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Combinatorial Sums:

Egorychev's Method of Coefficients and Riordan Arrays

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Master Thesis
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Abstract

G.P. Egorychev introduced a method which transforms combinatorial sums (e.g. sums involving binomial coefficients and also non-hypergeometric expressions arising in combinatorial context) into integrals. These integrals can be simplified using substitution or residue-calculus. With the help of this method one can compute combinatorial sums to which classical algorithms are not applicable. In this thesis we restrict to the residue functional instead of manipulating integral representations. We demonstrate among others how the Lagrange inversion rule can be applied to find closed forms for combinatorial sums. The special focus is laid on sums involving Stirling numbers and Bernoulli numbers that are not that easy to handle in comparison to sums over binomial coefficients. The latter sums can be handled e.g. with the application of Zeilberger's algorithm. A related notion that will be discussed and used are Riordan arrays, a concept which we also use to handle non-trivial sums.

Zusammenfassung

G.P. Egorychev hat eine Methode vorgestellt, welche kombinatorische Summen (z.B. Summen über Binomialkoeffizienten und nicht hypergeometrischen kombinatorischen Zahlen) in Integrale transformiert. Diese Integrale können dann mittels Substitution bzw. Residuen-Kalkül vereinfacht werden. Mit Hilfe dieses Verfahrens kann man geschlossene Formen für Summen berechnen, auf die klassische Algorithmen nicht anwendbar sind. In dieser Arbeit wird gezeigt wie zum Beispiel die Lagrange'sche Inversionsregel verwendet werden kann, um geschlossene Formen für kombinatorische Summen zu finden. Der Schwerpunkt ist auf Summen mit Stirlingzahlen und Bernoullizahlen gelegt, welche nicht so einfach zu handhaben sind wie vergleichsweise Summen über Binomialkoeffizienten (letztere können z.B. mit dem Zeilberger Algorithmus behandelt werden). Ein verwandtes Konzept, das beschrieben wird, ist das des Riordan Arrays welches auch verwendet werden kann, um nichttrivale Summen zu berechnen.

List of used Symbols

Symbol	Name	Defined on page
$\mathbb{N}$	set of nonnegative integers $\{0,1,2,3,...\}$	-
$\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$	set of integers, rational, real and complex numbers	-
$\mathbb{K}$	a field containing the field $\mathbb{Q}$ as a subfield	-
$\Re(z),\Im(z)$	real (resp. imaginary) part of $z = x + \mathrm{i}y$	-
$\langle x^n\rangle\ f(x)$	coefficient functional	12
$x^{\overline{k}}, x^{\underline{k}}$	rising (resp. falling) factorial	5
$\binom{n}{k}$	binomial coefficient	5
$S_1(n,k)$	Stirling numbers of the 1$^{\text{st}}$ kind	6
$S_2(n,k)$	Stirling numbers of the 2$^{\text{nd}}$ kind	6
$(B_n)_{n\geq 0}$	sequence of Bernoulli numbers	6
$(C_n)_{n\geq 0}$	a C-finite sequence	60
$(F_n)_{n\geq 0}$	sequence of Fibonacci numbers	56
$\delta(n,k)$	Kronecker delta function	6
$(H_n)_{n\geq 0}$	sequence of harmonic numbers	82
$\Gamma(x)$	Eulerian gamma function	90
$\mathbb{K}[x]$	ring of polynomials over $\mathbb{K}$	-
$\mathbb{K}[[x]]$	ring of formal power series over $\mathbb{K}$	7
$\mathbb{K}(x)$	field of rational functions over $\mathbb{K}$	-
$\mathbb{K}((x))$	field of formal Laurent series over $\mathbb{K}$	11
$\mathbb{K}_k((x))$	set of all formal Laurent series of order k	12
$D_x(a(x))$	formal derivative of $a(x)\in\mathbb{K}((x))$	12
CF	set of C-finite sequences over $\mathbb{K}$	60
PF	set of P-finite (or equiv. holonomic) sequences over $\mathbb{K}$	73
RA	set of Riordan arrays	37

Contents

Chapter 1

Introduction

1.1 Overview

This master thesis contributes to problem solving methods related to symbolic summation and generating functions of combinatorial sequences. During the last years several creative and effective approaches towards systematic treatments have been introduced. The Egorychev method is one such attempt.

One of the reasons why the Egorychev method is that powerful is that it takes advantage of the Lagrange inversion rule in a constructive way to derive a closed form for a generating function. With this application one can prove complicated identities such as Abel's identity

$$\sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k} = (a+b+n)^n \tag{1.1}$$

or Gould's identity

$$\sum_{k=0}^{r} \frac{r}{r-kq} \binom{r-qk}{k} \binom{p+kq}{n-k} = \binom{p+r}{n}. \tag{1.2}$$

A somewhat similar approach is the concept of the Riordan group that also applies the Lagrange inversion rule for proving combinatorial identities. Clever construction of Riordan arrays makes it easy to discover identities of similar type.

One of the reasons for this thesis was the interest of the author to compute sums that are not applicable to classical algorithms. The author has attended the Algorithmic Combinatorics Seminar of the RISC combinatorics group for several years and got an impression of the power of symbolic summation techniques and its utilization in several applications. In particular, the author wants to mention the work of Dr. Kauers, Dr. Schneider, Dr. Koutschan and the advisor of this thesis, Prof. Paule. The author wants to thank them for their enthusiasm and their always present will to give insights in their work and discussing until all details were understood. We will describe some of their work in the following chapters and also take a look on how their developments can be applied to solve symbolic summation problems.

This thesis focusses on combinatorial numbers such as Stirling numbers, Bernoulli numbers and binomial coefficients. Combinatorial numbers express some of the most fundamental properties of combinatorial objects in mathematics (as for instance the number of subsets with certain properties, ...) and are in fact not that trivial to handle. A goal of this master thesis is to present some approaches that can be used to derive closed form solutions for combinatorial sums. Also several concrete examples are given to see immediately how the machinery developed by Egorychev and Shapiro/Sprugnoli proves identities arising (among others) in combinatorial mathematics.

One might speculate that there is a way to automatize the Lagrange inversion rule on a computer algebra system which would lead to a new algorithmic method that assists in finding closed forms for combinatorial sums. Especially the inversion rule which boils the problem down to pattern matching has the potential to be implemented on some computer algebra system. Unfortunately this is only part of the work as we will see, because we need some preprocessing steps which is not that easy to automatize. Therefore we still need to go back to paper and pencil for particular problem classes.

1.2 Software packages we used

In this work the author used several software packages (most of them developed by the Algorithmic Combinatorics group of RISC Linz) for the Mathematica computer algebra system. We demonstrate how this packages can be used to solve symbolic summation problems. The following packages have been used in this thesis (in alphabetical order)

- M. Kauers: Stirling [Kau07]: a Mathematica package for computing recurrence equations of sums involving Stirling numbers or Eulerian numbers.
- C. Koutschan: Holonomic Functions [Kou09, Kou10]: a Mathematica package for dealing with multivariate holonomic functions, including closure properties, summation, and integration.

- C. Mallinger: Generating Functions [Mal96]: a Mathematica package for manipulations of univariate holonomic functions and sequences.
- P. Paule, M. Schorn [PS95]: the Paule/Schorn implementation of Gosper's and Zeilberger's algorithm in Mathematica.
- M. Petkovšek: Hyper [Pet98]: a Mathematica implementation of Petkovšek's algorithm Hyper.
- C. Schneider: Sigma [Sch01, Sch04, Sch07]: a Mathematica package for discovering and proving multi-sum identities.

The packages developed at RISC can be obtained from

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http://www.risc.jku.at/research/combinat/software/
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1.3 How to read the thesis

The thesis is divided into four chapters, which depend upon each other. In the following chapter 2 we clarify the notion of combinatorial numbers and recall their combinatorial interpretation. Further we go into details how to operate on formal power series and Laurent series. The central element in this thesis is the notion of residue functional. A detailed account will be devoted to its application on Laurent series. Finally we will investigate how this is related to the concept of residue appearing in complex analysis. Chapter 3 is devoted to the notion of Riordan arrays that closely relates to manipulating power series on coefficient level. We examine ways of characterizing Riordan arrays by its A- and Z-sequence. At the end of the day we want to apply the developed machinery to solve problems in the area of symbolic summation and computing closed forms for generating functions. This will be the main focus in chapter 4. This chapter will also provide some new aspects (to the author's opinion). We apply Egorychev's approach, to Gould's identity (1.2) and to a generalization of Abel's identity (4.12) that is based on Riordan arrays. The power of the Egorychev method on multi-sum identities is demonstrated on an American Mathematical Monthly problem (see section 4.2) that has been worked out (with input provided by the BSI Problems group, Bonn). An application of Mathematica packages is discussed to solve this kind of problems. As the name suggests, the next section 4.3 shows a recent American Mathematical Monthly problem that was solved by deriving a residue representation and evaluating it explicitly in closed form. During the examination of a symbolic sum involving Fibonacci numbers the author could generalize the shape of the summand to evaluate sums with binomial coefficients and C-finite sequences. This generalization (Thm. 4.4.5) includes an exercise of [Wil06, Ex. 4.16] as special case. Next we will show how to derive a residue representation for an explicit formula for Stirling numbers of the second kind (4.34) (that reflects the generating function). Stirling numbers are also subject in the identities (4.36), (4.37) that are out of scope for the known methods introduced in [PWZ96]. For proving an identity (equality of two binomial sums) that is needed for computing the generating function of Jacobi polynomials we will make use of the method of coefficients although Zeilberger's algorithm as well as the Snake Oil method would also be applicable. Finally we give a summation example involving harmonic numbers and present several ways to compute it in closed form. The final subsection is a remark on asymptotic analysis.

1.4 Acknowledgements

I joined the combinatorics group at RISC in spring 2008. Since that time I had the honor of learning from experts at this group both in theoretical lectures and practical research. My first thank undoubtedly goes to the leader of this group and the advisor of this thesis, Prof. Peter Paule. With his enthusiasm and his advices he handled to motivate me in writing this thesis. He also introduced the classic tools in his lectures and showed where still further research could be done. The author was financially supported by the Doctoral Program Computational Mathematics (W1214) whom I want to thank too. Further I want to thank Dr. Schneider for introducing me to his Sigma package, Dr. Koutschan for demonstrating his HolonomicFunctions package and Dr. Kauers for his inauguration on his Stirling package (and challenging exercises from his side that are part of this thesis). Dr. Kauers also was so kind to provide a LaTeX package for fancy typesetting of Mathematica source code listings. I want to express my acknowledgements to the faculty of RISC for teaching me that much in symbolic algorithms. They never got tired of answering my questions. Also among the PhD students I want to say thank you for helpful scientific (and non-scientific) discussions, especially to my friends Jakob Ablinger, Silviu Radu and Clemens Raab. Finally I must not forget to thank my family, who made all this possible.

Chapter 2

Preliminaries

2.1 Combinatorial notions

In chapter 6 of the book [GKP94] there is a comprehensive repertoire on combinatorial numbers. As mentioned in the introduction we will have a look at some of them to summarize the most important ones.

Definition 2.1.1 (Rising/falling factorials)
Let R be an arbitrary ring with unity, $x \in R, k \in \mathbb{N}$. We define the rising (resp. falling) factorial by (see [GKP94, p.47/48])

$$\begin{aligned} x^{\overline{k}} &= x(x+1)\dots(x+k-1), \quad k \geq 1, && (2.1)\\ x^{\underline{k}} &= x(x-1)\dots(x-k+1), \quad k \geq 1, && (2.2)\\ x^{\underline{0}} &= x^{\overline{0}} = 1. && (2.3) \end{aligned}$$

Definition 2.1.2 (Binomial coefficients)
Let R be a commutative ring containing $\mathbb{Q}$ and let $\lambda \in R$ and $k \in \mathbb{Z}$. Then

$$\begin{aligned} \binom{\lambda}{k} &:= \frac{\lambda^{\underline{k}}}{k!} = \frac{\lambda(\lambda-1)\dots(\lambda-k+1)}{k(k-1)\dots 1}, \quad k \geq 0, && (2.4)\\ \binom{\lambda}{k} &:= 0, \quad k < 0. && (2.5) \end{aligned}$$

For the case that $n, k \in \mathbb{N}$ we have the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}, \qquad (2.6)$$

and the usual convention that $\binom{n}{k} = 0$ when $k > n$.

Theorem 2.1.1 (Binomial theorem, [KP11], p. 87-89)
The binomial theorem states that for $n \in \mathbb{N}$ and any $a, b \in \mathbb{K}$ we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \tag{2.7}$$

As we have seen in Def. 2.1.2 it is also useful to consider the cases where $n \in -\mathbb{N}$. In this case we have for $n \in \mathbb{N}, a, b \in \mathbb{K}$:

$$(a+b)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k b^{-n-k} = \sum_{k=0}^{\infty} \frac{(-n)^{\underline{k}}}{k!} a^k b^{-n-k}. \tag{2.8}$$

Definition 2.1.3 (Stirling numbers of the 1st kind)
Let $n, k \in \mathbb{N}$. The signless Stirling numbers of the 1^{st} kind count the number of permutations of n objects with exactly k cycles. We will denote them by

$$S_1(n,k). \tag{2.9}$$

Definition 2.1.4 (Stirling numbers of the 2nd kind)
Let $n, k \in \mathbb{N}$. The symbol

$$S_2(n,k) \tag{2.10}$$

stands for the number of ways to partition a set of n objects into k nonempty subsets.

Definition 2.1.5 (Bernoulli numbers)
Let $n \in \mathbb{N}$. The sequence of Bernoulli numbers (see [FB07, p.114, Ex. 3])

$$(B_n)_{n\geq 0} \tag{2.11}$$

is recursively defined by

$$\begin{aligned} B_0 &= 1, \\ \sum_{k=0}^{n} \binom{n+1}{k} B_k &= 0, \quad n \geq 1. \end{aligned}$$

Definition 2.1.6 (The Kronecker symbol)
The Kronecker Symbol is given by

$$\delta(n,k) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \tag{2.12}$$

2.2 Manipulation of power series

In this section we summarize the most basic facts concerning power series. For a detailed treatment see [GCL92, Wil06, GKP94, KP11].

For any commutative ring R containing $\mathbb{Q}$ as a subring, the notation $R[[x]]$ denotes the set of all expressions of the form

$$A(x) \in R[[x]]: \; A(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_k \in R. \tag{2.13}$$

In other words $R[[x]]$ denotes the set of all formal power series in the indeterminate x over the ring R. We call $A(x) = \sum_{k=0}^{\infty} a_k x^k$ the *(ordinary) generating function* associated to the sequence $(a_k)_{k\geq 0}$.
The *order* $\mathrm{ord}(A(x))$ of a nonzero power series $A(x)$ is the least integer k such that $a_k \neq 0$. The exceptional case where $a_k = 0$ for all k is called the zero power series. It is common to define the order of the zero power series to be infinity. For a nonzero power series $A(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$ with $\mathrm{ord}(A(x)) = l$ the term $a_l x^l$ is called the low order term of $A(x)$, a_l is called the low order coefficient, and a_0, also written $A(0)$, is called the constant term.

2.2.1 Operations on formal power series

It is usual to define the binary operations of addition and multiplication in the set $R[[x]]$ as follows. If

$$a(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad b(x) = \sum_{k=0}^{\infty} b_k x^k,$$

then the power series addition is defined by

$$c(x) = a(x) + b(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_k = a_k + b_k, \quad k \geq 0.$$

Power series multiplication is defined by

$$d(x) = a(x) \cdot b(x) = \sum_{k=0}^{\infty} d_k x^k,$$

where

$$d_k = a_0 b_k + \cdots + a_k b_0, \quad k \geq 0.$$

If $\mathbb{K}$ a field, $(\mathbb{K}[[x]], +, \cdot)$ forms a commutative ring with unity $1 = 1 + 0 \cdot x + 0 \cdot x^2 + ...$

Lemma 2.2.1 $(\mathbb{K}[[x]], +, \cdot)$ *is an integral domain.*

Lemma 2.2.2 $(\mathbb{K}[[x]], +, \cdot)$ *is a principal ideal domain.*

Lemma 2.2.3
Let R be any commutative ring containing $\mathbb{Q}$ as a subring. The units (invertible elements) in $R[[x]]$ are all power series $A(x)$ whose constant terms $A(0)$ are units in the coefficient domain R.

Proof. If $a(x) = \sum_{k=0}^{\infty} a_k x^k$ is a unit in $R[[x]]$ then there must exist a power series $b(x) = \sum_{k=0}^{\infty} b_k x^k$ such that $a(x)b(x) = 1$. By the definitions of power series multiplication we must have

$$\begin{aligned}
1 &= a_0 b_0, \\
0 &= a_0 b_1 + a_1 b_0, \\
&\vdots \\
0 &= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0, \quad \text{etc.}
\end{aligned}$$

Thus, a_0 is a unit in R with $a_0^{-1} = b_0$. Conversely, if a_0 is a unit in R then the above equation can be solved for the b_k as follows:

$$\begin{aligned}
b_0 &= a_0^{-1}, \\
b_1 &= -a_0^{-1}(a_1 b_0), \\
&\vdots \\
b_n &= -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n b_0), \quad \text{etc.}
\end{aligned}$$

This way, we construct $b(x)$ such that $a(x)b(x) = 1$, so $a(x)$ is a unit in $R[[x]]$. □

Lemma 2.2.4 *If the coefficient domain R (an arbitrary ring with unity) is an integral domain then $R[[x]]$ is also an integral domain.*

Proof. We have to show that for given $a(x), b(x) \in R[[x]]$ such that

$$a(x) = \sum_{k=0}^{\infty} a_k x^k \neq 0, \qquad b(x) = \sum_{k=0}^{\infty} b_k x^k \neq 0,$$

where for $k \in \mathbb{N}$: $a_k, b_k \in R$, we have that their product $a(x)b(x)$ is not equal to zero. Because $a(x) \neq 0$ there exist $a_i \neq 0$ for some $i \in \mathbb{N}$ and similar $j \in \mathbb{N}$ such that $b_j \neq 0$. Let i, j be minimal with this property. The product $a(x)b(x)$ is given by

$$c(x) := \sum_{n=0}^{\infty} c_n x^n = a(x)b(x),$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \geq 0.$$

By assumption $a_k = 0$ for $k < i$ and $b_{n-k} = 0$ for $n - k < j$ or $k > n - j$. Therefore, for $n = i + j$ we find

$$c_{i+j} = \sum_{k=0}^{i+j} a_k b_{i+j-k} = a_i b_j.$$

The coefficient domain R is by assumption an integral domain and therefore free of zero divisors. a_i and b_j are nonzero, so c_{i+j} is nonzero. Therefore $c(x) \neq 0$. $\square$

Theorem and Definition 2.2.1 (General binomial theorem, [KP11], p.89)
For $\lambda \in R$, (R a commutative ring containing $\mathbb{Q}$ as a subring) the expression $(1+x)^\lambda$ does not have a meaning as formal power series. We still have

$$(1+x)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} x^n, \quad x \in \mathbb{K}, \ |x| < 1, \tag{2.14}$$

as analytic power series. Therefore it is reasonable to take

$$(1+x)^\lambda := \sum_{n=0}^{\infty} \binom{\lambda}{n} x^n \in R[[x]] \tag{2.15}$$

as the definition of the symbol $(1+x)^\lambda$. With this definition, we can prove, the multiplication law

$$(1+x)^\lambda (1+x)^\mu = (1+x)^{\lambda+\mu} \tag{2.16}$$

in $R[[x]]$. By applying the definition of product in $R[[x]]$ this is equivalent to the identity

$$\sum_{k=0}^{n} \binom{\lambda}{k} \binom{\mu}{n-k} = \binom{\lambda+\mu}{n} \quad n \geq 0, \lambda, \mu \in R, \tag{2.17}$$

that is known as Vandermonde convolution ([GKP94, p. 170, (5.27)]).

Now we consider the notion of limit in $\mathbb{K}[[x]]$. With this we can define composition of formal power series.

Definition 2.2.1 ([KP11], p.24)
A sequence $(a_k(x))_{k\geq 0}$ of formal power series in $\mathbb{K}[[x]]$ converges to another formal power series $a(x) \in \mathbb{K}[[x]]$ if the $a_k(x)$ get arbitrarily close to $a(x)$. Formally, $(a_k(x))_{k\geq 0}$ converges to $a(x)$ in $\mathbb{K}[[x]]$ if and only if

$$\lim_{k\to\infty} ord(a(x) - a_k(x)) = \infty,$$

i.e., if and only if

$$\forall n \in \mathbb{N}\ \exists k_0 \in \mathbb{N}\ \forall k \geq k_0 : ord(a(x) - a_k(x)) > n.$$

Definition 2.2.2 (Composition of power series)
Let $a(x) = \sum_{n=0}^{\infty} a_n x^n, b(x) \in \mathbb{K}[[x]]$ be such that $ord(b(x)) \geq 1$. Consider the sequence $(c_k(x))_{k\geq 0}$ defined by

$$c_k(x) = \sum_{j=0}^{k} a_j b(x)^j.$$

The composition of $a(x)$ and $b(x)$ is defined by

$$a(b(x)) := \sum_{n=0}^{\infty} a_n b(x)^n := \lim_{k\to\infty} c_k(x) = c(x). \tag{2.18}$$

The composition of power series is compatible with addition and multiplication as the following theorem shows.

Theorem 2.2.1 ([KP11], p.26)
For every fixed $u(x) \in \mathbb{K}[[x]]$ with $ord(u(x)) \geq 1$ the map

$$\begin{aligned} \Phi_u : \mathbb{K}[[x]] &\longrightarrow \mathbb{K}[[x]], \\ a(x) &\longmapsto a(u(x)), \end{aligned}$$

is a ring homomorphism.

Proof. See [KP11, p. 26, Thm. 2.6] $\square$

Note that $\frac{1}{x} \notin \mathbb{K}[[x]]$ but lies in its quotient field, i.e. $\frac{1}{x} \in \mathbb{K}((x))$, the field of formal Laurent series.

2.2.2 Formal Laurent series

By Lemma 2.2.1 $\mathbb{K}[[x]]$ is an integral domain. So one can construct the quotient field:

$$\mathbb{K}((x)) = \left\{ \frac{a(x)}{b(x)} : a(x), b(x) \in \mathbb{K}[[x]] \wedge b(x) \neq 0 \right\}.$$

We call this construction the field of formal Laurent series over $\mathbb{K}$.

Lemma 2.2.5

$$\mathbb{K}((x)) = \left\{ x^k c(x) : k \in \mathbb{Z}, c(x) \in \mathbb{K}[[x]] \right\}.$$

Proof. For one inclusion take $f(x) \in \mathbb{K}((x))$. Hence $f(x) = \frac{a(x)}{b(x)}$ for some $a(x), b(x) \in \mathbb{K}[[x]]$ and $b(x) \neq 0$. If we extract x^i such that

$$a(x) = \sum_{n=0}^{\infty} a_n x^n = x^i \underbrace{\sum_{n=0}^{\infty} \alpha_n x^n}_{=:\alpha(x)},$$

with $\alpha_0 \neq 0$ and do the same for

$$b(x) = \sum_{n=0}^{\infty} b_n x^n = x^j \underbrace{\sum_{n=0}^{\infty} \beta_n x^n}_{=:\beta(x)},$$

with $\beta_0 \neq 0$,we find by Lemma 2.2.3 a multiplicative inverse of $\beta(x)$. Hence the expression

$$\frac{a(x)}{b(x)} = \frac{x^i \alpha(x)}{x^j \beta(x)} = x^{i-j} \underbrace{\frac{\alpha(x)}{\beta(x)}}_{\in \mathbb{K}[[x]]}$$

is well defined and of the desired form.

For the other direction take $f(x) \in \left\{ x^k c(x) : k \in \mathbb{Z}, c(x) \in \mathbb{K}[[x]] \right\}$: by definition $f(x) = x^k c(x)$ for some $k \in \mathbb{Z}, c(x) \in \mathbb{K}[[x]]$. If $k \in \mathbb{N}_0$ we have:

$$x^k c(x) = x^k \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+k} = \sum_{n=k}^{\infty} c_{n-k} x^n,$$

and by setting $a(x) = \sum_{n=k}^{\infty} c_{n-k} x^n$ and $b(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots$ we have $x^k c(x) = \frac{a(x)}{b(x)} \in \mathbb{K}((x))$.

