucas numbers to the k-Fibonacci numbers.

Theorem 2.10 (Convolution theorem) *For $m \geq 1$, relation between the k-Lucas numbers and the k-Fibonacci numbers $L_{k,n+1}L_{k,m} + L_{k,n}L_{k,m-1} = (k^2 + 4)F_{k,n+m}$ is verified.*

Proof. By induction. For $m = 1$:

$$L_{k,n+1}L_{k,1} + L_{k,n}L_{k,0} = k\,L_{k,n+1} + 2L_{k,n} =$$
$$= L_{k,n+2} + L_{k,n} = F_{k,n+3} + 2F_{k,n+1} + F_{k,n-1}$$
$$= k\,F_{k,n+2} + 3F_{k,n+1} + F_{k,n+1} - k\,F_{k,n}$$
$$= k^2 F_{k,n+1} + k\,F_{k,n} + 4F_{k,n+1} - k\,F_{k,n} = (k^2 + 4)F_{k,n+1}$$

Let us suppose formula is true until $m - 1$:
$$L_{k,n+1}L_{k,m-1} + L_{k,n}L_{k,m-2} = (k^2 + 4)F_{k,n+m-1}.$$
Then

$$(k^2 + 4)F_{k,n+m} = (k^2 + 4)(k\,F_{k,n+m-1} + F_{k,n+m-2})$$
$$= k(L_{k,n+1}L_{k,m-1} + L_{k,n}L_{k,m-2}) + (L_{k,n+1}L_{k,m-2} + L_{k,n}L_{k,m-3})$$
$$= L_{k,n+1}L_{k,m} + L_{k,n}L_{k,m-1}$$

Particular cases:

- If $k = 1$, for both, the classical Lucas and classical Fibonacci sequences, formula $L_{n+1}L_m + L_nL_{m-1} = 5F_{n+m}$ is obtained.

- If $m = n + 1$, we obtain again formula 4: $L_{k,n+1}^2 + L_{k,n}^2 = (k^2 + 4)F_{k,2n+1}$

- In this last case, if $k = 1$, for the classical sequences it is obtained $L_{n+1}^2 + L_n^2 = 5F_{2n+1}$

- If $m = 1$, then it is $L_{k,n+1}L_{k,1} + L_{k,n}L_{k,0} = (k^2 + 4)F_{k,n+1}$
  $\to k\,L_{k,n+1} + 2L_{k,n} = (k^2 + 4)F_{k,n+1}$
  $\to L_{k,n+2} + L_{k,n} = (k^2 + 4)F_{k,n+1}$

  And, consecuently, changing n by $n - 1$ we obtain again formula (**??**)
  $L_{k,n+1} + L_{k,n-1} = (k^2 + 4)F_{k,n}$

Theorem 2.11 (D'Ocagne identity) *If* $m \geq n$: $L_{k,m}L_{k,n+1} - L_{k,m+1}L_{k,n} = (-1)^{n+1}(k^2 + 4)F_{k,m-n}$

Proof. By induction. For $n = 0$ and applying again formula (**??**)
$L_{k,m}L_{k,1} - L_{k,m+1}L_{k,0} = k\,L_{k,m} - 2L_{k,m+1}$
$= -(L_{k,m-1} + L_{k,m+1}) = -(k^2 + 4)F_{k,m}$
Let us suppose formula is true untill $n - 1$:
$L_{k,m}L_{k,n-1} - L_{k,m+1}L_{k,n-2} = (-1)^{n-1}(k^2 + 4)F_{k,m-(n-2)}$ and
$L_{k,m}L_{k,n} - L_{k,m+1}L_{k,n-1} = (-1)^n(k^2 + 4)F_{k,m-(n-1)}$.
Then
$L_{k,m}L_{k,n+1} - L_{k,m+1}L_{k,n} = L_{k,m}(k\,L_{k,n} + L_{k,n-1}) - L_{k,m+1}(k\,L_{k,n-1} + L_{k,n-2})$
$= k(L_{k,m}L_{k,n} - L_{k,m+1}L_{k,n-1}) + (L_{k,m}L_{k,n-1} - L_{k,m+1}L_{k,n-2})$
$= (-1)^n(k^2 + 4)[k\,F_{k,m-(n-1)} - F_{k,m-(n-2)}] = (-1)^{n+1}(k^2 + 4)F_{k,m-n}$
As a particular case, if $n = m - 1$, as $F_{k,1} = 1$, it is
$L_{k,m}^2 - L_{k,m+1}L_{k,m-1} = (-1)^m(k^2 + 4)$ and the Cassini identity is obtained

Theorem 2.12 (Fourth relation) *For* $n \in \mathcal{N}$, $L_{k,n}F_{k,n} = F_{k,2n}$

Proof. In [4], we have proved $F_{k,2n} = \frac{1}{k}(F_{k,n+1}^2 - F_{k,n}^2)$.
So, $F_{k,2n} = \frac{1}{k}(F_{k,n+1} + F_{k,n-1})(F_{k,n+1} - F_{k,n-1}) = L_{k,n}F_{k,n}$
This is a simple proof that $F_{k,2n}$ is multiple of $F_{k,n}$.

Corollary 2.13 (A new relation between these k-numbers) *For* $n \geq 2$, *relation* $\prod_{i=1}^{n-1} L_{k,2^i} = F_{k,2^n}$ *is verified.*

Proof. By induction. If $n = 2$, then $\prod_1^1 L_{k,2} = k = F_{k,2}$ (see table 1). Let us suppose formula is true for n: $\prod_{i=1}^{n-1} L_{k,2^i} = F_{k,2^n}$.