For $k \in -\mathbb{N}$ we get the desired form by setting $a(x) = c(x)$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n = x^{-k}$, i.e. $(b_n)_{n \geq 0} : (\delta(-k, n))_{n \geq 0}$. $\square$

This Lemma gives the justification to view a formal Laurent series as follows

$$\mathbb{K}((x)) = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\; a_n \in \mathbb{K} \;\wedge\;\; \text{finitely many } a_n \neq 0 \text{ where } n < 0 \right\}.$$

Notation: For $k \in \mathbb{Z}$:

$$\mathbb{K}_k((x)) := \{\, f \in \mathbb{K}((x)) \mid \operatorname{ord}(f(x)) = k\} \subset \mathbb{K}((x)). \tag{2.19}$$

Definition 2.2.3 (Coefficient functional)
Let

$$f(x) = \sum_{k \in \mathbb{Z}} f_k x^k \;\; \in \mathbb{K}((x)).$$

We will frequently use the notation

$$\langle x^n \rangle \; f(x) := f_n, \quad n \in \mathbb{Z}.$$

We define

$$f(0) := \langle x^0 \rangle \; f(x) = f_0.$$

An elementary property of this functional is given by

$$\langle x^n \rangle \; x^k f(x) = \langle x^{n-k} \rangle f(x), \qquad n, k \in \mathbb{Z}.$$

In subsection 2.2.4 we will give $\langle x^{-1} \rangle \; f(x)$ a special name.

2.2.3 Differentiation and integration

If R is a commutative ring (resp. a field) and $D : R \to R$ is such that

$$D(a+b) = D(a) + D(b), \tag{2.20}$$
$$D(a \cdot b) = D(a)b + aD(b) \tag{2.21}$$

for all $a, b \in R$, then D is called a (formal) derivation on R and the pair (R, D) is called a differential ring (resp. a differential field). Next we define the formal derivative on $\mathbb{K}((x))$.

Definition 2.2.4 (Formal derivative)
Let, for $k_0 \in \mathbb{Z}$,

$$a(x) = x^{-k_0} \sum_{k=0}^{\infty} a_k x^k \in \mathbb{K}((x)).$$

A derivation on the field of formal Laurent series is given by

$$\begin{array}{rcl} D_x : \mathbb{K}((x)) & \longrightarrow & \mathbb{K}((x)), \\ a(x) = x^{-k_0}\sum_{k=0}^{\infty} a_k x^k & \longmapsto & D_x(a(x)) := -k_0 x^{-k_0-1}\sum_{k=0}^{\infty} a_k x^k + x^{-k_0}\sum_{k=0}^{\infty} a_{k+1}(k+1)x^k. \end{array}$$

Notation: For $f(x) \in \mathbb{K}((x))$ we also write:

$$\begin{array}{rcl} f'(x) & := & D_x(f(x)), \\ f''(x) & := & D_x^2(f(x)) := D_x(D_x(f(x))), \\ f^{(k)}(x) & := & D_x^k(f(x)) := \underbrace{D_x(\dots D_x}_{k \text{ times}}(f(x))\dots), \quad \text{etc.} \end{array}$$

Note that this definition contains the ring of formal power series as special case by setting $k_0 = 0$, i.e. $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and

$$D_x(f(x)) = D_x\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} a_k k x^{k-1} = \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k. \tag{2.22}$$

We will not distinguish the symbol D_x for $\mathbb{K}((x))$ resp. $\mathbb{K}[[x]]$.

Lemma 2.2.6 $(\mathbb{K}[[x]], D_x)$ *is a differential ring.*

Lemma 2.2.7 $(\mathbb{K}((x)), D_x)$ *is a differential field.*

Lemma 2.2.8
Let $f(x), (f_n(x))_{n\geq 0}$ *be in* $\mathbb{K}((x))$ *such that* $\lim_{n\to\infty} ord(f_n(x)) = \infty$ *and*

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

Then:

- *For* $k \in \mathbb{Z}$
$$\langle x^k \rangle\ f(x) = \sum_{n=0}^{\infty} \langle x^k \rangle\ f_n(x)$$
-
$$D_x(f(x)) = \sum_{n=0}^{\infty} D_x(f_n(x))$$

Proof. By definition of the infinite sum we have that

$$\langle x^k \rangle\ f(x) \quad = \quad \langle x^k \rangle\ \sum_{n=0}^{\infty} f_n(x) = \langle x^k \rangle\ \lim_{N\to\infty} \sum_{n=0}^{N} f_n(x)$$

Now, because of our assumption that $\lim_{n\to\infty} \operatorname{ord}(f_n(x)) = \infty$, we know that there exists an index M such that for all $m \geq M$ we have that $\operatorname{ord}(f_m(x)) > k$. Hence we can split the sum.

$$\begin{aligned} &= \langle x^k \rangle \lim_{N\to\infty} \left(\sum_{n=0}^{M} f_n(x) + \sum_{n=M+1}^{N} f_n(x) \right) \\ &= \langle x^k \rangle \sum_{n=0}^{M} f_n(x) + \langle x^k \rangle \lim_{N\to\infty} \sum_{n=M+1}^{N} f_n(x) \end{aligned}$$

We have chosen M such that $\forall i \in \mathbb{N}$: $\operatorname{ord}(f_{M+i}(x)) > k$, hence the second sum does not contribute to $\langle x^k \rangle$. Now we use a linearity argument, i.e.,

$$\langle x^k \rangle \sum_{n=0}^{M} f_n(x) = \sum_{n=0}^{M} \langle x^k \rangle f_n(x),$$

which proves the theorem, by our choice of M. For the second statement, we plug in the definition of the infinite sum:

$$D_x(f(x)) = D_x \left(\sum_{n=0}^{\infty} f_n(x) \right) = D_x \left(\lim_{N\to\infty} \sum_{n=0}^{N} f_n(x) \right)$$

Now, because $\lim_{n\to\infty} \operatorname{ord}(f_n(x)) = \infty$ we know that for all $K \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$: $\operatorname{ord}(f_m(x) - f(x)) > K$. Hence, we find that

$$\begin{aligned} D_x \left(\lim_{N\to\infty} \sum_{n=0}^{N} f_n(x) \right) &= D_x \left(\lim_{N\to\infty} \left(\sum_{n=0}^{M} f_n(x) + \sum_{n=M+1}^{N} f_n(x) \right) \right) \\ &= D_x \left(\sum_{n=0}^{M} f_n(x) \right) + D_x \left(\lim_{N\to\infty} \left(\sum_{n=M+1}^{N} f_n(x) \right) \right) \end{aligned}$$

By our choice of M, we know that the second sum is of the form

$$\sum_{n=M+1}^{N} f_n(x) = x^{K+1} g(x), \quad g(x) \in \mathbb{K}[[x]].$$

If we differentiate this expression we get

$$D_x(x^{K+1}g(x)) = (K+1)x^K g(x) + x^{K+1}D_x(g(x)) \longrightarrow 0.$$

as K tends to infinity. Again the linearity property of the D_x operator,

$$D_x\left(\sum_{n=0}^{M} f_n(x)\right) = \sum_{n=0}^{M} D_x\left(f_n(x)\right),$$

proves our claim. □

Proposition 2.2.1 *The following rules hold for $f(x), g(x) \in \mathbb{K}((x)), n \in \mathbb{N}$:*

- $$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{D_x(f(x))g(x) - f(x)D_x(g(x))}{g(x)^2}, \quad g(x) \neq 0, \tag{2.23}$$

- $$D_x\left(f(x)^n\right) = n \cdot f(x)^{n-1}D_x(f(x)), \tag{2.24}$$

- *If $f(x) \in \mathbb{K}_0((x))$, $ord(g(x)) \geq 1$:*

$$\begin{aligned} D_x\left(f(g(x))\right) &= D_t(f(t))|_{t=g(x)}D_x(g(x)) \\ &= f'(g(x))g'(x) \end{aligned} \tag{2.25}$$

Proof. For the first statement we show that $D_x(1) = 0$. Indeed,

$$D_x(1) = D_x(1 \cdot 1) = 1 \cdot D_x(1) + 1 \cdot D_x(1) = D_x(1) + D_x(1) \Rightarrow D_x(1) = 0.$$

From this we find that:

$$\begin{aligned} 0 = D_x(1) = D_x\left(\frac{g(x)}{g(x)}\right) &= g(x)D_x\left(\frac{1}{g(x)}\right) + \frac{1}{g(x)}D_x(g(x)) \\ &\Rightarrow D_x\left(\frac{1}{g(x)}\right) = -\frac{1}{g(x)^2}D_x(g(x)) \end{aligned}$$

Now we can apply the product rule (2.21):

$$\begin{aligned} D_x\left(\frac{f(x)}{g(x)}\right) &= f(x)D_x\left(\frac{1}{g(x)}\right) + \frac{1}{g(x)}D_x(f(x)) \\ &= f(x)\left(-\frac{D_x(g(x))}{g(x)^2}\right) + \frac{1}{g(x)}D_x(f(x)) \\ &= \frac{D_x(f(x))g(x) - f(x)D_x(g(x))}{g(x)^2}. \end{aligned}$$

For the second statement we apply induction on n. The base cases $n = 0, 1$ are trivial from the definition. Now let us suppose that for fixed n we have that

$$D_x(f(x)^n) = nf(x)^{n-1}D_x(f(x))$$

We consider now

$$\begin{aligned} D_x(f(x)^{n+1}) &= D_x(f(x)^n f(x)) = f(x)^n D_x(f(x)) + f(x)D_x(f(x)^n) \\ &= f(x)^n D_x(f(x)) + nf(x)^n D_x(f(x)) \\ &= (n+1)f(x)^n D_x(f(x)), \end{aligned}$$

which proves the statement.

For the third statement let $f(x) = \sum_{k=0}^{\infty} f_k x^k$. By Lemma 2.2.8 we can exchange summation and limit. From this we find that

$$\begin{aligned} D_x(f(g(x)) &= D_x\left(\sum_{k=0}^{\infty} f_k g(x)^k\right) = D_x\left(\lim_{n\to\infty}\sum_{k=0}^{n} f_k g(x)^k\right) \\ &= \lim_{n\to\infty}\sum_{k=0}^{n} D_x(f_k g(x)^k) = \lim_{n\to\infty}\sum_{k=0}^{n} f_k D_x(g(x)^k) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f_k k g(x)^{k-1} D_x(g(x)) = D_x(g(x))\lim_{n\to\infty}\sum_{k=1}^{n} f_k k g(x)^{k-1} \\ &= D_t(f(t))|_{t=g(x)} D_x(g(x)) = g'(x)f'(g(x)), \end{aligned}$$

that proves the chain-rule. □

Example 2.2.1 *The exponential e^x is a shortcut notation for the formal power series*

$$e^{\alpha x} := \exp(\alpha x) := \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} x^k \in \mathbb{K}[[x]], \alpha \in \mathbb{K}.$$

It's formal derivative is given by

$$D_x(e^{\alpha x}) = D_x\left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} x^k\right) = \sum_{k=0}^{\infty} \frac{\alpha^{k+1}}{(k+1)!}(k+1)x^k = \alpha \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} x^k = \alpha e^{\alpha x}. \tag{2.26}$$

This power series satisfies:

$$\begin{aligned} e^{\alpha x} \cdot e^{\beta x} &= \left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} x^k\right)\left(\sum_{k=0}^{\infty} \frac{\beta^k}{k!} x^k\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k}\right) \\ &\overset{(2.7)}{=} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)^n}{n!} x^n = e^{(\alpha+\beta)x}, \quad \alpha, \beta \in \mathbb{K}. \end{aligned} \tag{2.27}$$

Similar relations hold for $(1+x)^\lambda \in R[[x]]$ (R a commutative ring containing $\mathbb{Q}$ as a subring). We do not carry out the proof in detail, but state the identity for sake of completeness (as we will need it in chapter 4):

$$D_x((1+x)^\lambda) = D_x\left(\sum_{k=0}^{\infty}\binom{\lambda}{k}x^k\right) = \lambda\cdot(1+x)^{\lambda-1}. \tag{2.28}$$

Definition 2.2.5 (Formal integral)
The formal integral of the formal power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{K}[[x]]$$

is defined by

$$\begin{aligned} \int_x : \mathbb{K}[[x]] \quad &\longrightarrow \quad \mathbb{K}[[x]], \\ f(x) = \sum_{k=0}^{\infty} a_k x^k \quad &\longmapsto \quad \int_x f(x) = \int_x \sum_{k=0}^{\infty} a_k x^k := \sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1} = \sum_{k=1}^{\infty}\frac{a_{k-1}}{k}x^k. \end{aligned}$$

Proposition 2.2.2 ([KP11], p.20, Thm. 2.3)
For all $a(x) \in \mathbb{K}[[x]]$:

- $$D_x\left(\int_x a(x)\right) = a(x), \tag{2.29}$$
- $$\int_x D_x a(x) = a(x) - a(0), \tag{2.30}$$
- $$\langle x^n\rangle\ a(x) = \frac{1}{n!}(D_x^n(a(x))|_{x=0}. \tag{2.31}$$

Proof. Throughout the proof, let $a(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{K}[[x]]$. The first statement is obtained as follows:

$$\begin{aligned} D_x\left(\int_x a(x)\right) &= D_x\left(\int_x \sum_{k=0}^{\infty} a_k x^k\right) = D_x\left(\sum_{k=1}^{\infty}\frac{a_{k-1}}{k}x^k\right) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k+1}(k+1)x^k = \sum_{k=0}^{\infty} a_k x^k = a(x). \end{aligned}$$

For the second statement we calculate

$$
\begin{aligned}
\int_x D_x(a(x)) &= \int_x D_x\left(\sum_{k=0}^{\infty} a_k x^k\right) = \int_x \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k \\
&= \sum_{k=0}^{\infty} \frac{a_{k+1}}{k+1}(k+1)x^{k+1} = \sum_{k=0}^{\infty} a_{k+1}x^{k+1} \\
&= \sum_{k=1}^{\infty} a_k x^k = a(x) - a(0).
\end{aligned}
$$

To prove (2.31) we proceed by induction on n. The base case $n = 0$ is easily checked:

$$\langle x^0 \rangle a(x) = a_0 = \frac{1}{0!}a(0).$$

Now suppose for some fixed $n \in \mathbb{N}$ we have that:

$$\langle x^n \rangle \; a(x) = \frac{1}{n!}D_x^n(a(x))|_{x=0}$$

The connection to the coefficient of x^{n+1} is given by the derivative as follows

$$
\begin{aligned}
\langle x^{n+1} \rangle \; a(x) &= \langle x^n \rangle \; D_x(a(x)) \cdot \frac{1}{n+1} \\
&= \frac{1}{n!}D_x^n\left(D_x(a(x))\right)|_{x=0}\frac{1}{n+1} \\
&= \frac{1}{(n+1)!}D_x^{n+1}(a(x))|_{x=0}.
\end{aligned}
$$

□

2.2.4 The concept of res

Definition 2.2.6 (res-functional)
If

$$C(x) = \sum_{k=-\infty}^{\infty} c_k x^k \in \mathbb{K}((x)),$$

then we define the formal residue ***res*** *of $C(x)$ to be the coefficient of x^{-1}, i.e.,*

$$\underset{x}{\boldsymbol{res}} \; C(x) := \langle x^{-1} \rangle \; C(x) = c_{-1}.$$

Remark: $\underset{x}{\mathbf{res}}\ A(x)$ is a purely formal operation; more precisely, a linear functional on the $\mathbb{K}$-vector space $\mathbb{K}((x))$. We will set this later into a context to the analytic meaning in complex analysis, but whenever we talk of application of the res-operator we mean extraction of the coefficient of x^{-1}.

Lemma 2.2.9 (Coefficient formula)
If

$$A(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathbb{K}[[x]]$$

is the generating function for the sequence $(a_k)_{k\geq 0}$ it follows that

$$a_k = \langle x^k \rangle A(x) = \underset{x}{\boldsymbol{res}}\ A(x)x^{-k-1}, \quad k \geq 0.$$

Lemma 2.2.10 *For all $A(x) \in \mathbb{K}((x))$: $\underset{x}{\boldsymbol{res}}\ D_x(A(x)) = 0$*

Proof. If we consider

$$A(x) = a_{-n_0}x^{-n_0} + ... + a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2 + ...$$

then

$$D_x(A(x)) = -n_0 a_{-n_0} x^{-n_0-1} + ... + (-1)a_{-1}x^{-2} + 0 + a_1 + 2a_2 x + ...$$

and hence $\langle x^{-1} \rangle D_x(A(x)) = 0$. □

Lemma 2.2.11 ([Ros97], p. 41, Lemma 48)
Let $g(x) \in \mathbb{K}_1((x))$, $n \in \mathbb{Z}$. Then:

$$\underset{x}{\boldsymbol{res}}\ \frac{D_x(g(x))}{g(x)^{n+1}} = \delta(n, 0).$$

Proof. For $n \neq 0$ we have

$$D_x(g(x))g(x)^{-n-1} = -\frac{1}{n} D_x(g(x)^{-n}),$$

which has residue zero by Lemma 2.2.10.

For $n = 0$, we write $g(x) = x/h(x)$ where $h(x) = \sum_{k=0}^{\infty} h_k x^k \in \mathbb{K}[[x]]$ such that $h_0 \neq 0$. By Lemma 2.2.3 such a representation of $g(x)$ exists. With this substitution, we have

$$\frac{D_x(g(x))}{g(x)} = \frac{1}{g(x)} \left(D_x \left(\frac{x}{h(x)} \right) \right) = \frac{h(x)}{x} \left(\frac{h(x) - xD_x(h(x))}{h(x)^2} \right) = \frac{1}{x} - \frac{D_x(h(x))}{h(x)}.$$

$1/h(x)$ is a well defined power series of order 0, the power series $D_x(h(x))$ has order ≥ 0, so the product $D_x(h(x))\frac{1}{h(x)}$ has order $\text{ord}(D_x(h(x)))+\text{ord}(1/h(x)) \geq 0$. Therefore

$$\underset{x}{\mathbf{res}} \frac{D_x(g(x))}{g(x)} = \underset{x}{\mathbf{res}} \left(\frac{1}{x} - \frac{D_x(h(x))}{h(x)}\right) = \underset{x}{\mathbf{res}} \left(\frac{1}{x}\right) + \underset{x}{\mathbf{res}} \left(\frac{D_x(h(x))}{h(x)}\right) = 1.$$

□

For computing with formal Laurent series we need the following important theorem

Theorem 2.2.2 (Lagrange, [Ros97], p. 38, Thm. 42)
Let $f(x), g(x) \in \mathbb{K}[[x]]$ formal power series with ord$(g(x)) = 1$ and let $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ be such that

$$f(x) = \sum_{n=0}^{\infty} a_n g(x)^n. \tag{2.32}$$

Then we have that

$$\begin{aligned} a_0 &= \langle x^0 \rangle\ f(x), & (2.33)\\ a_m &= \underset{x}{\boldsymbol{res}}\ f(x) D_x(g(x)) g(x)^{-m-1}, \quad m \geq 1. & (2.34) \end{aligned}$$

Proof. Several different proofs are given in [Ros97]. One of them goes like this:

The first statement on a_0 is immediate since $g(x)$ does not contribute. Hence, we have that

$$f(0) = \langle x^0 \rangle f(x) = a_0.$$

The statement on general a_m essentially relies on Lemma 2.2.11. If we multiply both sides of (2.32) by $D_x(g(x))g(x)^{-m-1}$ (for $m \in \mathbb{N}, m \geq 1$) we get by Lemma 2.2.8:

$$f(x) D_x(g(x)) g(x)^{-m-1} = \sum_{n=0}^{\infty} a_n \frac{D_x(g(x))}{g(x)^{m-n+1}}. \tag{2.35}$$

If we now apply the **res**-functional we find by Lemma 2.2.11 and Lemma 2.2.8 that

$$\begin{aligned} \underset{x}{\mathbf{res}}\ f(x) D_x(g(x)) g(x)^{-n-1} &= \underset{x}{\mathbf{res}} \sum_{n=0}^{\infty} a_n \frac{D_x(g(x))}{g(x)^{m-n+1}} \\ &= \sum_{n=0}^{\infty} a_n\, \underset{x}{\mathbf{res}}\ \frac{D_x(g(x))}{g(x)^{m-n+1}} \\ &= \sum_{n=0}^{\infty} a_n \delta(n, m) = a_m. \end{aligned}$$

□

In concrete examples (see section 4.1.2) we will find it useful, as in the proof of Lemma 2.2.11, to consider a power series of order 1 as a quotient of two power series as follows:

$$g(x) := \frac{x}{h(x)}, \tag{2.36}$$

where $h(x) \in \mathbb{K}_0((x))$. If we apply Thm. 2.2.2 to this choice, use the chain-rule in $\mathbb{K}[[x]]$ (see Proposition 2.2.2) and multiply both sides of (2.32) by $g(x)^{-m}$, we find by Lemma 2.2.8 that:

$$\begin{aligned} D_x(f(x)) &= \sum_{n=1}^{\infty} a_n n g(x)^{n-1} D_x(g(x)), \\ \frac{D_x(f(x))}{g(x)^m} &= \sum_{n=1}^{\infty} a_n n \frac{D_x(g(x))}{g(x)^{m-n+1}}. \end{aligned}$$

Now we apply on both sides the **res**-functional together with Lemma 2.2.8 and Lemma 2.2.11:

$$\begin{aligned} \operatorname*{\mathbf{res}}_x D_x(f(x))g(x)^{-m} &= \sum_{n=1}^{\infty} a_n n \operatorname*{\mathbf{res}}_x \frac{D_x(g(x))}{g(x)^{m-n+1}} \\ &= \sum_{n=1}^{\infty} a_n n \delta(m,n) \\ &= m \cdot a_m. \end{aligned}$$

If we finally expand $g(x)$ according to (2.36) we get that:

$$\begin{aligned} m \cdot a_m &= \operatorname*{\mathbf{res}}_x D_x(f(x))g(x)^{-m} \\ &= \langle x^{-1} \rangle \; D_x(f(x))h(x)^m x^{-m} \\ &= \langle x^{m-1} \rangle \; D_x(f(x))h(x)^m. \end{aligned}$$

Corollary 2.2.1
Under the assumptions (and with the notions) of Thm. 2.2.2 where $h(x) \in \mathbb{K}_0((x))$ and

$$g(x) = \frac{x}{h(x)},$$

we have for $m \geq 1$:

$$a_m = \operatorname*{\boldsymbol{res}}_x f(x) D_x(g(x)) g(x)^{-m-1} = \frac{1}{m} \langle x^{m-1} \rangle D_x(f(x)) h(x)^m. \tag{2.37}$$

Remark: An alternative derivation of Cor. 2.2.1 from Thm. 2.2.2 is straight-forward from the identity

$$0 = \langle x^{-1} \rangle \frac{1}{m} D_x \left(f(x) g(x)^{-m} \right), \quad m \geq 1. \tag{2.38}$$

Theorem 2.2.3 (Implicit function theorem, [KP11], Thm. 2.9, p. 33)
Let $a(x, y) \in \mathbb{K}[[x, y]]$ be such that

$$a(0,0) = 0 \quad \textit{and} \quad (D_y a)(0,0) \neq 0. \tag{2.39}$$

Then there exists a unique formal power series $f(x) \in \mathbb{K}[[x]]$ with $f(0) = 0$ such that $a(x, f(x)) = 0$.

Proof. See [KP11, p.34]. □

Theorem 2.2.4 (Existence of a unique compositional inverse)
Let $r(x) \in \mathbb{K}[[x]]$ with ord$(r(x)) = 1$. Then there exists a unique $R(x) \in \mathbb{K}[[x]]$ with ord$(R(x)) = 1$ such that

$$r(R(x)) = x. \tag{2.40}$$

Proof. In Thm. 2.2.3 set

$$a(x, y) := r(y) - x \in \mathbb{K}[[x, y]]. \tag{2.41}$$

Clearly, $a(0,0) = r(0) - 0 = 0$. The partial derivative $(D_y a)(x, y)$ is given by

$$(D_y a)(x, y) = (D_y r)(y), \text{ and } (D_y r)(0) \neq 0,$$

because $\operatorname{ord}(r(y)) = 1$. Hence, $(D_y a)(0,0) \neq 0$. By theorem 2.2.3 there exists some unique $R(x) \in \mathbb{K}[[x]]$ with $R(0) = 0$ such that

$$0 = a(x, R(x)) = r(R(x)) - x.$$

Finally, $\operatorname{ord}(r(x)) = 1$ and (2.40) imply that $\operatorname{ord}(R(x)) = 1$. □

Corollary 2.2.2 *For $r(x), R(x) \in \mathbb{K}[[x]]$ as in Thm.2.2.4:*

$$r(R(x)) = R(r(x)) = x. \tag{2.42}$$

Proof. For $R(x)$, as a consequence of Thm. 2.2.4, there exists $s(x) \in \mathbb{K}[[x]]$ with $\operatorname{ord}(s(x)) = 1$ such that

$$R(s(x)) = x. \tag{2.43}$$

Hence,

$$r(x) \stackrel{(2.43)}{=} r(R(s(x))) \stackrel{(2.40)}{=} s(x). \tag{2.44}$$

□

Notation: We will denote the unique compositional inverse $A(x) \in \mathbb{K}_1((x))$ of $a(x) \in \mathbb{K}_1((x))$ by

$$a^{\langle -1 \rangle}(x) := A(x). \tag{2.45}$$

In particular, in $\mathbb{K}[[x]]$:

$$a^{\langle -1 \rangle}(a(x)) = A(a(x)) = a(A(x)) = a(a^{\langle -1 \rangle}(x)) = x. \tag{2.46}$$

Example 2.2.2 *Consider the formal power series*

$$f(x) := \log(1+x) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \in \mathbb{K}_1((x)). \tag{2.47}$$

By Thm. 2.2.4 this power series has a unique compositional inverse. A detailed investigation of the proof (that involves the implicit function theorem Thm. 2.2.3) gives a way to construct this compositional inverse. We find that:

$$f^{\langle -1 \rangle}(x) = \exp(x) - 1 := \sum_{k=1}^{\infty} \frac{1}{k!} x^k \in \mathbb{K}_1((x)). \tag{2.48}$$

In particular, in $\mathbb{K}[[x]]$ we have the relation

$$\log(1 + (\exp(x) - 1)) = x, \tag{2.49}$$

and by Cor. 2.2.2 also that

$$\exp(\log(1+x)) - 1 = x \in \mathbb{K}[[x]]. \tag{2.50}$$

2.3 Rules for the res-functional

In the book [Ego84], the author comes up with several rules for computing with power series. In the following we will list this rules for operations on the coefficients of generating functions of the form $A(x) = \sum_k a_k x^k$. Most rules (with exception of the inversion rule and the change of variables) are simple consequences of Lemma 2.2.9.

Two formal power series coincide if and only if their coefficients are the same. Addition as we defined it is a $\mathbb{K}-$linear operation. Hence we get the two (trivial) rules

Rule 1 (Removal of res) *For $A(x), B(x) \in \mathbb{K}[[x]]$:*

$$A(x) = B(x)$$

if and only if

$$\underset{x}{\boldsymbol{res}}\, A(x)x^{-k-1} = \underset{x}{\boldsymbol{res}}\, B(x)x^{-k-1}, \quad k \geq 0.$$

Rule 2 (Linearity)
For $a(x), B(x) \in \mathbb{K}[[x]], \alpha, \beta \in \mathbb{K}$:

$$\alpha \operatorname*{\boldsymbol{res}}_{x} \; A(x)x^{-k-1} + \beta \operatorname*{\boldsymbol{res}}_{x} \; B(x)x^{-k-1} = \operatorname*{\boldsymbol{res}}_{x} \; (\alpha A(x) + \beta B(x))\, x^{-k-1}, \quad k \geq 0.$$

By Definition 2.2.2 we can, under certain conditions, compose two formal power series. This definition justifies

Rule 3 (Substitution)
Let $A(x) \in \mathbb{K}[[x]]$. If $z(t) = \sum_{k=1}^{\infty} a_k t^k \in \mathbb{K}_1((t))$, then

$$\sum_{k=0}^{\infty} z(t)^k \operatorname*{\boldsymbol{res}}_{x} \; A(x)x^{-k-1} = A(z(t)).$$

This relation remains valid, in the case where $A(x)$ is a polynomial and

$$z(t) = \sum_{k=-m}^{\infty} a_k t^k \in \mathbb{K}((t)),$$

with $a_{-m} \neq 0$ where m is a positive integer.

The theorem of Lagrange (Thm. 2.2.2) is the basis for numerous non-trivial applications.

Rule 4 (Inversion)
Given $f(x) \in \mathbb{K}[[x]], g(x) \in \mathbb{K}_1((x))$, and $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n g(x)^n.$$

Then Lagrange's theorem (Thm. 2.2.2) tells that the coefficient a_n is given by

$$a_n = \operatorname*{\boldsymbol{res}}_{x} \; f(x) D_x(g(x)) g(x)^{-n-1}, \quad n \geq 1. \tag{2.51}$$

Rule 5 (Change of variables under the res sign)
For $g(t) \in \mathbb{K}_1((t)), f(x) \in \mathbb{K}((x))$ we have that

$$\operatorname*{\boldsymbol{res}}_{x} \; f(x) = \operatorname*{\boldsymbol{res}}_{t} \; f(g(t)) \; D_t(g(t)),$$

where the symbol D_t denotes formal differentiation in $\mathbb{K}((t))$.

Proof. Let

$$f(x) = \sum_{k=-\infty}^{\infty} f_k x^k \in \mathbb{K}((x)), \quad g(t) = \sum_{k=1}^{\infty} g_k t^k \in \mathbb{K}_1((t)).$$

For the left hand side we find by Definition 2.2.6 that

$$\mathop{\mathbf{res}}_{x}\ f(x) = \langle x^{-1}\rangle\ f(x) = \langle x^{-1}\rangle \sum_{k=-\infty}^{\infty} f_k x^k = f_{-1}.$$

On the other side, we have that

$$\begin{aligned} \mathop{\mathbf{res}}_{t}\ f(g(t))D_t(g(t)) &= \mathop{\mathbf{res}}_{t} \sum_{k=-\infty}^{\infty} f_k g(t)^k D_t(g(t)) \\ &= \sum_{k=-\infty}^{\infty} f_k \mathop{\mathbf{res}}_{t}\ g(t)^k D_t(g(t)), \end{aligned}$$

where we used Lemma 2.2.8. Now we realize that $g(t)^k D_t(g(t))$ has the primitive function $g(t)^{k+1}/(k+1)$ for all $k \in \mathbb{Z}$, except $k = -1$. Hence, the last line is equivalent to

$$\sum_{k=-\infty}^{-2} f_k \mathop{\mathbf{res}}_{t}\ \frac{1}{k+1} D_t\left(g(t)^{k+1}\right) + f_{-1} \mathop{\mathbf{res}}_{t}\ \frac{D_t(g(t))}{g(t)} + \sum_{k=0}^{\infty} f_k \mathop{\mathbf{res}}_{t}\ \frac{1}{k+1} D_t\left(g(t)^{k+1}\right).$$

By Lemma 2.2.10 it follows that all terms except $f_{-1} \mathop{\mathbf{res}}_{t} \frac{D_t(g(t))}{g(t)}$ vanish and hence, by Lemma 2.2.11

$$\mathop{\mathbf{res}}_{t}\ f(g(t))D_t(g(t)) = f_{-1} \mathop{\mathbf{res}}_{t}\ \frac{D_t(g(t))}{g(t)} = f_{-1}.$$

□

Rule 6 (Differentiation)
Let $A(x) \in \mathbb{K}[[x]], k \in \mathbb{N}$:

$$k \mathop{\boldsymbol{res}}_{x}\ A(x)x^{-k-1} = \mathop{\boldsymbol{res}}_{x}\ D_x(A(x))x^{-k}.$$

Proof. For $k = 0$, the Rule is trivially true by Lemma 2.2.10. For $k \geq 1$ the left hand side is by Lemma 2.2.9:

$$k \mathop{\mathbf{res}}_{x}\ A(x)x^{-k-1} = k \cdot a_k.$$

The right hand side delivers:

$$\operatorname*{\mathbf{res}}_x D_x(A(x))x^{-k} = \langle x^{k-1}\rangle\ D_x(A(x))$$

$$= \langle x^{k-1}\rangle\ \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k = \langle x^{k-1}\rangle\ \sum_{k=1}^{\infty}ka_kx^{k-1} = k\cdot a_k.$$

□

Rule 7 (Integration)
Let $A(x)\in\mathbb{K}[[x]], k\in\mathbb{N}$:

$$\frac{1}{k+1}\operatorname*{\boldsymbol{res}}_x\ A(x)x^{-k-1} = \operatorname*{\boldsymbol{res}}_x\ \left(\int_x A(x)\,\mathrm{d}x\right)x^{-k-2}.$$

Proof. The left hand side is given by Lemma 2.2.9:

$$\frac{1}{k+1}\operatorname*{\mathbf{res}}_x\ A(x)x^{-k-1} = \frac{1}{k+1}a_k.$$

The right hand side delivers:

$$\operatorname*{\mathbf{res}}_x\ \left(\int_x A(x)\,\mathrm{d}x\right)x^{-k-2} = \langle x^{k+1}\rangle\left(\int_x A(x)\,\mathrm{d}x\right)$$

$$= \langle x^{k+1}\rangle\left(\int_x\sum_{k=0}^{\infty}a_kx^k\,\mathrm{d}x\right) = \langle x^{k+1}\rangle\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1} = \frac{1}{k+1}a_k.$$

□

2.4 Connection to complex analysis

In [FB07, p. 164, Def. 6.2] one finds

Definition 2.4.1 (Residue) *Let $a\in\mathbb{C}$ be a singularity of an analytic function f,*

$$f(z) = \sum_{n=-\infty}^{\infty}a_n(z-a)^n$$

its Laurent expansion in punctured neighborhood of a. The coefficient a_{-1} in this expansion is called ***residue*** *of f at a.*

Notation: $Res(f;a) = a_{-1}$

Following [FB07, p. 144, Cor. 5.2] we can express any function that is analytic on the domain

$$\mathcal{R} = \{z \in \mathbb{C} : r < |z-a| < R\}, \quad 0 \le r < R \le \infty,$$

by a Laurent series, which converges normal[1] in the annulus $\mathcal{R}$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \quad z \in \mathcal{R}.$$

Additionally this Laurent expansion is uniquely determined, by

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta-a|=\varrho} \frac{f(\zeta)}{(\zeta-a)^{n+1}} \,\mathrm{d}\zeta, \quad n \in \mathbb{Z}, r < \varrho < R.$$

As the special case $n = -1$ we have that

$$\operatorname{Res}(f;a) = a_{-1} = \frac{1}{2\pi i} \oint_{|\zeta-a|=\varrho} f(\zeta)\,\mathrm{d}\zeta.$$

Remark: In the following we will denote by

$$\oint_{\gamma} f(\zeta)\,\mathrm{d}\zeta$$

the path integral of f over a suitable curve γ being closed, piecewise smooth, and positive oriented (counterclockwise).

We will demonstrate how the residue representations in [Ego84] are connected with the Coefficient formula Lemma 2.2.9 and generating functions.

Example 2.4.1
The generating function of the binomial coefficient $\binom{n}{k}$, for fixed $n \in \mathbb{N}$, is given by

$$A(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n. \tag{2.52}$$

Because $\binom{n}{k} = 0$ for $k \in \mathbb{Z} : k < 0 \vee k > n$ we can extend the summation interval over all integers. From Lemma 2.2.9 we know that

$$\binom{n}{k} = \underset{x}{\boldsymbol{res}}\ (1+x)^n x^{-k-1}.$$

[1] A series of functions $f_0 + f_1 + f_2 + \dots$ where $f_n : D \subset \mathbb{C} \to \mathbb{C}, n \in \mathbb{N}$, is called normal convergent in D if for every $a \in D$ there exists a neighborhood U and nonnegative $(M_n)_{n\ge0}$ such that $|f_n(z)| \le M_n$ for all $z \in U \cap D$ and all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} M_n$ converges.

Using the residue integral, in complex analysis this can be rewritten as

$$\binom{n}{k} = \frac{1}{2\pi i} \oint_\gamma (1+x)^n x^{-k-1} \, \mathrm{d}x.$$

Sometimes we have to consider the exponential generating function as the following example shows.

Example 2.4.2
The exponential generating function of the sequence of Bernoulli numbers is given by

$$B(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}. \tag{2.53}$$

Hence, we find that

$$B_k = k! \cdot \underset{x}{\boldsymbol{res}} \; B(x) x^{-k-1} = k! \cdot \underset{w}{\boldsymbol{res}} \; (e^x - 1)^{-1} x^{-k},$$

or, in terms of complex analysis:

$$B_k = \frac{k!}{2\pi i} \oint_\gamma (e^x - 1)^{-1} x^{-k} \, \mathrm{d}x.$$

Remark: Note that the residue representations are not necessarily uniquely defined. The same applies to various possible representations of binomial coefficients. To cite [GKP94, p. 204] binomial coefficients are like chameleons, changing their appearance easily. The following two lemmas will state binomial coefficient identities and yield a different generating function than the classic binomial theorem.

Lemma 2.4.1 (Negating the upper index) *For $n, k \in \mathbb{N}$:*

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Proof.

$$\begin{aligned}
\binom{-n}{k} = \frac{(-n)^{\underline{k}}}{k!} &= \frac{(-n)(-n-1)(-n-2)...(-n-k+1)}{k!} \\
&= (-1)^k \frac{n(n+1)(n+2)...(n+k-1)}{k!} = (-1)^k \binom{n+k-1}{k}.
\end{aligned}$$

□

Lemma 2.4.2 *For $n \in \mathbb{N}$:*
$$\binom{2n}{n} = (-4)^n \binom{-1/2}{n}.$$

Proof. Following [GKP94, p. 186] we start by considering $n^{\underline{n}}(n-1/2)^{\underline{n}}$ for integer $n \geq 0$. We claim that
$$n^{\underline{n}}(n-1/2)^{\underline{n}} = \frac{(2n)^{\underline{2n}}}{4^n}, \tag{2.54}$$
which is obvious by expanding the left hand side according to its definition:
$$\begin{aligned} n^{\underline{n}}(n-1/2)^{\underline{n}} &= n(n-1/2)(n-1)(n-3/2)(n-2)\dots(1/2) \\ &= \frac{2n(2n-1)(2n-2)\dots(1)}{2^{2n}} \\ &= \frac{(2n)^{\underline{2n}}}{2^{2n}} = \frac{(2n)!}{2^{2n}}. \end{aligned}$$
If we now divide both sides of (2.54) by $n!^2$ we obtain the identity
$$\begin{aligned} \binom{n}{n}\binom{n-1/2}{n} &= \frac{1}{4^n}\binom{2n}{n} \\ \Longleftrightarrow 4^n\binom{n-1/2}{n} &= \binom{2n}{n} \end{aligned}$$
The result now follows by negating the upper index (Lemma 2.4.1) on the left hand side.
$$\binom{2n}{n} = 4^n\binom{n-1/2}{n} = (-4)^n\binom{1/2-n+n-1}{n} = (-4)^n\binom{-1/2}{n}$$
□

Remark: A more direct proof is obtained by computing the shift quotient $a(n+1)/a(n)$ for both sides of the identity in Lemma 2.4.2. Equality of these quotients reduces the proof of showing equality at the initial value $n = 0$. With the help of these lemmas we can now give a summary of several possible residue representations for selected combinatorial numbers as introduced in the beginning.

Remark: e^x is defined as in example 2.2.1, $\log(1-x)$ is the formal power series (see also example 2.2.2)
$$\log(1-x) := -\sum_{k=1}^{\infty} \frac{1}{k} x^k \in \mathbb{K}_1((x)).$$

binomial coefficient	$\binom{\alpha}{k}$	$\mathop{\mathbf{res}}\limits_x (1+x)^{\alpha} x^{-k-1}$	$\alpha \in \mathbb{K}, k \in \mathbb{N}$
binomial coefficient	$\binom{m+n-1}{n}$	$\mathop{\mathbf{res}}\limits_x (1-x)^{-m} x^{-n-1}$	$m, n \in \mathbb{N}$
binomial coefficient	$\binom{2n}{n}$	$\mathop{\mathbf{res}}\limits_x (1-4x)^{-1/2} x^{-n-1}$	$n \in \mathbb{N}$
exponential function	$\frac{\alpha^n}{n!}$,	$\mathop{\mathbf{res}}\limits_x e^{\alpha x} x^{-n-1}$	$\alpha \in \mathbb{K}, n \in \mathbb{N}$
Bernoulli numbers	B_n	$n! \mathop{\mathbf{res}}\limits_x (e^x - 1)^{-1} x^{-n}$	$n \in \mathbb{N}$
Stirling numbers of 1st kind	$S_1(n,k)$	$\frac{n!}{k!} \mathop{\mathbf{res}}\limits_x (-\log(1-x))^k x^{-n-1}$	$n, k \in \mathbb{N}$, $0 \le k \le n$
Stirling numbers of 2nd kind	$S_2(n,k)$	$\frac{n!}{k!} \mathop{\mathbf{res}}\limits_x (e^x - 1)^k x^{-n-1}$	$n, k \in \mathbb{N}$, $0 \le k \le n$

Table 2.1: Residue Representations, as in [Ego84]

Proof. [Identities in table 2.1] The first identity is a consequence of Thm. and Def. 2.2.1. For the second one we apply the binomial theorem Thm. 2.1.1 in connection with Lemma 2.4.1 (see also [GKP94, p. 199, eq. (5.56)]:

$$\sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n = \sum_{n=0}^{\infty} \binom{-m}{n} (-1)^n x^n = (1-x)^{-m},$$

and hence

$$\binom{m+n-1}{n} = \mathop{\mathbf{res}}\limits_x (1-x)^{-m} x^{-n-1}.$$

For the third identity we take again the binomial theorem Thm. 2.2.1 into account, combined with Lemma 2.4.2:

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^k = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-4)^n x^n = (1-4x)^{-1/2},$$

therefore

$$\binom{2n}{n} = \mathop{\mathbf{res}}\limits_x (1-4x)^{-1/2} x^{-n-1}.$$

The fourth identity goes back to the definition of e^x:

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x^n,$$

and is a simple application of Lemma 2.2.9:

$$\frac{\alpha^n}{n!} = \mathop{\mathbf{res}}_x \; e^{\alpha x} x^{-n-1}.$$

The fifth identity is derived in example 2.4.2. For proving the remaining identities for Stirling numbers we use without proof that the generating functions of Stirling numbers are given as follows ($k \in \mathbb{N}$ fixed):

$$\begin{aligned}
(-\log(1-x))^k &= k! \sum_{n=0}^{\infty} S_1(n,k) \frac{x^n}{n!}, \\
(e^x - 1)^k &= k! \sum_{n=0}^{\infty} S_2(n,k) \frac{x^n}{n!}.
\end{aligned}$$

Multiplying the identities by $\frac{n!}{k!}$ and applying Lemma 2.2.9 we get the desired identities:

$$\begin{aligned}
S_1(n,k) &= \frac{n!}{k!} \mathop{\mathbf{res}}_x \; (-\log(1-x))^k x^{-n-1}, \\
S_2(n,k) &= \frac{n!}{k!} \mathop{\mathbf{res}}_x \; (e^x - 1)^k x^{-n-1}.
\end{aligned}$$

$\square$

Chapter 3

The Riordan group

3.1 The Riordan array approach

In 1991, Louis Shapiro et. al. published the paper [SGWW91] in honor of John Riordan describing the concept of the Riordan group. We will use the definition from [Spr94] which is slightly different from Shapiro's original one.

Consider the infinite matrix $M = (m_{n,k})_{n,k\geq 0}$ with entries in $\mathbb{K}$. If we multiply the matrix M by the infinite vector $(1, x, x^2, ...) = (x^k)_{k\geq 0}$ from the left, we get an infinite row vector for the generating functions of the columns:

$$
\begin{aligned}
(1, x, x^2, x^3, \ \ldots) \quad \cdot \quad & \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \cdots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \cdots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \\
= \quad & (C_0(x), C_1(x), C_2(x), C_3(x), \ \ldots).
\end{aligned}
$$

If we can write each of these generating functions $(C_k(x))_{k\geq 0}$ in the form

$$C_k(x) = \sum_{n=0}^{\infty} m_{n,k} x^n = g(x)(x \cdot f(x))^k, \tag{3.1}$$

with $f(x), g(x) \in \mathbb{K}[[x]]$ such that $f(0) \neq 0$ and $g(0) \neq 0$, we call M a Riordan matrix. More precisely, we define:

Definition 3.1.1 *Let $f(x), g(x) \in \mathbb{K}[[x]]$ be such that $f(0) \neq 0$ and $g(0) \neq 0$. An infinite matrix $M = (m_{n,k})_{n,k\geq 0}$ with entries in $\mathbb{K}$ is called a Riordan array for $(g(x), f(x))$ if*

$$m_{n,k} = \langle x^n \rangle\ g(x)(x \cdot f(x))^k, \quad n, k \geq 0. \tag{3.2}$$

Notation: In this case we write

$$M = \mathcal{R}\left(g(x), f(x)\right). \tag{3.3}$$

Remark: As we can see, a Riordan array M always has to be a lower triangular matrix (because of the factor x^k which forces zero entries above the main diagonal).

For a Riordan array $(m_{n,k})_{n,k\geq 0} = \mathcal{R}\left(g(x), f(x)\right)$ consider the usual matrix vector product, as follows:

$$\begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \cdots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \cdots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix} \tag{3.4}$$

where the generating function of the resulting vector has the form

$$\begin{aligned} B(x) = \sum_{k=0}^{\infty} b_k x^k &= a_0 C_0(x) + a_1 C_1(x) + a_2 C_2(x) + \ldots \\ &= a_0 g(x) + a_1 g(x) x f(x) + a_2 g(x) x^2 f(x)^2 + \ldots \\ &= g(x)(a_0 + a_1 x f(x) + a_2 x^2 f(x)^2 + \ldots) \\ &= g(x) A(x f(x)). \end{aligned} \tag{3.5}$$

where $A(x) \in \mathbb{K}[[x]]$ is the generating function of the sequence $a = (a_0, a_1, a_2, \ldots)$. We summarize in form of a Lemma.

Lemma 3.1.1 *Let $\mathcal{R}(g(x), f(x))$ be a Riordan array. Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$, $B(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{K}[[x]]$. Then the matrix vector relation*

$$\mathcal{R}(g(x), f(x))\ (a_k)_{k\geq 0}^T = (b_n)_{n\geq 0}^T \tag{3.6}$$

is equivalent to

$$g(x) A(x f(x)) = B(x). \tag{3.7}$$

Example 3.1.1 (Pascal's triangle)
Consider the infinite matrix defined by

$$M = (m_{n,k})_{n,k\geq 0} = \left(\binom{n}{k}\right)_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.8}$$

The generating function of the first column is the geometric series, hence

$$C_0(x) = g(x) = \sum_{n=0}^{\infty} m_{n,0}x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{3.9}$$

The generating function of the second column can also be computed easily:

$$C_1(x) = g(x)(xf(x)) = \sum_{n=0}^{\infty} m_{n,1}x^n = \sum_{n=0}^{\infty} nx^n = \frac{1}{1-x}\left(\frac{x}{1-x}\right). \tag{3.10}$$

and already from this we can conjecture that $g(x) = f(x) = \frac{1}{1-x}$, and

$$M = (m_{n,k})_{n,k\geq 0} = \left(\binom{n}{k}\right)_{n,k\geq 0} = \mathcal{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right). \tag{3.11}$$

To prove the statement in general we have to prove that the generating function of the k-th's column is given by

$$C_k(x) = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^k, \quad k \geq 0. \tag{3.12}$$

by equation (3.1) and (3.2) this remains to proving that

$$\langle x^n \rangle \frac{1}{1-x}\left(\frac{x}{1-x}\right)^k = \binom{n}{k}, \quad n,k \geq 0. \tag{3.13}$$

This is also not too hard because

$$\langle x^n \rangle \frac{1}{1-x}\left(\frac{x}{1-x}\right)^k = \langle x^{n-k} \rangle \frac{1}{(1-x)^{k+1}} = \langle x^{n-k} \rangle (1-x)^{-k-1} \tag{3.14}$$

Now we have by the binomial theorem 2.1.1 that

$$(1-x)^{-k-1} = \sum_{n=0}^{\infty} \binom{-k-1}{n} (-1)^n x^n,$$

and hence,

$$\begin{aligned}\langle x^{n-k}\rangle (1-x)^{-k-1} &= (-1)^{n-k} \binom{-k-1}{n-k} \\ &= \binom{k+1+n-k-1}{n-k} = \binom{n}{n-k} = \binom{n}{k},\end{aligned}$$

where we used Lemma 2.4.1 and the elementary symmetry property.

It is remarkable, that $g(x)$ and $f(x)$ could also be guessed with the help of Mallinger's GeneratingFunctions package ([Mal96]), written in Mathematica, in the following way:

```
Mathematica 7.0 - Listing

In[1]:= << GeneratingFunctions.m
        GeneratingFunctions Package by Christian Mallinger © RISC Linz V 0.69 (28-Sep-2009)

In[2]:= PascalTriangle[n_] := Table[Table[Binomial[m, k], {k, 0, n}], {m, 0, n}];

In[3]:= GuessAE[Transpose[PascalTriangle[10]][[1]], g[x]]
Out[3]= {{1 + (-1 + x)g[x] == 0, g[0] == 1}, "ogf"}
In[4]:= GuessAE[Transpose[PascalTriangle[10]][[2]], f[x]]
Out[4]= {{-x^2 + (x - 2x^2 + x^3)f[x] == 0, f[0] == 0}, "ogf"}
```

As the name of the procedures already suggests, this is nothing but guessing of generating functions. It provides possible candidates for $f(x)$ and $g(x)$ which do not necessary need to correspond to the actual elements of the Riordan matrix.