Then

$$\begin{aligned}
\prod_{i=1}^{n} L_{k,2^i} &= \prod_{i=1}^{n-1} L_{k,2^i} \cdot L_{k,2^n} \\
&= F_{k,2^n} L_{k,2^n} = F_{k,2\cdot 2^n} = F_{k,2^{n+1}}
\end{aligned}$$

In particular, for both the classical Fibonacci and the classical Lucas sequences, it is $\prod_{i=1}^{n-1} L_{2^i} = F_{2^n}$

2.1 A new relation between the k-Lucas numbers

For
$n, \ r \geq 0$, *it is* $L_{k,n} L_{k,n+r} = L_{k,2n+r} + (-1)^n L_{k,r}$.

Proof. By induction. For $r = 0$ it is $L_{k,n}^2 = (\sigma_k^n + (-\sigma_k)^{-n})^2 = \sigma_k^{2n} + (-\sigma_k)^{-2n} + 2(-1)^n = L_{k,2n} + (-1)^n L_{k,0}$
Let us suppose formula is true untill $r - 1$:
$L_{k,n} L_{k,n+r-1} = L_{k,2n+r-1} + (-1)^n L_{k,r-1}$. Then:

$$\begin{aligned}
L_{k,n} L_{k,n+r} &= L_{k,n}(k\,L_{k,n+r-1} + L_{k,n+r-2}) \\
&= k(L_{k,2n+r-1} + (-1)^n L_{k,r-1}) + L_{k,2n+r-2} + (-1)^n L_{k,r-2} \\
&= k(L_{k,2n+r-1} + L_{k,2n+r-2}) + (-1)^n(k\,L_{k,r-1} + L_{k,r-2} \\
&= L_{k,2n+r} + (-1)^n L_{k,r}
\end{aligned}$$

Particular cases:

- If $r = 0$, then $L_{k,2n} = L_{k,n}^2 + 2(-1)^{n+1}$

- If $r = 1$, then $L_{k,n} L_{k,n+1} = L_{k,2n+1} + k(-1)^n$ and, consequently, $L_{k,2n+1} = L_{k,n} L_{k,n+1} + (-1)^{n+1} k$. In this case, if $k = 1$, for the classical Lucas numbers, relation $L_{2n+1} = L_n L_{n+1} + (-1)^{n+1}$ is verified.

- If $r = n$, then $L_{k,3n} = L_{k,n}(L_{k,n}^2 + 3(-1)^{n+1})$

From these two last equations, it is easy to obtain the two following formulas:

$$\text{If } n \text{ is odd} \qquad L_{k,m\cdot n} = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1-i}{i} L_{k,n}^{m-2i}$$

$$\text{If } n \text{ is even} \qquad L_{k,m\cdot n} = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1-i}{i} (-1)^i L_{k,n}^{m-2i}$$

Theorem 2.14 (Sum of the first k-Lucas numbers) *Sum of the n first k-Lucas numbers is* $\sum_{i=0}^{n} L_{k,i} = 1 + \frac{1}{k}(L_{k,n} + L_{k,n+1} - 2)$

Proof. As $L_{k,n} = F_{k,n+1} + F_{k,n-1}$ (equation (3)), and $\sum_{i=0}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1)$, it is

$$
\begin{aligned}
\sum_{i=0}^{n} L_{k,i} &= 2 + \sum_{i=1}^{n} L_{k,i} = 2 + \sum_{i=1}^{n}(F_{k,i-1} + F_{k,i+1}) \\
&= 2 + \frac{1}{k}(F_{k,n} + F_{k,n-1} - 1) + \frac{1}{k}(F_{k,n+2} + F_{k,n+1} - 1) - F_{k,1} \\
&= 1 + \frac{1}{k}(F_{k,n-1} + F_{k,n+1} + F_{k,n} + F_{k,n+2} - 2) \\
&= 1 + \frac{1}{k}(L_{k,n} + L_{k,n+1} - 2)
\end{aligned}
$$

In particular, if $k = 1$, then $\sum_{i=0}^{n} L_i = L_{n+2} - 1$

The only sequences of partial sums of k-Lucas numbers listed in OEIS are:

- For $k = 1$: $\{2, 3, 6, 10, 17, 28, 46, 75, \ldots\}$: A001610-$\{0\}$

- For $k = 2$: $\{2, 4, 10, 24, 58, 140, 338, \ldots\}$: A052542-$\{1\}$

Second sequence is simply twice the Pell numbers.

2.2 Generating function of the k-Lucas numbers

In this paragraph, the generating function for the k-Lucas sequences is given. As a result, k-Lucas sequences are seen as the coefficients of the corresponding generating function [13].

Let us suppose k-Fibonacci numbers are the coefficients of a potential series centered at the origin, and consider the corresponding analytic function $l_k(x)$. Function defined in such a way is called the generating function of the k-Lucas numbers. So,

$$
l_k(x) = L_{k,0} + L_{k,1}x + L_{k,2}x^2 + \ldots + L_{k,n}x^n + \ldots
$$

and then,

$$
\begin{aligned}
kx\, l_k(x) &= k\, L_{k,0}x + k\, L_{k,1}x^2 + k\, L_{k,2}x^3 + \ldots + k\, L_{k,n}x^{n+1} + \ldots \\
x^2 l_k(x) &= L_{k,0}x^2 + L_{k,1}x^3 + L_{k,2}x^4 + \ldots + L_{k,n}x^{n+2} + \ldots \\
&\rightarrow (1 - k\,x - x^2)l_k(x) = 2 - k\,x \\
&\rightarrow l_k(x) = \frac{2 - k\,x}{1 - k\,x - x^2}
\end{aligned}
$$

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