Next we want to compute the row sums of Pascal's triangle. This is equivalent to compute the sums

$$b_n := \sum_{k=0}^{n} \binom{n}{k}, \quad n \geq 0. \tag{3.15}$$

In matrix notation we compute the row sums by multiplying M from right with the vector $(1,1,1,...)^T$. This vector has the generating function

$$A(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{3.16}$$

Hence, by relation (3.5) we find that

$$\mathcal{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}, \tag{3.17}$$

i.e.,

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = g(x)A(xf(x)) \tag{3.18}$$

where we apply the insertion homomorphism $\Phi_{xf(x)}$ to $A(x) = (1-x)^{-1}$ as in Thm. 2.2.1,

$$= \frac{1}{1-x} \cdot \frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-2x}. \tag{3.19}$$

Therefore for the nth row we find that

$$b_n = \sum_{k=0}^{n} \binom{n}{k} = \langle x^n \rangle \ \frac{1}{1-2x} = 2^n, \quad n \geq 0. \tag{3.20}$$

The alternating row sum is multiplication of M by $(1,-1,1,-1,...)^T = ((-1)^k)_{k\geq 0}$. Again we get for $A(x)$ a geometric series

$$A(x) = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}, \tag{3.21}$$

and the relation

$$\mathcal{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right) \begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix}, \tag{3.22}$$

Again (3.5), and application of the insertion homomorphism $\Phi_{xf(x)}$ to $A(x) = (1+x)^{-1}$ gives

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = g(x)A(xf(x)) = \frac{1}{1-x} \frac{1}{1+\frac{x}{1-x}} = 1, \quad n \geq 0. \tag{3.23}$$

We get the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \delta(n,0), \quad n \geq 0. \tag{3.24}$$

3.2 Characterization of Riordan arrays

Let **RA** denote the set of all Riordan arrays over $\mathbb{K}$, i.e. the set of all infinite lower triangular matrices with entries in $\mathbb{K}$, that can be characterized in the way described in Def. 3.1.1. If $M_1 = \mathcal{R}(g(x), f(x)) \in \mathbf{RA}$ and $M_2 = \mathcal{R}(h(x), l(x)) \in \mathbf{RA}$ are two Riordan arrays, one might want to compute the usual matrix product (row by column - product) to obtain another Riordan array. So let

$$M_1 = (a_{n,k})_{n,k\geq 0}, \qquad M_2 = (b_{n,k})_{n,k\geq 0}, \tag{3.25}$$

such that

$$a_{n,k} = \langle x^n \rangle g(x)(xf(x))^k, \tag{3.26}$$
$$b_{n,k} = \langle x^n \rangle h(x)(xl(x))^k. \tag{3.27}$$

We compute the matrix product

$$M := (c_{n,k}) := M_1 \cdot M_2 = \mathcal{R}\left(g(x), f(x)\right) \cdot \mathcal{R}\left(h(x), l(x)\right), \tag{3.28}$$

this means, computing the entry $c_{n,k}$ as for matrices,

$$c_{n,k} = \sum_{j=0}^{\infty} a_{n,j} b_{j,k}. \tag{3.29}$$

Let $M = (M^{(0)}, M^{(1)}, M^{(2)}, ...)$, where

$$M^{(k)} = \begin{pmatrix} c_{0,k} \\ c_{1,k} \\ c_{2,k} \\ \vdots \end{pmatrix} = k\text{th column of } M \tag{3.30}$$

The generating function of the kth column is given by

$$\begin{aligned} M^{(k)}(x) &:= \sum_{n=0}^{\infty} c_{n,k} x^n = \sum_{n=0}^{\infty} x^n \left(\sum_{j=0}^{\infty} a_{n,j} b_{j,k} \right) = \sum_{j=0}^{\infty} b_{j,k} \left(\sum_{n=0}^{\infty} a_{n,j} x^n \right) \\ &= \sum_{j=0}^{\infty} b_{j,k} g(x)(xf(x))^j = g(x) \sum_{j=0}^{\infty} b_{j,k} (xf(x))^j \\ &\overset{\text{Lemma 3.1.1}}{=} g(x)h(xf(x))\left(xf(x)l(xf(x))\right)^k \end{aligned} \tag{3.31}$$

From this we can read off that the usual matrix product of Riordan arrays gives again a Riordan array as

$$\begin{aligned} \cdot : \quad \mathbf{RA} \times \mathbf{RA} &\longrightarrow \mathbf{RA}, \\ (\underbrace{\mathcal{R}(g(x), f(x))}_{M_1}, \underbrace{\mathcal{R}(h(x), l(x))}_{M_2}) &\longmapsto M_1 \cdot M_2 \end{aligned} \tag{3.32}$$

where

$$M_1 \cdot M_2 = \mathcal{R}(g(x)h(xf(x)), f(x)l(xf(x))) \tag{3.33}$$

Obviously, the operation $\cdot : \mathbf{RA} \times \mathbf{RA} \to \mathbf{RA}$ on the set of Riordan matrices is an associative binary operation.

It is also clear that the identity matrix $I := \mathcal{R}(1, 1)$ is the (right and left) neutral element w.r.t $\cdot$.

If we now want to find an inverse element w.r.t. our operation $\cdot$ we consider the product

$$\mathcal{R}(g(x), f(x)) \cdot \mathcal{R}(h(x), l(x)) = \mathcal{R}(g(x)h(xf(x)), f(x)l(xf(x))) = \mathcal{R}(1, 1). \tag{3.34}$$

The formal power series

$$F(x) := xf(x) \tag{3.35}$$

has order 1, and by Thm. 2.2.4 a unique compositional inverse $F^{\langle -1 \rangle}(x)$. Hence, we choose $h(x)$ and $l(x)$ such that

$$h(x) := \frac{1}{g(F^{\langle -1 \rangle}(x))}, \tag{3.36}$$

$$l(x) := \frac{1}{f(F^{\langle -1 \rangle}(x))}. \tag{3.37}$$

If we plug this in, we find that

$\mathcal{R}(g(x), f(x)) \cdot \mathcal{R}(h(x), l(x))$

$$\begin{aligned} &\overset{(3.32)}{=} \mathcal{R}(g(x)h(xf(x)), f(x)l(xf(x))) \\ &\overset{(3.35)}{=} \mathcal{R}(g(x)h(F(x)), f(x)l(F(x))) \\ &\overset{(3.36),(3.37)}{=} \mathcal{R}\left(g(x)\frac{1}{g(F^{\langle -1 \rangle}(F(x)))}, f(x)\frac{1}{f(F^{\langle -1 \rangle}(F(x)))}\right) \\ &\overset{(2.46)}{=} \mathcal{R}\left(g(x)\frac{1}{g(x)}, f(x)\frac{1}{f(x)}\right) = \mathcal{R}(1, 1). \end{aligned}$$

The unique (right- and left-)inverse of a Riordan array $\mathcal{R}(g(x), f(x))$, as we constructed it, will be denoted by $\mathcal{R}(g(x), f(x))^{-1}$.

Theorem 3.2.1
The set of Riordan matrices $\boldsymbol{RA}$ *with the operation* $\cdot$ *(short:* $(\boldsymbol{RA}, \cdot)$*) is group*

Let us examine, when an array $(d_{n,k})_{n,k\geq 0}$ is a Riordan array.

Theorem 3.2.2 ([HS09], p. 3963, Thm. 2.1)
An infinite lower triangular array $D = (d_{n,k})_{n,k\geq 0}$ *in* $\mathbb{K}$ *is a Riordan array if and only if a sequence* $A = (a_0 \neq 0, a_1, a_2, ...)$ *in* $\mathbb{K}^{\mathbb{N}}$ *exists such that for every* $n, k \in \mathbb{N}$ *the following relation holds:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \dots \tag{3.38}$$

Remark: The sum is actually finite since $d_{n,k} = 0$ *for* $k > n$.

Proof. ([Spr06, p. 58, Thm. 5.3.1]) $'' \Rightarrow''$: Let us suppose that $D = (d_{n,k})_{n,k\geq 0}$ is the Riordan array $\mathcal{R}(g(x), f(x))$, i.e.

$$d_{n,k} = \langle x^n \rangle g(x)(xf(x))^k, \tag{3.39}$$

and let us consider the Riordan array $\mathcal{R}(g(x)f(x), f(x))$. We define the Riordan array $\mathcal{R}(A(x), B(x))$ by the relation

$$\begin{aligned} \mathcal{R}(A(x), B(x)) &= \mathcal{R}(g(x), f(x))^{-1} \cdot \mathcal{R}(g(x)f(x), f(x)) \\ \Leftrightarrow \mathcal{R}(g(x), f(x)) \cdot \mathcal{R}(A(x), B(x)) &= \mathcal{R}(g(x)f(x), f(x)). \end{aligned}$$

Because $(\mathbf{RA}, \cdot)$ is a group, the Riordan array $\mathcal{R}(A(x), B(x))$ is well defined. We will later see, that $A(x)$ is the generating function of the sequence A. By performing the product on the left hand side we find that:

$$g(x)A(xf(x)) = g(x)f(x) \quad \text{and} \quad f(x)B(xf(x)) = f(x).$$

From the latter identity we get that $B(xf(x)) = 1 \Rightarrow B(x) = 1$. Therefore

$$\mathcal{R}(g(x), f(x)) \cdot \mathcal{R}(A(x), 1) = \mathcal{R}(g(x)f(x), f(x)).$$

The Riordan array on the left hand side is

$$\mathcal{R}(g(x), f(x)) \cdot \mathcal{R}(A(x), 1) = \mathcal{R}(g(x)A(xf(x)), f(x))$$

and its general element $f_{n,k}$ is

$$\begin{aligned} f_{n,k} &= \langle x^n \rangle\ g(x)A(xf(x))(xf(x))^k \\ &= \langle x^n \rangle \sum_{j=0}^{\infty} a_j (xf(x))^j g(x)(xf(x))^k \\ &\overset{Lem. 2.2.8}{=} \sum_{j=0}^{\infty} a_j \langle x^n \rangle\ g(x)(xf(x))^{k+j} \\ &\overset{(3.39)}{=} \sum_{j=0}^{\infty} a_j d_{n,k+j} \end{aligned}$$

The right hand side evaluates to:

$$\langle x^n \rangle \; g(x)f(x)(x(f(x))^k = \langle x^{n+1} \rangle \; xg(x)f(x)(xf(x))^k = d_{n+1,k+1}.$$

By equating these two quantities, we get the identity (3.38).

$'' \Leftarrow''$: If the first column of a Riordan matrix (i.e. the sequence of elements $(d_{k,0})_{k\geq 0}$) is given, relation (3.38) constructs the Riordan matrix recursively (by repeated application of (3.38)). Let $g(x)$ be the generating function of the first column (we assume $d_{0,0}$ is not zero), $A(x)$ the generating function of the sequence $A = (a_k)_{k\geq 0}$. Consider the functional equation (recall that $F(x) := xf(x)$)

$$f(x) = A(F(x)), \tag{3.40}$$

where $f(x) \in \mathbb{K}_0((x))$. Then, $F(x)$ has order 1, and by Thm. 2.2.4 a unique compositional inverse $F^{\langle -1 \rangle}(x)$ exists. In particular, (3.40) implies that

$$A(x) = f(F^{\langle -1 \rangle}(x)). \tag{3.41}$$

Therefore we can consider the Riordan array

$$\hat{D} := \mathcal{R}(g(x), f(x))$$

The generating function of the first column coincide by construction, the generating function for the kth column match by recurrence relation (3.38). $\square$

The sequence $A = (a_k)_{k\geq 0}$ is called the $A-$ sequence of the Riordan array $D = \mathcal{R}(g(x), f(x))$. As we have seen in the proof of the theorem, its generating function $A(x) = \sum_{k=0}^{\infty} a_k x^k$, satisfies the functional equation

$$f(x) = A(xf(x)), \tag{3.42}$$

and it only depends on $f(x)$.
Conversely, $A(x)$ can be determined by the relation:

$$A(x) = f(F^{\langle -1 \rangle}(x)), \text{ where } F(x) := xf(x). \tag{3.43}$$

Another type of characterization is obtained through the following observation

Theorem 3.2.3 ([Spr06] p. 58, Thm. 5.3.2)
Let $M := (d_{n,k})_{n,k\geq 0} = \mathcal{R}(g(x), f(x))$ be a Riordan array. Then a unique sequence $Z = (z_k)_{k\geq 0}$ exists such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \dots \tag{3.44}$$

Proof. Let $z_0 = \frac{d_{1,0}}{d_{0,0}}$. Now, due to the fact that $(d_{n,k})_{n,k\geq 0}$ is a lower triangular matrix, we can uniquely determine the value of z_1 by expressing $d_{2,0}$ in terms of the elements in row 1, i.e.

$$d_{2,0} = z_0 d_{1,0} + z_1 d_{1,1} \quad \Leftrightarrow \quad z_1 = \frac{d_{0,0}d_{2,0} - d_{1,0}^2}{d_{0,0}d_{1,1}}.$$

In the same way, we determine z_2 by expressing $d_{3,0}$ in terms of the elements in row 2, and by substituting the values just obtained for z_0 and z_1. By proceeding the same way, we determine the sequence Z in a unique way. □

The sequence Z is called the $Z-$sequence for the Riordan array. It characterizes column 0 except for the first element. Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ be the generating function of the A-sequence $(a_k)_{k\geq 0}$, $Z(t) = \sum_{k=0}^{\infty} z_k t^k$ the generating function of the Z-sequence $(z_k)_{k\geq 0}$. For $d_{0,0} \in \mathbb{K}\backslash\{0\}$, we can say that the triple

$$(d_{0,0}, A(t), Z(t))$$

completely characterizes a Riordan array. The next theorem is a way how to compute $g(x)$ given $f(x)$ and the $Z-$sequence of a Riordan array.

Theorem 3.2.4 ([MRSV97], p. 5, Thm. 2.3)
Let $M = (d_{n,k})_{n,k\geq 0} = \mathcal{R}(g(x), f(x))$ be a Riordan array and let $Z(t) = \sum_{n=0}^{\infty} z_n t^n$ be the generating function of the array's $Z-$sequence $(z_k)_{k\geq 0}$. Then:

$$g(x) = \frac{g(0)}{1 - xZ(xf(x))}. \tag{3.45}$$

Proof. By the preceding Theorem, the $Z-$sequence exists and is unique, and equation (3.44) is valid for every $n \in \mathbb{N}$. Relation (3.44) translates to

$$\begin{aligned} d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \ldots \\ \langle x^{n+1}\rangle g(x) &= \sum_{k=0}^{\infty} z_k \langle x^n\rangle g(x)(xf(x))^k \\ \langle x^n\rangle \frac{g(x) - g(0)}{x} &= \langle x^n\rangle g(x) Z(xf(x)) \end{aligned}$$

Because two power series are identical if and only if their coefficients coincide, we have equality above, because the last line holds for all $n \in \mathbb{N}$. Hence, we find that

$$\frac{g(x) - g(0)}{x} = g(x)Z(xf(x)) \Leftrightarrow g(x) = \frac{g(0)}{1 - xZ(xf(x))}.$$

□

Note: This relation can be inverted and this gives us a formula for the generating function of the $Z-$sequence ($F(x) := xf(x)$):

$$\frac{g(x) - g(0)}{xg(x)} = Z(xf(x)) \Rightarrow Z(y) = \frac{g(F^{\langle -1 \rangle}(x)) - g(0)}{F^{\langle -1 \rangle}(x) g(F^{\langle -1 \rangle}(x))}. \tag{3.46}$$

There is a non-trivial connection between the generating functions of the $A-$ and the $Z-$sequence and the functions $g(x)$ and $f(x)$. In particular the following holds:

Theorem 3.2.5 ([MRSV97], p. 6, Thm. 2.4)
Let $D = \mathcal{R}(g(x), f(x)) \in \boldsymbol{RA}$. Then $g(x) = f(x)$ if and only if $A(x) = g(0) + xZ(x)$

Proof. $'' \Leftarrow''$: Let us assume that $A(x) = g(0) + xZ(x)$ or what is the same $Z(x) = (A(x) - g(0))/x$. By the preceding theorem we have

$$g(x) = \frac{g(0)}{1 - xZ(xf(x))} = \frac{g(0)}{1 - (xA(xf(x)) - g(0)x)/(xf(x))} = \frac{g(0)xf(x)}{g(0)x} = f(x),$$

because, by (3.42) we have that $A(xf(x)) = f(x)$.

$'' \Rightarrow''$: By (3.45) and from the hypothesis $g(x) = f(x)$ we find that:

$$g(x) = \frac{g(0)}{1 - xZ(xf(x))} = \frac{g(0)}{1 - xZ(xg(x))}$$
$$\Leftrightarrow g(x) - xg(x)Z(xg(x)) = g(0).$$

Now we apply (3.42) and the hypothesis:

$$f(x) = A(xf(x)) \Rightarrow g(x) = A(xg(x)),$$

to obtain the identity:

$$A(xg(x)) = g(0) + xg(x)Z(xg(x)).$$

or, with $G(x) := xg(x)$:

$$A(G(x)) = g(0) + G(x)Z(G(x)).$$

Setting $x = G^{\langle -1 \rangle}(x)$ gives the desired equality. □

Chapter 4

Application to Symbolic Summation

4.1 The Identities of Abel and Gould

Wilf and Zeilberger provide an algorithm for proving summation identities of the form

$$\sum_k \text{summand}(n,k) = \text{answer}(n), \quad n \geq 0. \tag{4.1}$$

where summand(n,k) and answer(n) are nice.

For a given sum

$$f(n) = \sum_k F(n,k), \tag{4.2}$$

where F is doubly hypergeometric (that is both $F(n{+}1,k)/F(n,k)$ and $F(n,k{+}1)/F(n,k)$ are rational functions of n and k) every proper hypergeometric term $F(n,k)$ satisfies a k-free recurrence [PWZ96, Thm. 4.4.1, p. 65]. A proper hypergeometric term can be written in the form

$$P(n,k)\frac{\prod_{i=0}^{u}(a_in+b_ik+c_i)!}{\prod_{i=0}^{v}(u_in+v_ik+w_i)!}x^k, \tag{4.3}$$

where $P(n,k) \in \mathbb{K}[n,k], x \in \mathbb{K}, a_n, b_n, u_n, v_n \in \mathbb{N}, u, v \in \mathbb{N}$.

So there exist $I, J \in \mathbb{N}$ and polynomials $a_{i,j}(n)$ such that the recurrence

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{i,j}(n)F(n-j,k-i) = 0 \tag{4.4}$$

holds at every point (n,k) where $F(n,k) \neq 0$. Further there are bounds for I, J given.

However, there are combinatorial sums that are not doubly hypergeometric in the sense defined above. For instance, if we try to prove the identity of **Abel** ([GKP94, p. 202, (5.64)])

$$\sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k} = (a+b+n)^n, \quad a,b \in \mathbb{K}, n \geq 0. \tag{4.5}$$

with the Paule/Schorn implementation of Zeilberger's algorithm [PS95] we get the negative answer

```
Mathematica 7.0 - Listing

In[1]:= << zb.m
        Fast Zeilberger Package by Peter Paule and Markus Schorn (enhanced by Axel Riese)
        © RISC Linz  V 3.54 (02/23/05)

        (* We are looking for a recurrence in n of order 1 *)

In[2]:= Zb[Binomial[n, k]a(a + k)^(k−1)(b + n − k)^(n−k), {k, 0, n}, n, 1]
        Zb::badfac :   The factor (b − k + n)^(−k+n) cannot be handled
```

Similar difficulties arise by trying to prove the identity of **Gould** ([GKP94, p. 202, (5.62)])

$$S(n) := \sum_{k=0}^{n} \left(\binom{r-qk}{k} + q\binom{r-qk-1}{k-1} \right) \binom{p+qk}{n-k} = \binom{p+r}{n}, \tag{4.6}$$

where $p, q, r \in \mathbb{N}, n \geq 0$. If we expand the binomial coefficient due to (2.6), and do some simplification we have

$$S(n) = \sum_{k=0}^{n} \frac{r}{r-qk} \binom{r-qk}{k} \binom{p+qk}{n-k}, \quad n \geq 0, \tag{4.7}$$

where we assume that $r - qk \neq 0$ for any choice of $r, q, k \in \mathbb{N}$.

If we use Zeilberger's algorithm to compute a recurrence for $S(n)$ we get

```
Mathematica 7.0 - Listing

In[1]:= Zb[r/(r − kq) Binomial[r − kq, k]Binomial[p + kq, n − k], {k, 0, n}, n, 1]
        Zb::intlin :   p + kq is not integer-linear in k
```

If we try to solve the specialized problem where we set $q = 0$, we get the summation problem

$$\sum_{k=0}^{n} \binom{r}{k} \binom{p}{n-k} = \binom{p+r}{n}, \quad p, r \in \mathbb{N}, n \geq 0, \tag{4.8}$$

which is Vandermonde's convolution formula (2.17). Zeilberger's algorithm produces the order 1 recurrence

```
Mathematica 7.0 - Listing

In[2]:= Zb [Binomial[r, k]Binomial[p, n − k], {k, 0, n}, n, 1]
        If 'n' is a natural number, then:
Out[2]= {(−n + p + r)SUM[n] + (−1 − n)SUM[1 + n] == 0}
```

If we set $q = 1$ we do not succeed in finding a order 1 recurrence, but a recurrence of order 2:

```
Mathematica 7.0 - Listing

In[3]:= Zb [ r/(r − k) Binomial[r − k, k]Binomial[p + k, n − k], {k, 0, n}, n, 2]
        If 'n' is a natural number, then:
Out[3]= {(n − p − r)(1 + n − p − r)SUM[n] + (3 + 2n − r)(1 + n − p − r)SUM[1 + n] + (2 + n)(2 + n − r)SUM[2 + n] == 0}
```

If we plug in further values for q we get higher order for the recurrences obtained. In particular for the values up to 3 we get:

Value for q	Order of Recurrence for $S(n)$	Computation Time
0	1	0.015 s
1	2	0.125 s
2	4	16.411 s
3	6	5581.14 s $\approx$ 93 min

This is not really satisfactory because the complexity of solving the problem depends on the input parameter q. For every fixed integer value q we have that

$$F(n,k) = r \cdot \frac{(r - qk - 1)!(p + qk)!}{k!(n-k)!(r + q - qk)!(p - n + 2qk)!} \tag{4.9}$$

is a proper hypergeometric term, and therefore $f(n) = \sum_k F(n,k)$ satisfies a k-free recurrence. In the following, we present ways of how to compute the sum for general q.

4.1.1 Applying the Egorychev Method

Example 4.1.1 (Abel's Identity) [1] *Because of* $\binom{n}{k} = 0$ *for* $k > n$ *we can extend the summation interval to the nonnegative integers. As a preprocessing step we need to do some algebraic manipulations.*

$$\sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k} = \sum_{k=0}^{\infty} \binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k}$$
$$= \sum_{k=0}^{\infty} \binom{n}{k} (a+k-k)(a+k)^{k-1}(b+n-k)^{n-k}$$
$$= n! \sum_{k=0}^{\infty} \frac{(a+k)^k - k(a+k)^{k-1}}{k!} \frac{(b+n-k)^{n-k}}{(n-k)!},$$

where we have expanded the binomial coefficient as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. *The first part involves*

$$\frac{(a+k)^k - k(a+k)^{k-1}}{k!} = \frac{(a+k)^k}{k!} + \frac{(a+k)^{k-1}}{(k-1)!},$$

that is valid for $k \geq 1$.

We will need to take care of this, when we apply the inversion rule 4.

$$= \quad n! \sum_{k=1}^{\infty} \left(\frac{(a+k)^k}{k!} + \frac{(a+k)^{k-1}}{(k-1)!} \right) \frac{(b+n-k)^{n-k}}{(n-k)!}$$
$$= \quad n! \sum_{k=1}^{\infty} \left(\underset{u}{\boldsymbol{res}} \, (e^{(a+k)u} u^{-k-1}) - \underset{u}{\boldsymbol{res}} \, (e^{(a+k)u} u^{-k}) \right) \underset{w}{\boldsymbol{res}} \, e^{(b+n-k)w} w^{-n+k-1}$$
$$= \quad n! \, \underset{w}{\boldsymbol{res}} \, e^{(b+n)w} w^{-n-1} \underbrace{\sum_{k=1}^{\infty} (we^{-w})^k \underset{u}{\boldsymbol{res}} \, ((1-u)e^{(a+k)u} u^{-k-1})}_{f(w)}.$$

Now we can apply the Lagrange Inversion Formula (see Thm.2.2.2) and by pattern matching we find for $k \geq 1$*:*

$$\begin{aligned} \underset{u}{\boldsymbol{res}} \, (1-u)e^{(a+k)u} u^{-k-1} &= \underset{u}{\boldsymbol{res}} \, f(u) D_u((ue^{-u}))(ue^{-u})^{-k-1} \\ &= \underset{u}{\boldsymbol{res}} \, f(u)(1-u)e^{ku} u^{-k-1}, \end{aligned}$$

[1] Example 2.1.6 [Ego84, p. 48]

and we can take $f(u) = e^{au}$. Finally we get for the original sum

$$\begin{aligned} &= n! \underset{w}{\boldsymbol{res}}\ e^{(b+n)w} w^{-n-1} \underbrace{\sum_{k=1}^{\infty} (we^{-w})^k \underset{u}{\boldsymbol{res}}\ ((1-u)e^{(a+k)u} u^{-k-1})}_{=e^{aw}} \\ &= n! \underset{w}{\boldsymbol{res}}\ e^{(a+b+n)w} w^{-n-1} = (a+b+n)^n. \end{aligned}$$

Example 4.1.2 (Gould's Identity) [2] *We start the computation by our original summand*

$$\sum_{k=0}^{n} \left(\binom{r-qk}{k} + q\binom{r-kq-1}{k-1} \right) \binom{p+qk}{n-k}.$$

Reasoning similar as above on $\binom{p+qk}{n-k}$ we can extend the summation interval to range over all nonnegative integers. Applying the substitutions from Table 2.1 we get

$$\begin{aligned} &= \sum_{k=0}^{\infty} \left(\binom{r-qk}{k} + q\binom{r-kq-1}{k-1} \right) \binom{p+qk}{n-k} \\ &= \sum_{k=0}^{\infty} \left(\underset{u}{\boldsymbol{res}}\ (1+u)^{r-kq} u^{-k-1} + q \underset{u}{\boldsymbol{res}}\ ((1+u)^{r-kq-1} u^{-k}) \right) \underset{w}{\boldsymbol{res}}\ ((1+w)^{p+qk} w^{-n+k-1}) \\ &= \underset{w}{\boldsymbol{res}}\ (1+w)^p w^{-n-1} \underbrace{\sum_{k=0}^{\infty} (w(1+w)^q)^k \underset{u}{\boldsymbol{res}}\ ((1+u)^{r-kq-1} u^{-k-1} (1+u+qu))}_{f(w)}. \end{aligned}$$

As in Abel's identity we now have to apply the Lagrange Inversion Formula (Thm.2.2.2) to compute the sum:

$$\begin{aligned} \underset{u}{\boldsymbol{res}}\ (1+u+qu)(1+u)^{r-kq-1} u^{-k-1} &= \underset{u}{\boldsymbol{res}}\ f(u) D_u(u(1+u)^q)(u(1+u)^q)^{-k-1} \\ &\overset{!}{=} \underset{u}{\boldsymbol{res}}\ f(u)(1+u+qu)(1+u)^{-kq-1} u^{-k-1}, \end{aligned}$$

and we can take $f(u) = (1+u)^r$; therefore the original sum equals

$$\begin{aligned} &= \underset{w}{\boldsymbol{res}}\ (1+w)^p w^{-n-1} \underbrace{\sum_{k=0}^{\infty} (w(1+w)^q)^k \underset{u}{\boldsymbol{res}}\ ((1+u)^{r-kq-1} u^{-k-1} (1+u+qu))}_{=(1+w)^r} \\ &= \underset{w}{\boldsymbol{res}}\ (1+w)^{p+r} w^{-n-1} = \binom{p+r}{n}. \end{aligned}$$

[2] See [Ego84, p. 80, exercise 2.4.3.e Hagen's identity]

4.1.2 Applying the Riordan Array paradigm

The problem was investigated in [Spr95]. For computation Sprugnoli constructs an Riordan Array where we can read off the solution.

Theorem 4.1.1 ([Spr95], p. 218, Thm. 3.1)
Let $(m_{n,k})_{n,k\geq 0} = \mathcal{R}(g(x), f(x))$ be a Riordan array and let $h(x)$ be the generating function of a sequence $(h_k)_{k\geq 0}$. If $(\hat{h}_k)_{k\geq 0}$ is the sequence, whose generating function is

$$\hat{h}_{(f)}(x) := h(F^{\langle -1\rangle}(x)), \text{ where } F(x) = xf(x)$$

then:

$$\sum_{k=0}^{\infty} m_{n,k}\hat{h}_k = \langle x^n\rangle\ g(x)h(x) \tag{4.10}$$

Proof. We follow the proof given in [Spr95, p. 218].

From Lemma 3.1.1 we have for the sequence $(\hat{h}_k)_{k\geq 0}$:

$$\begin{aligned}\sum_{k=0}^{\infty} m_{n,k}\hat{h}_k &= \langle x^n\rangle\ g(x)\hat{h}_f(xf(x)) \\ &= \langle x^n\rangle\ g(x)h(F^{\langle -1\rangle}(xf(x))) \\ &= \langle x^n\rangle\ g(x)h(x).\end{aligned}$$

□

Example 4.1.3 *We apply Thm. 4.1.1 to the Riordan array*

$$D = \mathcal{R}(g(x), f(x)) = \mathcal{R}\left(e^{(b+n)x}, e^{-x}\right)$$

with $h(x) = e^{ax}$, where $a, b \in \mathbb{K}$. Note that $h(x)$ is the generating function of the sequence $(a^k/k!)_{k\geq 0}$. We get that

$$m_{n,k} = \langle x^n\rangle g(x)(xf(x))^k = \langle x^n\rangle\ e^{(b+n)x}(xe^{-x})^k \overset{(2.27)}{=} \langle x^{n-k}\rangle\ e^{(b+n-k)x} = \frac{(b+n-k)^{n-k}}{(n-k)!}.$$

Because of $\hat{h}_{(f)}(x) = h(F^{\langle -1\rangle}(x))$ we have $\hat{h}_0 = \hat{h}_{(f)}(0) = h_0 = 1$, and

$$\hat{h}_{(f)} = \sum_{n=0}^{\infty} \hat{h}_k(xe^{-x})^k = h(x).$$

Consequently, by the Lagrange inversion formula given in Cor. 2.2.1, for $k \geq 1$:

$$\hat{h}_k = \frac{1}{k}\langle x^{k-1}\rangle\ h'(x)e^{kx} \overset{(2.26)}{=} \frac{a}{k}\langle x^{k-1}\rangle e^{ax}e^{kx} \overset{(2.27)}{=} \frac{a}{k}\langle x^{k-1}\rangle e^{(a+k)x} = a\frac{(a+k)^{k-1}}{k!} \tag{4.11}$$

By formula (4.10) we have that

$$\begin{aligned}\sum_{k=0}^{\infty} m_{n,k}\hat{h}_k = \sum_{k=0}^{\infty} a\frac{(a+k)^{k-1}}{k!}\frac{(b+n-k)^{n-k}}{(n-k)!} \quad &= \quad \langle x^n\rangle\ g(x)h(x) \\ &\overset{(2.27)}{=} \quad \langle x^n\rangle\ e^{(a+b+n)x} = \frac{(a+b+n)^n}{n!},\end{aligned}$$

which can be rewritten as

$$\sum_{k=0}^{n}\binom{n}{k}a(a+k)^{k-1}(b+n-k)^{n-k} = (a+b+n)^n.$$

Example 4.1.4 (Generalized Abel's identity) [3] *If one instead considers the Riordan array*

$$D = \mathcal{R}(g(x), f(x)) = \mathcal{R}\left(e^{(b+dn)x}, e^{-dx}\right)$$

and $h(x) = e^{ax}$, where $a, b, d \in \mathbb{K}, n \in \mathbb{N}$, one gets as general entry

$$m_{n,k} = \langle x^n\rangle\ e^{(b+dn)x}(xe^{-dx})^k \overset{(2.27)}{=} \langle x^{n-k}\rangle\ e^{(b+dn-dk)x}\frac{(b+d(n-k))^{n-k}}{(n-k)!}.$$

Similar as above, we have that $\hat{h}_0 = \hat{h}_{(f)}(0) = h_0 = 1$, and for $k \geq 1$:

$$\hat{h}_k = \frac{1}{k}\langle x^{k-1}\rangle\ h'(x)e^{kx} \overset{(2.26)}{=} \frac{a}{k}\langle x^{k-1}\rangle e^{ax}e^{kdx} \overset{(2.27)}{=} \frac{a}{k}\langle x^{k-1}\rangle e^{(a+kd)x} = a\frac{(a+kd)^{k-1}}{k!}.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} m_{n,k}\hat{h}_k &= \sum_{k=0}^{\infty} a\frac{(a+dk)^{k-1}}{k!}\frac{(b+d(n-k))^{n-k}}{(n-k)!} \\ &= \langle x^n\rangle\ g(x)h(x) \overset{(2.27)}{=} \langle x^n\rangle\ e^{(a+b+dn)x} = \frac{(a+b+dn)^n}{n!}.\end{aligned}$$

The resulting identity can rewritten as

$$\sum_{k=0}^{n}\binom{n}{k}a(a+dk)^{k-1}(b+d(n-k))^{n-k} = (a+b+dn)^n, \tag{4.12}$$

which is not present in [Spr95].

[3] See [Rio68, p. 18, Equ. (13)]

Example 4.1.5 *Consider the Riordan array*

$$D = \mathcal{R}(g(x), f(x)) = \mathcal{R}\left((1+x)^p, (1+x)^q\right), \quad p, q \in \mathbb{N},$$

and $h(x) = (1+x)^r$*, where* $r \in \mathbb{N}$*. By Thm. 4.1.1 we obtain that*

$$m_{n,k} = \langle x^n \rangle\ (1+x)^p (x(1+x)^q)^k = \langle x^{n-k} \rangle\ (1+x)^{p+qk} = \binom{p+qk}{n-k}.$$

Furthermore, $\hat{h}_0 = \hat{h}_{(f)}(0) = h_0 = 1$ *and*

$$\begin{aligned}\hat{h}_k &= \frac{1}{k}\langle x^{k-1} \rangle\ h'(x) f(x)^{-k} = \frac{1}{k}\langle x^{k-1} \rangle\ D_x((1+x)^r)((1+x)^q)^{-k} \\ &\overset{(2.28),(2.16)}{=} \frac{r}{k}\langle x^{k-1} \rangle\ (1+x)^{r-1-qk} = \frac{r}{k}\binom{r-1-qk}{k-1} = \frac{r}{r-qk}\binom{r-qk}{k}\end{aligned}$$

So we finally find

$$\sum_{k=0}^{n} \frac{r}{r-qk}\binom{r-qk}{k}\binom{p+qk}{n-k} = \langle x^n \rangle\ g(x)h(x) \overset{(2.16)}{=} \langle x^n \rangle\ (1+x)^{p+r} = \binom{p+r}{n}.$$

4.2 Multi-Sum Identities

The machinery developed by Egorychev is not restricted to one single summation quantifier as the following American Mathematical Monthly Problem shows.

Example: The American Mathematical Monthly, Problem 11033.[4]
Proposed by *M.N. Deshpande and R.M. Welukar, Institute of Science, Nagpur, India.*
Let

$$P(m, n, r) := \sum_{k=0}^{r} (-1)^k \binom{m+n-2(k+1)}{n}\binom{r}{k} \tag{4.13}$$

Let m, n and r be integers such that $0 \le r \le n \le m-2$. Show that $P(m, n, r)$ is positive and that

$$\sum_{r=0}^{n} P(m, n, r) = \binom{m+n}{n} \tag{4.14}$$

We start by considering the inner sum $P(m, n, r)$. The summation over k can be extended to range over the nonnegative integers because the binomial coefficient forces the summand

[4]The American Mathematical Monthly, Vol. 110, No. 8 (Oct., 2003), p. 742

to vanish identically for $k > r$. Afterwards we can replace the binomial coefficient by our residue functional.

$$\begin{aligned}
P(m,n,r) &= \sum_{k=0}^{r}(-1)^k \binom{m+n-2(k+1)}{n}\binom{r}{k} = \sum_{k=0}^{\infty}(-1)^k \binom{m+n-2(k+1)}{n}\binom{r}{k} \\
&= \sum_{k=0}^{\infty}(-1)^k \mathop{\mathbf{res}}_{u}\ (1+u)^{m+n-2-2k}u^{-n-1}\ \mathop{\mathbf{res}}_{w}\ (1+w)^r w^{-k-1} \\
&= \mathop{\mathbf{res}}_{u}\ (1+u)^{m+n-2}u^{-n-1}\ \sum_{k=0}^{\infty}(-(1+u)^{-2})^k \mathop{\mathbf{res}}_{w}\ (1+w)^r w^{-k-1}.
\end{aligned}$$

after pulling out the factors not depending on the summation index. Now we can take into account the substitution rule 3. The remaining sums simplifies to

$$\sum_{k=0}^{\infty}(-(1+u)^{-2})^k \mathop{\mathbf{res}}_{w}\ (1+w)^r w^{-k-1} \quad = \quad \left(1-\frac{1}{(1+u)^2}\right)^r.$$

This gives us for the inner sum the residue representation

$$P(m,n,r) = \mathop{\mathbf{res}}_{u}\ (1+u)^{m+n-2-2r}u^r(2+u)^r u^{-n-1}. \tag{4.15}$$

The BSI Problems Group, Bonn, Germany claimed in their solution[5] of the problem that the original sum $P(m,n,r)$ can be rewritten in the following way

$$\sum_{k=0}^{r}(-1)^k \binom{m+n-2(k+1)}{n}\binom{r}{k} = \sum_{k=0}^{m-2}\binom{r}{k}\binom{n-r+m-2-k}{n-r}. \tag{4.16}$$

If this is the case we have proven that $P(m,n,r)$ is indeed positive since we are adding up non-negative quantities. (Note that by the assumptions on m,n,r the quantity $n-r+m-2-k$ for $0 \le k \le m-2$ is always positive and so is the binomial coefficient (i.e. Lemma 2.4.1 is never invoked)).

We will present two proofs: The first one takes into account that the summands are both hypergeometric. Hence, with the help of Zeilberger's algorithm we can find recurrence relations for the free variables r, m, n of both sums. If they coincide (up to a constant multiple) it remains to check initial values to ensure equivalence.

[5]The American Mathematical Monthly, Vol. 112, No. 5 (May, 2005), p. 471, "Expansion By Inclusion-Exclusion"

Mathematica 7.0 - Listing

(* We are looking for a recurrence in r of order 2 *)

In[1]:= $\mathbf{Zb[(-1)^k Binomial[m+n-2(k+1), n]Binomial[r, k], \{k, 0, r\}, r, 2]}$
If 'r' is a natural number, then:

Out[1]= $\{4(n-r)(1+r)\mathrm{SUM}[r] + (14-4m-13n+2mn+n^2+22r-4mr-8nr+8r^2)\mathrm{SUM}[1+r] - (-5+m+n-2r)(-4+m+n-2r)\mathrm{SUM}[2+r] == 0\}$

In[2]:= $\mathbf{Zb[Binomial[r, k]Binomial[n-r+m-2-k, n-r], \{k, 0, m-2\}, r, 2]}$
If '-2 + m' is a natural number and none of {-2 + n - r, r} is a negative integer, then:

Out[2]= $\{-4(n-r)(1+r)\mathrm{SUM}[r] + (-14+4m+13n-2mn-n^2-22r+4mr+8nr-8r^2)\mathrm{SUM}[1+r] + (-5+m+n-2r)(-4+m+n-2r)\mathrm{SUM}[2+r] == 0\}$

(* We are looking for a recurrence in m of order 2 *)

In[3]:= $\mathbf{Zb[(-1)^k Binomial[m+n-2(k+1), n]Binomial[r, k], \{k, 0, r\}, m, 2]}$
If 'r' is a natural number, then:

Out[3]= $\{(-1+m+n-2r)\mathrm{SUM}[m] + (1+n)\mathrm{SUM}[1+m] - m\mathrm{SUM}[2+m] == 0\}$

In[4]:= $\mathbf{Zb[Binomial[r, k]Binomial[n-r+m-2-k, n-r], \{k, 0, m-2\}, m, 2]}$
If '-2 + m' is a natural number and none of {n - r, r} is a negative integer, then:

Out[4]= $\{(1-m-n+2r)\mathrm{SUM}[m] + (-1-n)\mathrm{SUM}[1+m] + m\mathrm{SUM}[2+m] == 0\}$

(* We are looking for a recurrence in n of order 2 *)

In[5]:= $\mathbf{Zb[(-1)^k Binomial[m+n-2(k+1), n]Binomial[r, k], \{k, 0, r\}, m, 2]}$
If 'r' is a natural number, then:

Out[5]= $\{(-1+m+n-2r)\mathrm{SUM}[n] + (-1-2m-3n+4r)\mathrm{SUM}[1+n] + 2(2+n-r)\mathrm{SUM}[2+n] == 0\}$

In[6]:= $\mathbf{Zb[Binomial[r, k]Binomial[n-r+m-2-k, n-r], \{k, 0, m-2\}, m, 2]}$
If '-2 + m' is a natural number and none of {n - r, r} is a negative integer, then:

Out[6]= $\{(1-m-n+2r)\mathrm{SUM}[n] + (1+2m+3n-4r)\mathrm{SUM}[1+n] - 2(2+n-r)\mathrm{SUM}[2+n] == 0\}$

Another way is to derive an residue representation of "their" sum. If they coincide we have proven that they express the same value. First we notice that by assumption $r \leq m-2$ and therefore the summand vanishes for $k > r$. So we first change the bounds of our summation and replace the second binomial coefficient.

$$\begin{aligned}
\sum_{k=0}^{m-2}\binom{r}{k}\binom{n-r+m-2-k}{n-r} &= \sum_{k=0}^{r}\binom{r}{k}\binom{n-r+m-2-k}{n-r} \\
&= \sum_{k=0}^{r}\binom{r}{k}\operatorname*{\mathbf{res}}_u\ (1+u)^{n+m-r-2-k}u^{-n+r-1} \\
&= \operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-r-2}u^{-n+r-1}\ \sum_{k=0}^{r}\binom{r}{k}(1+u)^{-k}.
\end{aligned}$$

Now apply the binomial theorem and get

$$\begin{aligned}\operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-r-2}u^{-n+r-1}\ \sum_{k=0}^{r}\binom{r}{k}(1+u)^{-k} &= \operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-r-2}u^{-n+r-1}\ \left(1+\frac{1}{1+u}\right)^r\\ &= \operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-2-2r}u^r\ \ (2+u)^r\, u^{-n-1},\end{aligned}$$

in accordance with (4.15). This proves identity (4.16). We note that identity (4.16) could be derived by reading our proof of (4.16) backwards.

Finally we note that the identity (4.14) we want to prove here is in fact not very hard once we plug in (4.15). We pull out the factors not depending on r and remain with a simple geometric series.

$$\begin{aligned}S(m,n) := \sum_{r=0}^{n} P(m,n,r) &= \sum_{r=0}^{n}\operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-2-2r}u^r(2+u)^r u^{-n-1}\\ &= \operatorname*{\mathbf{res}}_u\ (1+u)^{m+n-2}u^{-n-1}\sum_{r=0}^{n}\left(\frac{u(2+u)}{(1+u)^2}\right)^r.\end{aligned}$$

The geometric series evaluates to

$$\sum_{r=0}^{n}\left(\frac{u(2+u)}{(1+u)^2}\right)^r = (1+u)^2 - u^{n+1}(2+u)^{n+1}(1+u)^{-2n},$$

and hence

$$S(m,n) = \operatorname*{\mathbf{res}}_u\ (1+u)^{m+n}u^{-n-1} + \operatorname*{\mathbf{res}}_u\ (1+u)^{m-n-2}(2+u)^{n+1} = \binom{m+n}{n} + 0,$$

because by assumption $m \geq n+2$.

□

4.3 Another Mathematical Monthly Problem

Manuel Kauers together with Sheng-Lang Ko came in their work to meet the sum

$$S(n) := (-1)^n \sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}S_1(n+k,k). \tag{4.17}$$

It was posed as an Mathematical Monthly problem (The American Mathematical Monthly, Problem 11545, Vol. 118, No. 1 (Jan. 2011), p.84) to find a simple closed form for the sum. Again we will apply the Egorychev method to give a simple closed form solution.

As in the previous section we start by extending the summation interval to go over all nonnegative integers. This can be done, because the binomial coefficient $\binom{2n}{n+k} = 0$ for $k > n$. So from now on let us consider the infinite version of the sum

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} S_1(n+k,k) = (-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{2n}{n+k} S_1(n+k,k).$$

We start by expanding the binomial coefficient and by replacing $S_1(n+k,k)$ according to its residue representations. We recall the generating function

$$\sum_{k=0}^{\infty} \frac{S_1(k,m)}{k!} x^k = \frac{(-\log(1-x))^m}{m!}. \tag{4.18}$$

Identity (4.18) gives us the residue representation

$$S_1(n+k,k) = \frac{(n+k)!}{k!} \operatorname*{\mathbf{res}}_u \; (-\log(1-u))^k u^{-n-k-1}. \tag{4.19}$$

With cancellation we get the representation

$$(-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{2n}{n+k} S_1(n+k,k)$$

$$\begin{aligned} &= (-1)^n (2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!(n-k)!} \frac{(n+k)!}{k!} \operatorname*{\mathbf{res}}_u \; (-\log(1-u))^k u^{-n-k-1} \\ &= (-1)^n \frac{(2n)!}{n!} \operatorname*{\mathbf{res}}_u \; u^{-n-1} \sum_{k=0}^{\infty} \binom{n}{k} \left(-\frac{\log(1-u)}{u} \right)^k. \end{aligned}$$

This sum can now be simplified by the binomial theorem 2.1.1 and rule 3 (note that the generating function is a polynomial) and we find that

$$S(n) = (-1)^n \frac{(2n)!}{n!} \operatorname*{\mathbf{res}}_u \; \left(1 - \frac{\log(1-u)}{u} \right)^n u^{-n-1}.$$

Plugging in the series representation of the logarithm and after some simplification we find

$$\begin{aligned}(-1)^n \frac{(2n)!}{n!} \underset{u}{\mathbf{res}} \left(1 - \frac{\log(1-u)}{u}\right)^n u^{-n-1} &= (-1)^n \frac{(2n)!}{n!} \underset{u}{\mathbf{res}} \left(-\frac{u}{2} - \frac{u^2}{3} - \frac{u^3}{4} - \dots\right)^n u^{-n-1} \\ &= \frac{(2n)!}{n!} \underset{u}{\mathbf{res}}\ u^{-1} \left(\frac{1}{2} + \frac{u}{3} + \frac{u^2}{4} + \dots\right)^n \\ &= \frac{(2n)!}{n!} \langle u^0 \rangle \left(\frac{1}{2} + \frac{u}{3} + \frac{u^2}{4} + \dots\right)^n,\end{aligned}$$

which gives

$$S(n) = \frac{(2n)!}{n!2^n}. \tag{4.20}$$

Remark: A fraction free representation would be over the double factorial notion. In particular we define that for $n \in \mathbb{N}$:

$$n!! := \begin{cases} n \cdot (n-2) \dots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd}, \\ n \cdot (n-2) \dots 6 \cdot 4 \cdot 2 & n > 0 \text{ even}, \\ 1 & n = -1, 0. \end{cases} \tag{4.21}$$

Lemma 4.3.1 *For $n \in \mathbb{N}$:*

$$\frac{(2n)!}{n!2^n} = (2n-1)!!. \tag{4.22}$$

Proof. Both sides satisfy the recurrence

$$f(n+1) - (2n+1)f(n) = 0, \quad n \geq 0.$$

Initial values match. □

Summarizing we have proven

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} S_1(n+k, k) = \frac{(2n)!}{n!2^n} = (2n-1)!!. \tag{4.23}$$

4.4 Symbolic Sums involving C-finite sequences

In this section we will simplify the sum

$$F(n) := \sum_{k=0}^{n} \binom{n}{k} F_k, \quad n \geq 0. \tag{4.24}$$

where $(F_k)_{k\geq 0}$ denotes the sequence of Fibonacci numbers defined by the recurrence

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_n &= F_{n-1} + F_{n-2}, \quad n \geq 2. \end{aligned}$$

Afterwards we will extend this example to work for arbitrary C-finite sequences. (A precise definition will follow.)

As we have seen in the example involving Pascal's triangle, the binomial coefficients can be represented by the Riordan array

$$M = (m_{nk})_{n,k\geq 0} = \left(\binom{n}{k}\right)_{n,k\geq 0} = \mathcal{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right),$$

Furthermore by Lemma 3.1.1 we have for any sequence $A = (a_0, a_1, a_2, ...)$ with generating function $A(x)$ that

$$\mathcal{R}(g(x), f(x))A(x) = g(x)A(xf(x)),$$

which in case of the binomial coefficient gives

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \langle x^n \rangle \, \frac{1}{1-x} A\left(\frac{x}{1-x}\right), \tag{4.25}$$

which is known as Euler's transform. We recall

Lemma 4.4.1 *The generating function of the sequence of Fibonacci numbers is given by*

$$A(x) = \sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}.$$

Then, using Lemma 4.4.1 together with the insertion homomorphism $\Phi_{x/(1-x)}$ 2.2.1, one obtains

Corollary 4.4.1 *The value of*

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} F_k$$

is given by

$$F(n) = \langle x^n \rangle \, \frac{1}{1-x} A\left(\frac{x}{1-x}\right) = \langle x^n \rangle \, \frac{x}{1-3x+x^2}.$$

One might find this answer not very satisfactory since the extraction of a certain coefficient might be a cumbersome task. But in the case where we have to extract a coefficient from a rational function we have the following result

Theorem 4.4.1 ([GKP94], p. 340, Rational Expansion Theorem)
Let $R(z) \in \mathbb{K}[[z]]$ *with*

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z), Q(z) \in \mathbb{K}[z]$ *such that* $Q(z) = q_0(1-\rho_1 z)\dots(1-\rho_l z), \rho_1, ...\rho_l \in \mathbb{K}$ *pairwise distinct, and* $\deg(P(z)) < l$*. Then*

$$\langle z^n \rangle \ R(z) = a_1\rho_1^n + \cdots + a_l\rho_l^n, \quad n \geq 0,$$

where

$$a_k = -\rho_k \frac{P(1/\rho_k)}{(D_x Q)(1/\rho_k)}, \quad 1 \leq k \leq l. \tag{4.26}$$

Example 4.4.1 (Contd.) *Let* $P(x) = x, Q(x) = 1 - 3x + x^2$*. From Corollary 4.4.1 we know that*

$$\sum_{k=0}^{n} \binom{n}{k} F_k = \langle x^n \rangle \ \frac{P(x)}{Q(x)}, \quad n \geq 0.$$

Following Thm. 4.4.1, we have

$$Q(x) = 1 - 3x + x^2 = \left(1 - \frac{2}{3-\sqrt{5}}x\right)\left(1 - \frac{2}{3+\sqrt{5}}x\right),$$

so we set $\rho_1 = \frac{2}{3-\sqrt{5}}$ *and* $\rho_2 = \frac{2}{3+\sqrt{5}}$*, and we calculate*

$$\begin{aligned} a_1 &= -\rho_1 \frac{P(1/\rho_1)}{(D_x Q)(1/\rho_1)} = \frac{2}{-3+\sqrt{5}} \frac{P\left(\frac{3-\sqrt{5}}{2}\right)}{(D_x Q)\left(\frac{3-\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}}, \\ a_2 &= -\rho_2 \frac{P(1/\rho_2)}{(D_x Q)(1/\rho_2)} = \frac{-2}{3+\sqrt{5}} \frac{P\left(\frac{3+\sqrt{5}}{2}\right)}{(D_x Q)\left(\frac{3+\sqrt{5}}{2}\right)} = -\frac{1}{\sqrt{5}}, \end{aligned}$$

and we finally find

$$\sum_{k=0}^{n} \binom{n}{k} F_k = \langle x^n \rangle \ \frac{x}{1-3x+x^2} = \frac{1}{\sqrt{5}}\left(\left(\frac{2}{3-\sqrt{5}}\right)^n - \left(\frac{2}{3+\sqrt{5}}\right)^n\right), \quad n \geq 0.$$

Proof. [Proof of Thm. 4.4.1, see [GKP94]]

Let $a_1, ..., a_l$ be as defined in (4.26). Formula (4.26) holds if $R(z) = P(z)/Q(z)$ is equal to

$$S(z) = \frac{a_1}{1-\rho_1 z} + \cdots + \frac{a_l}{1-\rho_l z}.$$

And we can prove that $R(z) = S(z)$ by showing that the function $T(z) = R(z) - S(z)$ is not infinite as $z \to 1/\rho_k$ for all $k \in \{1, \dots, l\}$. For this will show that the rational function $T(z)$ is never infinite; hence $T(z)$ must be a polynomial. We also can show that $T(z) \to 0$ as $z \to \infty$; hence $T(z)$ must be zero.

Let $\alpha_k = 1/\rho_k$. To prove that

$$\lim_{z \to \alpha_k} T(z) \neq \infty,$$

it suffices to show that $\lim_{z\to\alpha_k}(z-\alpha_k)T(z) = 0$, because $T(z)$ is a rational function of z. Thus we want to show that

$$\lim_{z\to\alpha_k}(z-\alpha_k)R(z) = \lim_{z\to\alpha_k}(z-\alpha_k)S(z).$$

The right-hand limit equals $\lim_{z\to\alpha_k} a_k(z-\alpha_k)/(1-\rho_k z) = -a_k/\rho_k$, because $(1-\rho_k z) = -\rho_k(z-\alpha_k)$ and $(z-\alpha_k)/(1-\rho_j z) \to 0$ for $j \neq k$. The left-hand limit is

$$\lim_{z\to\alpha_k}(z-\alpha_k)\frac{P(z)}{Q(z)} = P(\alpha_k)\lim_{z\to a_k}\frac{z-\alpha_k}{Q(z)} = \frac{P(\alpha_k)}{(D_x Q)(\alpha_k)},$$

by l'Hôspital's rule. Thus the theorem is proved.

□

In general the roots of a polynomial are not distinct. For the case that we have got a root of multiplicity > 1, we use the following theorem.

Theorem 4.4.2 ((General Version), [GKP94], p. 341)
Let $R(z) \in \mathbb{K}[[z]]$ *with*

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z), Q(z) \in \mathbb{K}[z]$*such that* $Q(z) = q_0(1-\rho_1 z)^{d_1}\dots(1-\rho_l z)^{d_l}, \rho_1, ..., \rho_l \in \mathbb{K}$ *pairwise distinct,* $d_1, ..., d_l \in \mathbb{N}\backslash\{0\}$ *and* $\deg(P(z)) < l$*. Then*

$$\langle z^n \rangle \; R(z) = f_1(n)\rho_1^n + \cdots + f_l(n)\rho_l^n, \quad n \geq 0,$$

where each $f_k(n)$ *is a polynomial of degree* $d_k - 1$ *with leading coefficient*

$$a_k = (-\rho_k)^{d_k}\frac{P(1/\rho_k)d_k}{(D_x^{d_k}Q)(1/\rho_k)} = \frac{P(1/\rho_k)}{(d_k-1)!q_0\prod_{j\neq k}(1-\rho_j/\rho_k)^{d_j}}. \tag{4.27}$$

Example 4.4.2 *Consider*

$$rat(x) := \frac{3x^4+2x^3+x^2+1}{-4+4x+7x^2-6x^3-4x^4+2x^5+x^6}$$

We compute

$$-4+4x+7x^2-6x^3-4x^4+2x^5+x^6 = -4\left(1+\frac{x}{2}\right)^2(1-x)^3(1+x),$$

and hence $q_0=-4, \rho_1=-\frac{1}{2}, d_1=2, \rho_2=1, d_2=3, \rho_3=-1, d_3=1$. *The numbers* ρ_i *are pairwise distinct,* $\deg(P(z)) = 4 < d_1+d_2+d_3 = 6$. *We compute the coefficients* a_k *by formula (4.27). By the theorem we get*

$$\langle x^n \rangle\ rat(x) = \underbrace{\left(\frac{37}{108}n + c_1\right)(-2)^{-n} + \left(-\frac{7}{36}n^2 + c_2 n + c_3\right) - \frac{3}{8}(-1)^n}_{f_n}$$

This gives us an Ansatz for the general shape of our expression. If we now compute the first 3 *values (corresponding to* 3 *unknowns) of the Taylor expansion we get*

$$\frac{3x^4+2x^3+x^2+1}{-4+4x+7x^2-6x^3-4x^4+2x^5+x^6} = -\frac{1}{4} - \frac{1}{4}x - \frac{15}{16}x^2 + \mathcal{O}(x^3),$$

and we can read off $f_0 = f_1 = -1/4$ *and* $f_2 = -15/16$. *Equipped with this additional information we can now set up a linear system of equations*

$$\begin{pmatrix} 1 & 0 & 1 \\ -1/2 & 1 & 1 \\ 1/4 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1/8 \\ -7/27 \\ 19/432 \end{pmatrix}$$

that has the unique solution $(11/36, 2/27, -13/72)^T$. *Hence, we find that*

$$\langle x^n \rangle\ rat(x) = \left(\frac{37}{108}n + \frac{11}{36}\right)(-2)^{-n} + \left(-\frac{7}{36}n^2 + \frac{2}{27}n - \frac{13}{72}\right) - \frac{3}{8}(-1)^n.$$

Example 4.4.3 (Application: non-congruent triangles)
In [APR01] the following application is described: What is the number of non-congruent triangles with prescribed perimeter $n \in \mathbb{N} \setminus \{0\}$ *and sides* a, b, c *of positive integer length? The triangles are described via the conditions on the sides*

$$\begin{aligned} n &= a+b+c, \\ 1 &\leq a \leq b \leq c, \end{aligned}$$

and the conditions that we are examining triangles, i.e. the triangle inequalities:

$$\begin{aligned} a &\geq b+c, \\ b &\geq a+c, \\ c &\geq a+b. \end{aligned}$$

If T_n denotes the number of such tuples, then it is derived in [APR01] that the generating function is given by

$$\sum_{n=0}^{\infty} T_n x^n = \frac{x^3}{(1-x^2)(1-x^3)(1-x^4)} = x^3 + x^5 + x^6 + 2x^7 + x^8 + 3x^9 + \dots \tag{4.28}$$

If we apply the theorem we get the explicit formula

$$\begin{aligned} T_k \;=\;& \frac{1}{48}\Bigg(\frac{16}{3}\Big(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big)^k - (3+3\mathrm{i})(-\mathrm{i})^k - (3-3\mathrm{i})(\mathrm{i})^k + \frac{1}{6}\Big(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big) \\ & \cdot\Bigg(8 - 8\Big(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big) + 9\Big(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big)\Bigg) + \frac{16}{3}\Big(-\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\Big)^k + \frac{1}{3}\Bigg(1+8\Big(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big) \\ & -8\Big(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big)\Bigg)k + k^2 - 3(-1)^k\Bigg(\frac{1}{18}\Bigg(19+8\Big(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big) - 8\Big(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\Big) + k\Bigg)\Bigg), \end{aligned}$$

where $\mathrm{i}^2 = -1$, *or after further simplification (i.e. rewriting complex numbers to trigonometric functions)*

$$\begin{aligned} T_k \;=\;& \frac{1}{288}\Bigg(-1 + 6k(3+k) - 36\cos\left(\frac{k\pi}{2}\right) + 64\cos\left(\frac{2k\pi}{3}\right) - 9(3+2k)\cos(k\pi) \\ & -36\sin\left(\frac{k\pi}{2}\right) - 9\mathrm{i}(3+2k)\underbrace{\sin(k\pi)}_{=0}\Bigg), \quad k \geq 0. \end{aligned}$$

Note: *The trigonometric functions are coding periodicity, e.g.,* $\cos(k\pi) = (-1)^k$.

The general case

Definition 4.4.1 (C-Finite sequence) *Let $\alpha_1, \alpha_2, ..., \alpha_d$ be elements in $\mathbb{K}$, $d \geq 1$, and $\alpha_d \neq 0$. The sequence $(f_n)_{n\geq 0}$ is C-finite if and only if*

$$f_{n+d} + \alpha_1 f_{n+d-1} + \alpha_2 f_{n+d-2} + \cdots + \alpha_d f_n = 0, \quad n \geq 0.$$

We will abbreviate the set of C-finite sequences by ***CF****, i.e.,*

$$\begin{aligned} \boldsymbol{CF} := \{(f_0, f_1, ...) \in \mathbb{K}^{\mathbb{N}} \quad | \quad & \exists\, (\alpha_1, ..., \alpha_d) \in \mathbb{K}^d,\ \alpha_d \neq 0 \\ & \forall n \geq 0 : f_{n+d} + \alpha_1 f_{n+d-1} + \cdots + \alpha_d f_n = 0\} \end{aligned}$$

The sequence of Fibonacci numbers $(F_k)_{k\geq 0}$ is $\in$ **CF** by the choice $\alpha_1 = \alpha_2 = -1$. C-finite sequences have the nice property that their generating function can be expressed a rational function. In particular we have the following theorem:

Theorem 4.4.3 ([Sta86], p. 202, [KP11], p. 74) *Let $\alpha_1, \alpha_2, ..., \alpha_d$ be elements in $\mathbb{K}$, $d \geq 1$ and $\alpha_d \neq 0$. The following conditions on a function $f : \mathbb{N} \to \mathbb{K}$ are equivalent:*

- $$\sum_{n\geq 0} f_n x^n = \frac{P(x)}{Q(x)}$$
  where $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d \in \mathbb{K}[x]$ and $P(x) \in \mathbb{K}[x]$ is a polynomial in x of degree less than d.

- *For all $n \geq 0$,*
  $$f_{n+d} + \alpha_1 f_{n+d-1} + \alpha_2 f_{n+d-2} + \cdots + \alpha_d f_n = 0.$$

- *For all $n \geq 0$,*
  $$f_n = \sum_{i=1}^{k} P_i(n)\gamma_i^n$$
  where
  $$1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d = \prod_{i=1}^{k} (1 - \gamma_i x)^{d_i}$$
  and the γ_i's are distinct elements in $\bar{\mathbb{K}}$, the algebraic closure of $\mathbb{K}$, and the $P_i(x)$ are polynomials in $\mathbb{K}[x]$ of degree less than d_i.

Proof. See [Sta86, p. 203] □

Theorem 4.4.4 *Let $(C_k)_{k\geq 0} \in$ **CF**. Then:*

$$\sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} \binom{n}{k} C_k \right)$$

is a rational function.

Proof. The binomial coefficients correspond to the Riordan array

$$D = \mathcal{R}\left(\frac{1}{1-x}, \frac{1}{1-x}\right)$$

and hence, by (4.25) we find that

$$\sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} \binom{n}{k} C_k \right) = \frac{1}{1-x} C\left(\frac{x}{1-x}\right)$$

where $C(x) \in \mathbb{K}[[x]]$ is the generating function of the sequence $(C_k)_{k\geq 0}$. Because of the assumption that $(C_k)_{k\geq 0} \in \mathbf{CF}$ we know that $C(x)$ is a rational function. Composition of $C(x)$ by $x/(1-x)$ and multiplication by $1/(1-x)$ is again a rational function. $\square$

Theorem 4.4.5 *Let $(C_k)_{k\geq 0} \in \boldsymbol{CF}$.*

- *If $a, b \in \mathbb{N}$ such that $1 \leq b \leq a+1$:*

$$\sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} \binom{m+n+ak}{m+bk} C_k \right) \in \mathbb{K}(x).$$

- *If $a \in \mathbb{N}, b \in \mathbb{Z}$ such that $-1 \leq b \leq a-1$:*

$$\sum_{m=0}^{\infty} x^m \left(\sum_{k=0}^{m} \binom{n+ak}{m+bk} C_k \right) \in \mathbb{K}(x).$$

Proof. The proof proceeds with the same steps as the proof of the previous theorem. If we take $n \in \mathbb{N}$ fixed and consider

$$\begin{aligned}
\langle x^n \rangle \; \frac{1}{(1-x)^{m+1}} \left(\frac{x^{b-a}}{(1-x)^b} \right)^k &= \langle x^n \rangle \; \frac{x^{bk-ak}}{(1-x)^{m+1+bk}} \\
&= \langle x^{n-bk+ak} \rangle \; (1-x)^{-m-1-bk} \\
&= (-1)^{n-bk+ak} \binom{-m-1-bk}{n-bk+ak} \\
&= \binom{m+n+ak}{n-bk+ak} \\
&= \binom{m+n+ak}{m+bk}, \quad m, k \geq 0.
\end{aligned}$$

Similar, for $m \in \mathbb{N}$ fixed:

$$\begin{aligned}
\langle x^m \rangle \; (1+x)^n \left(\frac{x^{-b}}{(1+x)^{-a}} \right)^k &= \langle x^m \rangle \; (1+x)^n \frac{x^{kb}}{(1+x)^{-ka}} \\
&= \langle x^{m+kb} \rangle \; (1+x)^{n+ak} \\
&= \binom{n+ak}{m+bk}, \quad n, k \geq 0.
\end{aligned}$$

This suggests to consider the Riordan arrays

$$\begin{aligned}
\left(\binom{m+n+ak}{m+bk}\right)_{n,k\geq 0} &= \mathcal{R}\left(\frac{1}{(1-x)^{m+1}}, \frac{x^{b-a-1}}{(1-x)^b}\right) \\
\left(\binom{n+ak}{m+bk}\right)_{m,k\geq 0} &= \mathcal{R}\left((1+x)^n, \frac{x^{-b-1}}{(1+x)^{-a}}\right).
\end{aligned}$$

To prove that this are indeed Riordan arrays, we convince ourselves (by standard manipulations as above), that by the assumptions on $a, b \in \mathbb{Z}$ all formal power series involved have non-zero constant term.

$$\begin{aligned}
\langle x^0\rangle \frac{1}{(1-x)^{m+1}} &= 1 \neq 0, \\
\langle x^0\rangle \frac{x^{b-a-1}}{(1-x)^b} &= \binom{a}{b-1} \neq 0, \\
\langle x^0\rangle (1+x)^n &= 1 \neq 0, \\
\langle x^0\rangle \frac{x^{-b-1}}{(1+x)^{-a}} &= \binom{a}{b+1} \neq 0.
\end{aligned}$$

We find by (3.7) for $a, b \in \mathbb{N}$ such that $1 \leq b \leq a+1$:

$$\sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} \binom{m+n+ak}{m+bk} C_k\right) = \frac{1}{(1-x)^{m+1}} C\left(\frac{x^{b-a}}{(1-x)^b}\right), \tag{4.29}$$

resp. for $a \in \mathbb{N}, b \in \mathbb{Z}$ such that $-1 \leq b \leq a-1$:

$$\sum_{m=0}^{\infty} x^m \left(\sum_{k=0}^{m} \binom{n+ak}{m+bk} C_k\right) = (1+x)^n C(x^{-b}(1+x)^a). \tag{4.30}$$

Again we take into account that composition and multiplication by rational functions keeps the sum in the field of rational functions. □

In Connection with Thm. 4.4.2 we can always compute symbolically the value of

$$\sum_{k=0}^{\{n,m\}} \binom{n+ak}{m+bk} C_k,$$

for any $(C_k)_{k\geq 0} \in \mathbf{CF}$ and an appropriate choice of $m, n \in \mathbb{N}$, in the way illustrated above.

Noteworthy is also that, by Thm. 4.4.3, the sequence

$$\left(A_{\{m,n\}}\right)_{\{m,n\}\geq 0} = \left(\sum_{k=0}^{\{m,n\}} \binom{n+ak}{m+bk} C_k\right)_{\{m,n\}\geq 0},$$

with $(C_k)_{k\geq 0} \in \mathbf{CF}$, also satisfies a C-finite recurrence.

Example 4.4.4 ([Wil06], p.162, Ex. 4.16)
If two sequences $(f_n)_{n\geq 0}$ and $(c_k)_{k\geq 0}$ are connected by the equations

$$f_n = \sum_k \binom{n+k}{m+2k} c_k, \quad n \geq 0,$$

where $m \geq 0$ is fixed, then their opsgf's are connected by

$$F(x) = \frac{x^m}{(1-x)^{m+1}} C\left(\frac{x}{(1-x)^2}\right)$$

Proof. Let

$$F(x) = \sum_{n=0}^{\infty} f_n x^n, \quad C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$\langle x^n \rangle F(x) = f_n = \sum_k \binom{n+k}{m+2k} c_k = \sum_{k=0}^{n-m} \binom{n+k}{m+2k} c_k,$$

because of the "support" of the binomial coefficient. By (4.29) we have that

$$\begin{aligned}
\langle x^n \rangle \frac{x^m}{(1-x)^{m+1}} C\left(\frac{x}{(1-x)^2}\right) &= \langle x^{n-m} \rangle \frac{1}{(1-x)^{m+1}} C\left(\frac{x^{2-1}}{(1-x)^2}\right) \\
&= \sum_{k=0}^{n-m} \binom{n+k}{m+2k} c_k.
\end{aligned}$$

Note that the sequence $(c_k)_{k\geq 0}$ does not has to be necessarily in **CF**. This restriction allows us to compute the value in a symbolical fashion (because then we have a rational generating function) but has not to be necessarily the case. □

As we have seen in this discussion the case where the sequence of coefficients satisfy a C-finite recurrence can always be fully solved. In particular we can always give an explicit formula for the general term c_k.

The author was pointed to the work of Koutschan [Kou09, Kou10] who examined applications of the holonomic systems approach. He developed the Mathematica package HolonomicFunctions which computes the annihilator of the sum $\sum_{k=0}^{n}\binom{n}{k}F_k$. We will present how to proceed in Mathematica.

```
Mathematica 7.0 - Listing

In[1]:= <<HolonomicFunctions.m
        HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.3 (25.01.2010)

In[2]:= Annihilator[Sum[Binomial[n, k] * Fibonacci[k], {k, 0, n}], S[n]]
Out[2]= {S_n^2 - 3S_n + 1}
```

Hence, if we denote $F(n) = \sum_{k=0}^{n}\binom{n}{k}F_k$ we know from this that

$$F(n+2) - 3F(n+1) + F(n) = 0$$

holds for $n \geq 0$. If we compute initial values we can invoke Thm. 4.4.3 to compute a closed form for the sum in question.

4.5 An explicit formula for Stirling numbers

From the combinatorial interpretation of Stirling numbers of the second kind, one find a recurrence relation for these numbers. The Stirling numbers of the second kind satisfy the recurrence [Wil06, p. 17]

$$S_2(n,k) \;=\; S_2(n-1,k-1) + kS_2(n-1,k). \tag{4.31}$$

Proof. Take a set with n elements and highlight one particular element, say for instance the last one. To obtain a partition of the n element set into k blocks we can partition the $n-1$ element set (where we excluded our highlighted element) into k blocks and place the last element into any of these blocks in $kS_2(n-1,k)$ ways, or we can put the last element in a block by itself and partition the $n-1$ element set into $k-1$ blocks in $S_2(n-1,k-1)$ ways. So the total number of ways is given by (4.31)

□

This recurrence is valid in $\mathbb{Z}^2$ with the exceptional point $(n,k) \neq (0,0)$ where we have that $S_2(0,0) = 1$. (Note that $S_2(n,k) := 0$ if $n \cdot k < 0$) Following the derivation in [Wil06] we define the generating function

$$f_k(x) = \sum_{n=0}^{\infty} S_2(n,k)x^n, \tag{4.32}$$

and find by the defining recurrence immediately that

$$f_k(x) = xf_{k-1}(x) + kxf_k(x), \quad k \geq 1, f_0(x) = 1, \tag{4.33}$$

leading us to the recurrence

$$f_k(x) = \frac{x}{1-kx} f_{k-1}(x), \quad k \geq 1, f_0(x) = 1,$$

and finally the evaluation

$$f_k(x) = \sum_{n=0}^{\infty} S_2(n,k)x^n = \frac{x^k}{(1-x)(1-2x)(1-3x)\dots(1-kx)}. \tag{4.34}$$

Wilf now performs partial fraction decomposition to extract the coefficient of x^n (which can be compared to what we did in an earlier section, but is a little more involved because we deal with a symbolic parameter k rather than a concrete number). After some lengthy computation he comes up with the representation

$$S_2(n,k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}, \quad n,k \geq 0. \tag{4.35}$$

Special cases of these formulas are of interest. Namely if we set $k = 2$ we calculate the sum directly (in fact, the summation quantifier would be an overkill :-)) we get the nice formula

$$\sum_{r=1}^{2} (-1)^{2-r} \frac{r^n}{r!(2-r)!} = -1 + \frac{2^n}{2!0!} = 2^{n-1} - 1.$$

About this we could have reasoned combinatorially, because if a set of $n > 0$ elements is divided into two nonempty subsets one subset contains the last element and the other subset the first $n-1$ objects. There are 2^{n-1} ways to choose the subset because the $n-1$ objects are either inside the subset or not. But we mustn't put all of those objects in it, because we want to end up with two nonempty sets. Therefore we subtract 1

$$S_2(n,2) = 2^{n-1} - 1, \quad n \in \mathbb{N} : n > 0.$$

In the introductory section we learned that the Stirling numbers of the second kind are subject to the residue representation

$$S_2(n,k) = \frac{n!}{k!} \operatorname*{\mathbf{res}}_{w} \quad (e^w - 1)^k w^{-n-1}.$$

If we look at the sum representation we find that we can extend the summation interval because r^n will not contribute if r is set to 0 or if $r > k$. So let us manipulate the sum by some extensions

$$\begin{aligned}\sum_{r=1}^{k}(-1)^{k-r}\frac{r^n}{r!(k-r)!} &= \frac{1}{k!}\sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r}r^n \\ &= \frac{n!}{k!}\sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r}\frac{r^n}{n!},\end{aligned}$$

Now we can replace the exponential factor by its residue representation involving the exponential function and pull out factors not depending on the summation index

$$\begin{aligned}\frac{n!}{k!}\sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r}\frac{r^n}{n!} &= \frac{n!}{k!}\sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r} \operatorname*{\mathbf{res}}_{u} \quad e^{ru}u^{-n-1} \\ &= \frac{n!}{k!} \operatorname*{\mathbf{res}}_{u} \quad u^{-n-1} \sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r}\ e^{ru}.\end{aligned}$$

The remaining sum can be simplified with the help of the binomial theorem giving us the residue representation

$$\frac{n!}{k!} \operatorname*{\mathbf{res}}_{u} \quad u^{-n-1} \sum_{r=0}^{\infty}\binom{k}{r}(-1)^{k-r}\ e^{ru} = \frac{n!}{k!} \operatorname*{\mathbf{res}}_{u} \quad (e^u - 1)^k u^{-n-1}.$$

Hence, we have proven that Wilf's sum has the same residue representation as the Stirling numbers of the second kind and therefore its correctness (without the use of the partial fraction decomposition or the rational expansion theorem).

4.6 Further non-hypergeometric examples

The sequence of Bernoulli Nos. $(B_k)_{k\geq 0}$ causes problem's for computing combinatorial sums. Similar the Stirling Nos. of both kinds are not nice in the sense defined at the beginning of this chapter . Therefore we need other methods for computing sums involving this kind of numbers.

Progress in this direction was made by Kauers [Kau07], who provided an algorithm for computing sums where

$$\text{summand}(n, m, k) = \text{hyp}(n, m, k) \cdot S(an + bk, cn + dk),$$

where $S(n, k)$ are either Stirling numbers of the first or second kind, or Eulerian numbers of the first or second kind, $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = \pm 1$ and $\text{hyp}(n, m, k)$ is a proper hypergeometric term (i.e. a product of binomials, factorials, exponentials and polynomials).

The essential idea is that one considers bivariate operators of the form

$$\sum_{i,j \in \mathbb{Z}} p_{i,j}(n, k) N^i K^j,$$

with $p_{i,j} \in C(n, k)$, C a field of characteristic zero. These operators act in the following way on sequences $f : \mathbb{Z}^2 \to C$

$$\left(\sum_{i,j \in \mathbb{Z}} p_{i,j}(n, k) N^i K^j \right) \cdot f(n, k) = \sum_{i,j \in \mathbb{Z}} p_{i,j}(n, k) f(n + i, k + j), \quad n, k \in \mathbb{Z}.$$

The set of this operators is denoted by $C(n, k)\langle N, K\rangle$. What is essential is that for a given bivariate sequence $f : \mathbb{Z}^2 \to C$ the set

$$\{ \ Q \in C(n, k)\langle N, K\rangle : Q \cdot f \equiv 0 \ \},$$

forms a left ideal (called the annihilator ideal) of the ring $C(n, k)\langle N, K\rangle$. If one now considers an ideal $\mathfrak{a} \trianglelefteq C(n, k)\langle N, K\rangle$ one can show that under certain assumptions that $\mathfrak{a} \cap C(n, m_1, ..., m_r)\langle N, K, M_1, ..., M_r\rangle \neq \{0\}$ or in other words the summand satisfies a nontrivial recurrence relation whose coefficients are free of k. With the help of this one is able to solve definite and indefinite summation problems. We do not want to go into details here, but refer the interested reader to the work of Kauers.

With the help of the developed package Kauers was able to prove most identities arising in [GKP94, p. 265, Table 265] with exception of (6.28) and (6.29) that are not of the desired form. The Egorychev approach is able to derive this identities by the use of the substitution rule.

Example 4.6.1 ([GKP94], p. 265, Equ. (6.28))
We prove the identity

$$\sum_{k=0}^{n} S_2(k,l)S_2(n-k,m)\binom{n}{k} = \binom{l+m}{l}S_2(n,l+m), \tag{4.36}$$

that involves the non-hypergeometric Stirling numbers of second kind. As in a preceding example we expand the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ *and pull out the terms not depending on k. Further we extend the summation interval and substitute the Stirling numbers according to Table 2.1.*

$$\begin{aligned}
&= n! \sum_{k=0}^{n} S_2(k,l)S_2(n-k,m)\frac{1}{k!(n-k)!} \\
&= n! \sum_{k=0}^{\infty} \frac{k!}{l!} \mathop{\boldsymbol{res}}_{w} \left((e^w-1)^l w^{-k-1}\right) \frac{(n-k)!}{m!} \mathop{\boldsymbol{res}}_{u} \left((e^u-1)^m u^{-n+k-1}\right) \frac{1}{k!(n-k)!} \\
&= \frac{n!}{l!m!} \mathop{\boldsymbol{res}}_{u} \left((e^u-1)^m u^{-n-1}\right) \sum_{k=0}^{\infty} u^k \mathop{\boldsymbol{res}}_{w} \left((e^w-1)^l w^{-k-1}\right).
\end{aligned}$$

Now we can apply the substitution rule and expand by $(l+m)!$

$$\begin{aligned}
&= \frac{n!}{l!m!} \mathop{\boldsymbol{res}}_{u} \left((e^u-1)^m u^{-n-1}\right) \underbrace{\sum_{k=0}^{\infty} u^k \mathop{\boldsymbol{res}}_{w} \left((e^w-1)^l w^{-k-1}\right)}_{=(e^u-1)^l} \\
&= \frac{n!}{l!m!}\frac{(l+m)!}{(l+m)!} \mathop{\boldsymbol{res}}_{u} (e^u-1)^{l+m} u^{-n-1} \\
&= \binom{l+m}{l}\frac{n!}{(l+m)!} \mathop{\boldsymbol{res}}_{u} (e^u-1)^{l+m} u^{-n-1} = \binom{l+m}{l} S_2(n,l+m).
\end{aligned}$$

[GKP94, p. 265, eq. (6.29)] is essentially the same but involves (signless) Stirling numbers of the first kind instead the second kind. One could take this derivation and replace $e^{\{u,w\}} - 1$ by $-\log(1-\{u,w\})$.

Exercise 2.4.9 (f) in [Ego84, p. 85] asks for the proof of a similar identity, namely

$$\sum_{k=m-r}^{n-r} \binom{n}{k} S_2(n-k,r)S_2(k,m-r) = \binom{m}{r} S_2(n,m), \quad r \le m \le n. \tag{4.37}$$

The calculation is straightforward and we do not want to present it in full detail. But the author wants to point out that with the help of the Egorychev method one might gets a handle on a more general class than what Kauers calls a proper Stirling-like term.

For handling Stirling Number identities, it is pointed out in [Spr94] that the Riordan array approach is also applicable.

Lemma 4.6.1 *For $k \in \mathbb{N}$ fixed:*

$$\begin{aligned} (-\log(1-x))^k &= k! \sum_{n=0}^{\infty} S_1(n,k) \frac{x^n}{n!}, &(4.38)\\ (e^x-1)^k &= k! \sum_{n=0}^{\infty} S_2(n,k) \frac{x^n}{n!}. &(4.39)\end{aligned}$$

Let us consider the Riordan array

$$M := (m_{n,k}), \quad \text{where} \quad m_{n,k} := S_1(n,k)\frac{k!}{n!}.$$

Then, the Riordan matrix M will look like $M = (M^{(0)}, M^{(1)}, M^{(2)}, ...)$, where

$$M^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \text{first column of } M, \tag{4.40}$$

and

$$M^{(k)} = \begin{pmatrix} S_1(0,k)\frac{k!}{0!} \\ S_1(1,k)\frac{k!}{1!} \\ S_1(2,k)\frac{k!}{2!} \\ \vdots \end{pmatrix} = k\text{th column of } M, \tag{4.41}$$

By Lemma 4.6.1, the generating function of the kth column is given by $(-\log(1-x))^k$. We can reason similar for the Stirling numbers of the second kind, and hence we have found the Riordan arrays

$$\begin{aligned} \left(S_1(n,k)\frac{k!}{n!}\right)_{n,k\geq 0} &= \mathcal{R}\left(1, -\frac{1}{x}\log(1-x)\right), &(4.42)\\ \left(S_2(n,k)\frac{k!}{n!}\right)_{n,k\geq 0} &= \mathcal{R}\left(1, \frac{1}{x}(e^x-1)\right). &(4.43)\end{aligned}$$

To see how easy this will work with Riordan arrays we will illustratively show an example.

Example 4.6.2 ([Spr94])

$$\sum_{k=0}^{n} S_2(n,k)k!\frac{(-1)^k}{k+1} = B_n \tag{4.44}$$

We start by rewriting the original summand as follows:

$$\sum_{k=0}^{n} S_2(n,k)k!\frac{(-1)^k}{k+1} = n! \sum_{k=0}^{n} S_2(n,k)\frac{k!}{n!}\frac{(-1)^k}{k+1} \tag{4.45}$$

As noted above, we can identify by the Stirling numbers of the second kind by the Riordan array

$$\left(S_2(n,k)\frac{k!}{n!}\right)_{n,k\geq 0} = \mathcal{R}(g(x), f(x)) = \mathcal{R}\left(1, \frac{1}{x}(e^x - 1)\right).$$

The sequence $\big((-1)^k/(k+1)\big)_{k\geq 0}$ *has the generating function*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}x^k = \frac{1}{x}\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}x^{k+1} = \frac{1}{x}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}x^k = \frac{1}{x}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k = \frac{\log(1+x)}{x}. \tag{4.46}$$

Hence, by Lemma 3.1.1 and (2.49) we find that:

$$B(x) = g(x)A(xf(x)) = \frac{\log(1 + (e^x - 1))}{e^x - 1} = \frac{x}{e^x - 1}$$

and therefore

$$n! \sum_{k=0}^{n} S_2(n,k)\frac{k!}{n!}\frac{(-1)^k}{k+1} = n!\ \langle x^n \rangle \frac{x}{e^x - 1} = B_n.$$

where the last step is a consequence from (2.53).

4.7 Symbolic Sums involving holonomic sequences

As we have seen in the last section, we sometimes might have to extract the coefficient of x^n of a generating function that is not rational. In particular in examples involving Stirling numbers we might get generating functions that contain logarithmic resp. exponential factors. In this case we know by Thm. 4.4.3 that the coefficients will not satisfy a C-finite recurrence. We start again by a concrete example and generalize afterwards.

Example 4.7.1 *Suppose we want to extract*

$$\langle x^n \rangle \; f(x) := \frac{x - \log(1+x) + x\log(1+x)}{x^2}.$$

Again we take Mallinger's package in action to obtain information about the behaviour of the sequence. For this purpose we start by computing the first 10 *values of the Taylor expansion of the series.*

Mathematica 7.0 - *Listing*

In[1]:= $\textbf{CoefficientList}\left[\textbf{Series}\left[\frac{\boldsymbol{x} - \textbf{Log}[\textbf{1}+\boldsymbol{x}] + \boldsymbol{x}\textbf{Log}[\textbf{1}+\boldsymbol{x}]}{\boldsymbol{x}^{\textbf{2}}}, \{\boldsymbol{x}, \textbf{0}, \textbf{15}\}\right], \boldsymbol{x}\right]$

Out[1]= $\left\{\frac{3}{2}, -\frac{5}{6}, \frac{7}{12}, -\frac{9}{20}, \frac{11}{30}, -\frac{13}{42}, \frac{15}{56}, -\frac{17}{72}, \frac{19}{90}, -\frac{21}{110}, \frac{23}{132}\right\}$

No obvious pattern is visible from the coefficient list. We have to work a little more to get a closed form for general f_n. If we try Mallinger's procedures we might obtain a recurrence relation with polynomial coefficients:

Mathematica 7.0 - *Listing*

In[2]:= $\textbf{GuessRE}\left[\left\{\frac{\textbf{3}}{\textbf{2}}, -\frac{\textbf{5}}{\textbf{6}}, \frac{\textbf{7}}{\textbf{12}}, -\frac{\textbf{9}}{\textbf{20}}, \frac{\textbf{11}}{\textbf{30}}, -\frac{\textbf{13}}{\textbf{42}}, \frac{\textbf{15}}{\textbf{56}}, -\frac{\textbf{17}}{\textbf{72}}, \frac{\textbf{19}}{\textbf{90}}, -\frac{\textbf{21}}{\textbf{110}}, \frac{\textbf{23}}{\textbf{132}}\right\}, \boldsymbol{a}\,[\boldsymbol{n}]\right]$

Out[2]= $\left\{\left\{(-1-n)a[n] + a[1+n] + (4+n)a[2+n] == 0, a[0] == \frac{3}{2}, a[1] == -\frac{5}{6}\right\}, \text{"}ogf\text{"}\right\}$

The initial values are actually no surprise. But the author of this work guesses from the list of coefficients no person would have come up with this recurrence relation.

How to solve this recurrence equation? There is a built in Mathematica function that can handle C-finite recurrences and also recurrences of this type. Indeed, if one tries

Mathematica 7.0 - *Listing*

In[3]:= $\textbf{RSolve}\left[\left\{(\textbf{-1}-\boldsymbol{n})\boldsymbol{a}[\boldsymbol{n}] + \boldsymbol{a}[\textbf{1}+\boldsymbol{n}] + (\textbf{4}+\boldsymbol{n})\boldsymbol{a}[\textbf{2}+\boldsymbol{n}] == \textbf{0}, \boldsymbol{a}[\textbf{0}] == \frac{\textbf{3}}{\textbf{2}}, \boldsymbol{a}[\textbf{1}] == -\frac{\textbf{5}}{\textbf{6}}\right\}, \boldsymbol{a}[\boldsymbol{n}], \boldsymbol{n}\right]$

Out[3]= $\left\{\left\{a[n] \to \frac{(-1)^n(3+2n)}{(n^2+3n+2)}\right\}\right\}$

one gets already a closed form for the general term. An alternative is to apply Petkovšek's algorithm Hyper (see [Pet98] and [PWZ96]) for finding hypergeometric solutions to such equations (we will come back to this).

The general case
In general it will be the case that the order of the guessed recurrence for the coefficient sequence will be higher than 2. Let us develop the theory for higher order equations.

Definition 4.7.1 (See [Mal96], p. 10, Def. 1.3.1)
A sequence $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is holonomic (or P-recursive) if and only if $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients (holonomic recurrence equation), i.e., there are polynomials $p_0, p_1, ..., p_d \in \mathbb{K}[x], p_d \neq 0$, such that for all $n \in \mathbb{N}$:

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0 \tag{4.47}$$

We will call d the order and $\max(\deg(p_0(n), ..., p_d(n)))$ the degree of the recurrence.

*Similar as before we denote the set of holonomic sequences by **PF** standing for P-finite.*

As it turns out such recurrences does not only appear here but also in cases of definite and indefinite hypergeometric summation (as in Zeilberger's creative telescoping respectively in Gosper's algorithm). Therefore mathematicians started to investigate the problem of finding solutions to this kind of equations. We distinguish 3 kind of solutions.

Definition 4.7.2 *A sequence $(a_n)_{n\geq 0}$ will be called*

- *polynomial over $\mathbb{K}$ if there is a polynomial $f(x) \in \mathbb{K}[x]$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$ large enough*
- *rational over $\mathbb{K}$ if there is a rational function $f(x) \in \mathbb{K}(x)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$ large enough*
- *a hypergeometric term over $\mathbb{K}$ if there is a rational function $r(x) \in \mathbb{K}(x)$ such that $a_{n+1} = r(n)a_n$ for all $n \in \mathbb{N}$ large enough*

Petkovšek [Pet98] provides an algorithm not only for finding polynomial solutions but also for computing hypergeometric solutions. The algorithm is inspired by Gosper's algorithm [Gos78] that relies on the fact that any rational function $r(x) \in \mathbb{K}(x)$ can be represented as

$$r(x) = Z\frac{A(x)}{B(x)}\frac{C(x+1)}{C(x)}$$

where $\gcd(A(x), B(x+k)) = 1$ for every non-negative integer k. With the help of this he was able to formulate an algorithm which takes

Input: Polynomials $p_i(n)$ over $\mathbb{K}$ for $i = 0, 1, ..., d$; an extension field $\mathbb{F}$ of $\mathbb{K}$

and produces

Output: A hypergeometric solution $(a_n)_{n\geq 0}$ of

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0$$

over $\mathbb{F}$ if it exists.

Example 4.7.2 *Let us try Petkovšek's algorithm to solve the recurrence of Example 4.7.1. Hyper is implemented in his Mathematica package*[6] *Hyper.m:*

Mathematica 7.0 - *Listing*

```
In[1]:= <<Hyper.m
In[2]:= Hyper[(-1 - n)a[n] + a[1 + n] + (4 + n)a[2 + n] == 0, a[n], Solutions → All]
```

Out[2]= $\left\{\frac{1+n}{3+n}, -\frac{(1+n)(5+2n)}{(3+n)(3+2n)}\right\}$

The two results obtained are actually not the bases for the sequences a_n but the shift quotient

$$\frac{y_{n+1}}{y_n}.$$

But this order 1 recurrence can now be solved by unfolding the recurrence equation

$$\frac{y_{n+1}^{(1)}}{y_n^{(1)}} = \frac{1+n}{3+n} \Rightarrow y_n^{(1)} = \frac{c_1}{2+3n+n^2},$$

and analogously

$$\frac{y_{n+1}^{(2)}}{y_n^{(2)}} = -\frac{(1+n)(5+2n)}{(3+n)(3+2n)} \Rightarrow y_n^{(2)} = \frac{2(-1)^n(3+2n)c_2}{3(1+n)(2+n)}.$$

Taking initial values into account we find that $c_1 = 0$ and $c_2 = \frac{3}{2}$ giving us the same result as the Mathematica procedure

$$a_n = \frac{(-1)^n(3+2n)}{2+3n+n^2}.$$

[6] available online at `http://vega.fmf.uni-lj.si/~petkovsek/distrib.m`, accessed 07.06.2010

4.8 Symbolic Sums involving trigonometric functions

It is pointed out in Egorychev's work, that one is able to compute symbolic sums involving trigonometric terms. In fact, there is no magic behind it, because of Euler's identity

$$e^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x). \tag{4.48}$$

Manipulation of this formula give the well known representations of sine and cosine

$$\begin{aligned} \cos(x) &= \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}, && (4.49)\\ \sin(x) &= \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2}. && (4.50) \end{aligned}$$

Another way of viewing trigonometric functions is to look to the real part (resp. imaginary part) of Euler's exponential function. The idea is to replace any appearance of sine and cosine by

$$\cos(x) = \Re(e^{\mathrm{i}x}), \qquad \sin(x) = \Im(e^{\mathrm{i}x}).$$

With the help of this substitutions many identities can be traced back to the binomial theorem. However, in general this will not suffice to compute the sums of interest. For the identities in [Ego84, par. 2.4.6] additional knowledge on the trigonometric functions is necessary. As a reminder we state here without proof summary records and the trigonometric Pythagorean theorem

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \sin(y)\cos(x), && (4.51)\\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y), && (4.52)\\ \sin(x)^2 + \cos(x)^2 &= 1. && (4.53) \end{aligned}$$

Further we remark, that not only the Egorychev method is able to handle this kinds of sums, but also by the Riordan array approach as pointed out in [Spr07] (in fact the calculation was inspired by [Spr07]). We present two ways to prove the identity[7]

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos(kx) = (-2)^n \cos\left(\frac{n(x+\pi)}{2}\right) \left(\sin\left(\frac{x}{2}\right)\right)^n. \tag{4.54}$$

[7] This example appears as exercise 2.4.6 (b) in [Ego84, p. 81]

4.8.1 Applying the Egorychev Method

As described at the beginning of this section, we replace the disturbing $\cos(kx)$ term by the complex exponential function

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cos(kx) &= \sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\cos(kx)\\ &= \sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\Re(e^{\mathrm{i}kx})\\ &= \Re\left(\sum_{k=0}^{\infty}\binom{n}{k}(-e^{\mathrm{i}x})^k\right).\end{aligned}$$

The remaining sum can now be simplified with the help of the binomial theorem

$$\begin{aligned}\Re\left(\sum_{k=0}^{\infty}\binom{n}{k}(-e^{\mathrm{i}x})^k\right) &= \Re\left(1-e^{\mathrm{i}x}\right)^n\\ &= \Re((1-\cos(x)-\mathrm{i}\sin(x))^n).\end{aligned}$$

But how to extract the real part now? Here we need additional knowledge as stated. Let us rewrite the 1 and expand the sine and cosine as described

$$\begin{aligned}\Re\left((1-\cos(x)-\mathrm{i}\sin(x))^n\right) &= \Re\left((\underbrace{\sin\left(\frac{x}{2}\right)^2+\cos\left(\frac{x}{2}\right)^2}_{=1}-(\underbrace{\cos\left(\frac{x}{2}\right)^2-\sin\left(\frac{x}{2}\right)^2}_{=\cos(x)})-\mathrm{i}\sin(x))^n\right)\\ &= \Re\left((2\sin\left(\frac{x}{2}\right)^2-\mathrm{i}\cdot\underbrace{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}_{=\sin(x)})^n\right)\\ &= \Re\left(2^n\sin\left(\frac{x}{2}\right)^n(\sin\left(\frac{x}{2}\right)-\mathrm{i}\cos\left(\frac{x}{2}\right))^n\right).\end{aligned}$$

Now we take into account the theorem of Moivre stating that

$$(\cos(x)+\mathrm{i}\sin(x))^n=\cos(nx)+\mathrm{i}\sin(nx),\quad n\in\mathbb{Z}. \tag{4.55}$$

But first we need some preprocessing because the imaginary unit i appears at the cosine. This can be done by some special cases of the summary records. Namely set $y=\frac{\pi}{2}$ in (4.51) to obtain

$$\sin\left(x+\frac{\pi}{2}\right)=\sin(x)\cos\left(\frac{\pi}{2}\right)+\cos(x)\sin\left(\frac{\pi}{2}\right)=\cos(x), \tag{4.56}$$

and similar

$$\cos\left(x+\frac{\pi}{2}\right) = \cos(x)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right) = -\sin(x)\,. \tag{4.57}$$

Equipped with this, we rewrite our problem to

$$\Re\left(2^n \sin\left(\frac{x}{2}\right)^n (\sin\left(\frac{x}{2}\right) - \mathrm{i}\cos\left(\frac{x}{2}\right))^n\right) \;=\; \Re\left(2^n \sin\left(\frac{x}{2}\right)^n (-\cos\left(\frac{x}{2}+\frac{\pi}{2}\right) - \mathrm{i}\sin\left(\frac{x}{2}+\frac{\pi}{2}\right))^n\right)$$

$$= \Re\left((-2)^n \sin\left(\frac{x}{2}\right)^n \left(\cos\left(\frac{n(\pi+x)}{2}\right) + \mathrm{i}\sin\left(\frac{n(\pi+x)}{2}\right)\right)\right).$$

From this we can read off the real part easily and hence, we have proven our desired sum.

4.8.2 Applying the Riordan Array paradigm

As we have shown earlier, the Riordan array paradigm recognizes this formula as a special case of Euler's transformation rule (4.25). Namely if we set

$$a_k = \left(-e^{\mathrm{i}x}\right)^k,$$

we find the generating function (the geometric series)

$$A(t) = \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} (-te^{\mathrm{i}x})^k = \frac{1}{1+te^{\mathrm{i}x}}.$$

Now Euler's transformation rule tells us that

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \langle t^n \rangle \; \frac{1}{1-t} A\left(\frac{t}{1-t}\right),$$

that gives in our case

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos(kx) = \Re\left(\sum_{k=0}^{n} \binom{n}{k} (-e^{\mathrm{i}x})\right)$$

$$= \Re\left(\langle t^n \rangle \; \frac{1}{1-t} \cdot \frac{1}{1+e^{\mathrm{i}x}\frac{t}{1-t}}\right) = \Re\left(\langle t^n \rangle \; \frac{1}{1-t(1-e^{\mathrm{i}x})}\right) = \Re((1-e^{\mathrm{i}x})^n).$$

Now we can reason exact the same way as we did in our Egorychev style solution to obtain the identity.

Remark: Whenever we are able to compute our sum this way we get actually two identities. Namely, by considering the imaginary part of our equation we get the identity with the sine replaced by the cosine for free. In particular we have proven two identities

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cos(kx) = (-2)^n\cos\left(\frac{n(x+\pi)}{2}\right)\left(\sin\left(\frac{x}{2}\right)\right)^n, \quad n\geq 0, \tag{4.58}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sin(kx) = (-2)^n\sin\left(\frac{n(x+\pi)}{2}\right)\left(\sin\left(\frac{x}{2}\right)\right)^n, \quad n\geq 0. \tag{4.59}$$

4.9 An Identity for Jacobi polynomials $\mathbf{P}_n^{(\alpha,\beta)}(x)$

Exercise 4.15 in [Wil06] asks the reader to derive a closed form for the generating function of the Jacobi polynomials. In the following we will show how to make use of the Egorychev and of the Snake Oil method to derive a simple closed form.

The Jacobi polynomials are solutions to the Jacobi differential equation [AAR99, p. 297, (6.3.9)]

$$(1-x^2)y''(x)+[\beta-\alpha-(\alpha+\beta+2)x]y'(x)+n(n+\alpha+\beta+1)y(x)=0. \tag{4.60}$$

The first few polynomials are given by

$$\begin{aligned}
P_0^{(\alpha,\beta)}(x) &= 1\\
P_1^{(\alpha,\beta)}(x) &= \frac{1}{2}[2(\alpha+1)+(\alpha+\beta+2)(x-1)]\\
P_2^{(\alpha,\beta)}(x) &= \frac{1}{8}[4(\alpha+1)(\alpha+2)+4(\alpha+\beta+3)(\alpha+2)(x-1)+(\alpha+\beta+3)(\alpha+\beta+4)(x-1)^2]
\end{aligned}$$

A different definition would be over the Rodrigues formula [AAR99, p. 300, Remark 6.4.1]

$$P_n^{(\alpha,\beta)}(x)=\frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{(-2)^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right], \quad n\geq 0. \tag{4.61}$$

In the following we will give the generating function of Jacobi polynomials. We will derive that

$$\sum_n P_n^{(\alpha,\beta)}(x)\ t^{n+\alpha+\beta}=\frac{(1+x-2t)^{n+\alpha}}{2^n(x-1)^\alpha(1-t)^{n+1}}.$$

That solves Wilf's example [Wil06, Ex. 4.1.15, p. 161]. Following Wilf's proposal, we proceed in 3 steps to show this.

Example 4.9.1 ([Wil06], p. 161, Ex. 4.15)
For all $m, n, q \geq 0$, we have

$$\sum_r \binom{m}{r}\binom{n-r}{n-r-q}(t-1)^r = \sum_r \binom{m}{r}\binom{n-m}{n-r-q} t^r.$$

We will derive a residue representation for both sums in question and show that they are the same. This proves the claim. Let us first look at the sum at the left hand side.

The convention in Wilf's book is that the summation index r ranges over the integers. So the first step, extending the summation interval precipitates. In the next step we replace the binomial coefficients by their residue representations

$$\begin{aligned}\sum_r \binom{m}{r}\binom{n-r}{n-r-q}(t-1)^r &= \sum_r \operatorname*{\mathbf{res}}_w\ (1+w)^m w^{-r-1}\ \operatorname*{\mathbf{res}}_u\ (1+u)^{n-r}u^{-n+r+q-1}\ (t-1)^r \\ &= \operatorname*{\mathbf{res}}_u\ (1+u)^n u^{-n+q-1} \sum_r \left(\frac{(t-1)u}{(1+u)}\right)^r \operatorname*{\mathbf{res}}_w\ (1+w)^m w^{-r-1}.\end{aligned}$$

Now the substitution rule applies and we find that

$$\sum_r \binom{m}{r}\binom{n-r}{n-r-q}(t-1)^r = \operatorname*{\mathbf{res}}_u\ (1+u)^{n-m}(1+ut)^m u^{-n+q-1}. \tag{4.62}$$

For the right hand side start again by replacing the binomial coefficients

$$\begin{aligned}\sum_r \binom{m}{r}\binom{n-m}{n-r-q} t^r &= \sum_r \operatorname*{\mathbf{res}}_w\ (1+w)^m w^{-r-1}\ \operatorname*{\mathbf{res}}_u\ (1+u)^{n-m}u^{-n+r+q-1} t^r \\ &= \operatorname*{\mathbf{res}}_u\ (1+u)^{n-m}u^{-n+q-1} \sum_r (ut)^r\ \operatorname*{\mathbf{res}}_w\ (1+w)^m w^{-r-1}.\end{aligned}$$

Again, with the help of the substitution rule we find that

$$\sum_r \binom{m}{r}\binom{n-m}{n-r-q} t^r = \operatorname*{\mathbf{res}}_u\ (1+u)^{n-m}(1+ut)^m u^{-n+q-1}. \tag{4.63}$$

The next step in Wilf's exercise states that the *Jacobi polynomials* may be defined for $n \geq 0$, by

$$P_n^{(\alpha,\beta)}(x) = \sum_k \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k. \tag{4.64}$$

One should now use the result of part (a) to show also that

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j. \tag{4.65}$$

To see, why this equation holds, we start by considering (4.64) and pulling out constants

$$\sum_k \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k = \left(\frac{x-1}{2}\right)^n \sum_k \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x+1}{x-1}\right)^k.$$

By matching with the sum (4.63) and the assignment of the parameters

$$\begin{aligned} m &= n+\alpha, \\ r &= k, \\ n &= 2n+\alpha+\beta, \\ q &= n+\alpha+\beta, \\ t &= \frac{x+1}{x-1}, \end{aligned}$$

we find, with the help of part (a) (observe that $t-1 = \frac{2}{x-1}$), that this sum is equal to

$$\begin{aligned} & \left(\frac{x-1}{2}\right)^n \sum_k \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x+1}{x-1}\right)^k \\ = \; & \left(\frac{x-1}{2}\right)^n \sum_k \binom{n+\alpha}{k}\binom{2n+\alpha+\beta-k}{n-k}\left(\frac{2}{x-1}\right)^k \\ = \; & \sum_k \binom{n+\alpha}{k}\binom{n+\alpha+\beta+(n-k)}{n-k}\left(\frac{x-1}{2}\right)^{n-k}. \end{aligned}$$

If we now reverse the summation interval (this is replacing k by $n-k$, a bijective mapping on the summation interval) we get the desired identity

$$\begin{aligned} = \; & \sum_k \binom{n+\alpha}{n-k}\binom{n+\alpha+\beta+k}{k}\left(\frac{x-1}{2}\right)^k \\ = \; & \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j. \end{aligned}$$

Finally sub exercise (c) wants the reader to use part (b) and a dash of the Snake Oil to show that

$$P_n^{(\alpha,\beta)} = 2^{-n}(x-1)^{-\alpha}\,\langle t^{n+\alpha+\beta}\rangle\,\left\{\frac{(1+x-2t)^{n+\alpha}}{(1-t)^{n+1}}\right\}. \tag{4.66}$$

To show this, we consider the generating function for Jacobi polynomials

$$\begin{aligned}
\sum_n P_n^{(\alpha,\beta)}(x)\ t^{n+\alpha+\beta} &= \sum_n t^{n+\alpha+\beta}\left(\sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j\right) \\
&= \sum_n \left(\sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j\right) t^{n+\alpha+\beta}.
\end{aligned}$$

The next step in the Snake Oil method tells to interchange order of summation and replace the summation variable. Note that we are actually not summing on n but as we shall see we are summing on $n+\alpha+\beta+j$. Therefore we can pull out the binomial coefficient although it depends on n.

$$\begin{aligned}
&= \sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2t}\right)^j \sum_n \binom{n+\alpha+\beta+j}{j} t^{n+\alpha+\beta+j} \\
&= \sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2t}\right)^j \sum_s \binom{s}{j} t^s.
\end{aligned}$$

Hence, together with the elementary generating function

$$\sum_r \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}, \quad k \geq 0. \tag{4.67}$$

we find that

$$\sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2t}\right)^j \sum_s \binom{s}{j} t^s = \frac{1}{1-t}\sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2(1-t)}\right)^j.$$

Because this "trick" worked so well, let's try it once more. We add a factor of $\left(\frac{x-1}{2(1-t)}\right)^\alpha$ and apply afterwards the binomial theorem.

$$\begin{aligned}
\frac{1}{1-t}\sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2(1-t)}\right)^j &= \frac{1}{1-t}\left(\frac{x-1}{2(1-t)}\right)^{-\alpha}\sum_j \binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2(1-t)}\right)^{j+\alpha} \\
&= \frac{1}{1-t}\left(\frac{x-1}{2(1-t)}\right)^{-\alpha}\sum_s \binom{n+\alpha}{s}\left(\frac{x-1}{2(1-t)}\right)^{s} \\
&= \frac{1}{1-t}\left(\frac{x-1}{2(1-t)}\right)^{-\alpha}\left(1+\frac{x-1}{2(1-t)}\right)^{n+\alpha}.
\end{aligned}$$

Further simplification leads to the closed form

$$\sum_n P_n^{(\alpha,\beta)}(x)\ t^{n+\alpha+\beta} = \frac{(1+x-2t)^{n+\alpha}}{2^n(x-1)^\alpha(1-t)^{n+1}}.$$

4.10 An Example with Harmonic Numbers

Section 6.4 in [GKP94] talks about Harmonic Summation. This are sums that may involve harmonic numbers, defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k}, \quad n \geq 1, \tag{4.68}$$

$$H_0 := 0 \tag{4.69}$$

The authors show in skillful ways how to prove the identities such as

$$\sum_{k=0}^{n-1} H_k = nH_n - n \tag{4.70}$$

$$\sum_{k=0}^{n-1} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4} \tag{4.71}$$

The method that works for their purposes is the concept of summation by parts, a summation analogue of integration by parts. In their words, [GKP94, (6.69)] reads as

$$\sum_a^b u(x)\Delta v(x)\delta x = u(x)v(x)|_a^b - \sum_a^b v(x+1)\Delta u(x)\delta x$$

where $\Delta u(x) := u(x+1) - u(x)$. Equipped with this knowledge they prove

$$\sum_{k=0}^{n-1} \binom{k}{m} H_k = \binom{n}{m+1}\left(H_n - \frac{1}{m+1}\right) \tag{4.72}$$

which includes (4.70) and (4.71) as special cases.

4.10.1 Solution by the Sigma package

In his doctoral thesis [Sch01] Schneider started to develop the Sigma package for symbolic summation. The algorithm due to Karr can be seen as the discrete analogue to Risch's algorithm for indefinite integration. The essential ingredients to his method are

- Telescoping
- Creative telescoping
- Recurrence solving

We do not want to go into the technical details of difference field theory here, but refer the interested reader to [Sch04, Sch07], which present various applications including non trivial examples from particle physics. What we want to present here is the way how to compute sum (4.72) with the help of Schneider's package Sigma[8].

Mathematica 7.0 - Listing

In[1]:= $<<$ **Sigma.m**
Sigma - A summation package by Chasten Schneider - ©RISC Linz - v 0.8 (1/05/10)

In[2]:= $\boldsymbol{mysum} = \mathbf{SigmaSum[SigmaBinomial}[\boldsymbol{k}, \boldsymbol{m}]\mathbf{SigmaHNumber}[\mathbf{1}, \boldsymbol{k}], \{\boldsymbol{k}, \mathbf{0}, \boldsymbol{n}-\mathbf{1}\}];$

In[3]:= $\boldsymbol{res} = \mathbf{SigmaReduce}[\boldsymbol{mySum}]$

Out[3]= $-H_m + \left(\frac{-m+n}{(-1-m)(1+m)} + \frac{(-1-n)H_n}{-1-m}\right)\binom{n}{m} + \sum_{k=0}^{m} H_k \binom{k}{m}$

(* Since Binomial[k,m]=0 if k<m (this check is not built in), we get: *)

In[4]:= $\boldsymbol{res} = \boldsymbol{res}\ /.\ \sum_{\boldsymbol{k}=\mathbf{0}}^{\boldsymbol{m}} \boldsymbol{H_k}\binom{\boldsymbol{k}}{\boldsymbol{m}} \to \boldsymbol{H_m}$

Out[4]= $\left(\frac{-m+n}{(-1-m)(1+m)} + \frac{(-1-n)H_n}{-1-m}\right)\binom{n}{m}$

4.10.2 A Guessing try

One possible way to evaluate sums involving harmonic numbers is to guess a recurrence equation that can be solved. This is in general not the best way, because we can not assume that we will find one. But for (4.70) we are lucky. We will present how to proceed in Mathematica by using Mallinger's package.

Mathematica 7.0 - Listing

In[1]:= **Table [Sum[HarmonicNumber[*k*], {*k*, 0, *n*}], {*n*, 0, 15}]**

[8] Thanks to Dr. Schneider who pointed this way to me

```
Out[1]= {0, 1, 5/2, 13/3, 77/12, 87/10, 223/20, 481/35, 4609/280, 4861/252, 55991/2520, 58301/2310, 785633/27720, 811373/25740, 835397/24024, 1715839/45045}
In[2]:= GuessRE[%, a[n]]
Out[2]= {{(2 + n)^2 a[n] - (1 + n)(5 + 2n)a[1 + n] + (1 + n)(2 + n)a[2 + n] == 0, a[0] == 0, a[1] == 1}, "ogf"}
In[3]:= RSolve[%[[1]], a[n], n] // FullSimplify
Out[3]= {{a[n] → (1 + n)(-1 + HarmonicNumber[1 + n])}}
```

We found the (optimistic) guess that

$$\sum_{k=0}^{n} H_k = (1+n)(-1+H_{n+1}).$$

There are now several ways for verifying that this guess is indeed true. Let's try by induction on n. The case $n=0$ is indeed trivial because

$$\sum_{k=0}^{0} H_k = H_0 = 0 = (1+0)(-1+1)$$

Now let us suppose the identity holds for n and go to $n+1$. We find that

$$\begin{aligned}
\sum_{k=0}^{n+1} H_k &= \sum_{k=0}^{n} H_k + H_{n+1} \\
&= (1+n)(-1+H_{n+1}) + H_{n+1} \\
&= -(1+n) + (n+2)H_{n+1}.
\end{aligned}$$

We finally find that

$$\begin{aligned}
-(1+n) + (n+2)H_{n+1} + 0 &= -(n+2) + (n+2)H_{n+1} + \frac{n+2}{n+2} \\
&= -(n+2) + (n+2)\left(H_{n+1} + \frac{1}{n+2}\right) \\
&= (n+2)(-1+H_{n+2}).
\end{aligned}$$

That proves our claim.

$\square$

4.10.3 Applying the Egorychev method

Now we will use again the method of coefficients and generating functions to derive result (4.70). Before we need knowledge about the generating function of harmonic numbers.

Lemma 4.10.1 *The generating function of the harmonic numbers is given by*

$$\sum_{n=0}^{\infty} H_n z^n = -\frac{\log(1-z)}{1-z} \tag{4.73}$$

Proof. An equivalent definition of harmonic numbers is given by the recurrence

$$\begin{aligned} H_0 &= 0 \\ H_n &= \frac{1}{n} + H_{n-1}, \quad n \geq 1. \end{aligned}$$

From the second line we find that

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad n \geq 0.$$

Now

$$\begin{aligned} H(z) = \sum_{n=0}^{\infty} H_{n+1} z^{n+1} &= \sum_{n=0}^{\infty} H_n z^{n+1} + \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\ \Leftrightarrow H(z) &= zH(z) - \log(1-z) \\ \Leftrightarrow H(z) &= -\frac{1}{1-z} \log(1-z) \end{aligned}$$

□

By Lemma 2.2.9, we know now that the harmonic numbers have the residue representation

$$H_k = \operatorname*{\mathbf{res}}_z \; -\frac{\log(1-z)}{1-z} z^{-k-1}, \quad k \geq 0. \tag{4.74}$$

If we now calculate sum (4.70) we find that

$$\begin{aligned} \sum_{k=0}^{n} H_k &= \sum_{k=0}^{n} \operatorname*{\mathbf{res}}_z \; -\frac{\log(1-z)}{1-z} z^{-k-1} \\ &= \operatorname*{\mathbf{res}}_z \; -\frac{\log(1-z)}{1-z} \frac{1}{z} \sum_{k=0}^{n} z^{-k} \\ &= \operatorname*{\mathbf{res}}_z \; -\frac{\log(1-z)}{1-z} \frac{z^{-n-1} - 1}{1-z} \\ &= \operatorname*{\mathbf{res}}_z \; -\frac{\log(1-z)}{(1-z)^2} z^{-n-1} - \operatorname*{\mathbf{res}}_z \frac{\log(1-z)}{(1-z)^2} \end{aligned}$$

By computing a series expansion of the second residue we find

$$f(z) = \frac{\log(1-z)}{(1-z)^2} = -z - \frac{5}{2}z^2 - \frac{13}{3}z^3 + \mathcal{O}(z^4),$$

or in other words $f(z) \in \mathbb{K}_1((z))$. By Lemma 2.2.9 we now find that

$$\sum_{k=0}^{n} H_k = \langle z^n \rangle \left(-\frac{\log(1-z)}{(1-z)^2} \right), \quad n \geq 0. \tag{4.75}$$

In other words, we have the generating function

$$\sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} H_k \right) = -\frac{\log(1-z)}{(1-z)^2}. \tag{4.76}$$

Note that a closed form for the generating function (4.76) could also be derived in a more elementary way. Namely by taking Lemma 4.10.1 and the following elementary Lemma

Lemma 4.10.2 ([Mal96], p. 26, Cor. 1.4.6 (c))

$$\frac{1}{1-z} \sum_{k=0}^{\infty} a_k z^k = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} a_k \right)$$

Proof.

$$\frac{1}{1-z} \sum_{k=0}^{\infty} a_k z^k = \left(\sum_{k=0}^{\infty} z^k \right) \left(\sum_{k=0}^{\infty} a_k z^k \right) = \sum_{n=0}^{\infty} (a_k \cdot 1)\, z^n$$

□

If we now set $a_k = H_k$ we get immediately

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} H_k \right) z^n = \frac{1}{1-z} \sum_{k=0}^{\infty} H_k z^k = -\frac{\log(1-z)}{(1-z)^2}$$

An interesting thing is, that this generating function is holonomic (i.e. a solution to an ordinary differential equation with polynomial coefficients). The logarithm function $f(z) := -\log(1-z)$ satisfies the differential equation

$$1 - (1-z) f'(z) = 0, \quad f(0) = 0$$

The factor $(1-z)^{-2}$ is solution to the algebraic equation

$$(1-z)^2 f(z) = 1, \quad f(0) = 1$$

With this knowledge and the help of closure properties we are able to compute a differential equation for the product (and therefore the generating function (4.76)). An ordinary differential equation translates to a recurrence relation for the sequence of coefficients. This is implemented in Mallinger's package.

Mathematica 7.0 - Listing

In[1]:= $\mathbf{DECauchy[\{1-(1-z)f'[z]==0, f[0]==0\}, \{(1-z)^2 f[z]==1, f[0]==1\}, f[z]]}$

Out[1]= $\{1+2(1-2z+z^2)f[z]+(-1+3z-3z^2+z^3)f'[z]==0, f[0]==0\}$

In[2]:= $\mathbf{re := DE2RE[\{1+z(1-2z+z^2)f[z]+(-1+3z-3z^2+z^3)f'[z]==0, f[0]==0\}, f[z], a[n]]}$

Out[2]= $\{(2+n)^2 a[n]-(2+n)(7+3n)a[n+1]+(2+n)(8+3n)a[n+2]-(2+n)(3+n)a[n+3]==0,$
$a[0]==0, a[1]==1, a[2]==\frac{5}{2}\}$

In[3]:= $\mathbf{re[[1,1]]\ /.\ a[n\_]\to(1+n)(-1+\mathrm{HarmonicNumber}[1+n])\ //\ \mathrm{FullSimplify}}$

Out[3]= 0

In[4]:= $\mathbf{re[[2]]\ /.\ a[n\_]\to(1+n)(-1+\mathrm{HarmonicNumber}[1+n])}$

Out[4]= True

4.10.4 Solution by the HolonomicFunctions package

Closure properties allow us to compute a differential equation resp. a recurrence relation for the generating function

$$f(z)=-\frac{\log(1-z)}{(1-z)^2}.$$

Koutschan's HolonomicFunctions package[9] is able to derive the differential equation almost automatically. Let us demonstrate how this is performed

Mathematica 7.0 - Listing

In[1]:= $\mathbf{Annihilator[-Log[1-z]/(1-z)^2, Der[z]]}$

Out[1]= $\{(1-2z+z^2)D_z^2+(-5+4z)D_z+4\}$

In[2]:= $\mathbf{ApplyOreOperator[First[\%], f[z]]==0}$

Out[2]= $4f[z]+(-5+5z)f'[z]+(1-2z+z^2)f''[z]==0$

In[3]:= $\mathbf{DFiniteDE2RE[\%\%, z, n]}$

Out[3]= $\{(2+3n+n^2)S_n^2+(-5-7n-2n^2)S_n+(4+4n+n^2)\}$

[9] Thanks to Dr. Koutschan for demonstrating examples

Hence, $f(z)$ satisfies the differential equation

$$(1-2z+z^2)f''(z)+(-5+5z)f'(z)+4f(z)=0$$

and on coefficient level

$$(2+3n+n^2)f(n+2)+(-5-7n-2n^2)f(n+1)+(4+4n+n^2)f(n)=0$$

But how are this differential equations related? In fact they are equivalent, except that the second is a homogenous differential equation. In particular, if we differentiate Mallinger's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(1+2(1-2z+z^2)f[z]+(-1+3z-3z^2+z^3)f'[z]=0\right)$$
$$\to(-1+z)(4f[z]+(-1+z)(5f'[z]+(-1+z)f''[z]))=0$$

which is (up to multiplication by $(-1+z)$) exactly Koutschan's differential equation. The reason for this is that if $f(z)\in\mathbb{K}[[z]]$ the linear space spanned by

$$\langle\{\ f^{(k)}(z)\mid k\in\mathbb{N}\ \}\rangle_{\mathbb{K}(z)}=\left\langle\left\{\ \frac{\mathrm{d}^k}{\mathrm{d}z^k}f(z)\mid k\in\mathbb{N}\ \right\}\right\rangle_{\mathbb{K}(z)}$$

is a finite dimensional subspace of $\mathbb{K}((z))$ over $\mathbb{K}(z)$. Hence there are many ways of describing the same object. As an example consider (for $k\in\mathbb{N}$) the differential equation

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}f(z)-f(z)=0,$$

and the initial condition $\left(\frac{\mathrm{d}^m}{\mathrm{d}z^m}f(z)\right)|_{z=0}=C_0$ where $0\leq m\leq k-1, C_0\in\mathbb{K}$. This system of differential equations has the unique solution

$$f(z)=C_0e^z,$$

although there are infinitely many equations describing $f(z)$. With an appropriate choice of initial values we can ensure uniqueness of the result.

4.10.5 Application of change of variables

In the text examples of [Ego84], Egorychev demonstrates his method by proving the identity

$$S(n):=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k=\frac{1}{n}.$$

In fact he proceeded the same lines as we did and arrives at the residue representation

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k = \operatorname*{\mathbf{res}}_{u} \frac{(1+u)^{n-1}\log(1-u)}{u^{n+1}}.$$

But now a miracle happens. If one substitutes

$$w = \frac{u}{1+u} \in \mathbb{K}_1((u)) \leftrightarrow u = \frac{w}{1-w},$$

according to rule 5 we get

$$\operatorname*{\mathbf{res}}_{u} \frac{(1+u)^{n-1}\log(1-u)}{u^{n+1}} = -\operatorname*{\mathbf{res}}_{w} \frac{\log(1-w)}{w^{n+1}},$$

or in other words the coefficient of w^n in the series expansion of $-\log(1-w)$. But this is known explicitly in closed form. Namely we find that

$$S(n) = -\operatorname*{\mathbf{res}}_{w} \frac{\log(1-w)}{w^{n+1}} = \frac{1}{n}.$$

4.11 Analytic aspects

One reason, why formal power series are that powerful is that we can look at them from different points of view. So far we have manipulated them in a purely formal way. Let us now suppose, we have given a generating function in closed form and we want to extract information about the asymptotic growth of the coefficient sequence. That is, we want to give an estimate about the size of the n'th element of the sequence $(f_n)_{n\geq 0}$ that has the generating function $f(z)$. In general [FS09, p. 226] this will look like

$$\langle z^n \rangle\ f(z) = \langle z^n \rangle\ \sum_{n=0}^{\infty} f_n z^n = f_n = A^n \theta(n), \tag{4.77}$$

where we call A^n the exponential growth part and $\theta(n)$ the subexponential part. In [FS09, p. 227] the two main principles for extracting asymptotic information about the sequence are described as follows

- **First Principle of Coefficient Asymptotics** The location of a function's singularities dictates the exponential growth (A^n) of its coefficients

- **Second Principle of Coefficients Asymptotics** The nature of a function's singularities determines the associate subexponential factor ($\theta(n)$)

The exponential factor A is related to the radius of convergence of a series. As it turns out, the rate of growth is given by

$$A = \frac{1}{\rho} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f_n|}}, \tag{4.78}$$

where ρ is the first singularity encountered along the positive real axis ([FS09, p. 226]). With the help of complex analysis one is able to determine formulas for the asymptotic growth of the coefficients in the power series expansion. We consider the case where we do not find an explicit closed form as it might be the case in examples involving logarithmic factors.

Theorem 4.11.1 ([FS09], p.385, Thm. VI.2) *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in the function*

$$f(z) = (1-z)^{-\alpha} \left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{\beta}$$

admits for a large n a full asymptotic expansion in descending powers of $\log(n)$*,*

$$f_n = \langle z^n \rangle \; f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log(n))^{\beta} \left[1 + \frac{C_1}{\log(n)} + \frac{C_2}{\log(n)^2} + \dots\right]$$

where

$$C_k = \binom{\beta}{k} \; \Gamma(\alpha) \; \frac{d^k}{ds^k}\frac{1}{\Gamma(s)}|_{s=\alpha}$$

Proof. The proof uses complex integration methods and is a consequence of the preceding theorem VI.1 in [FS09]. For the full details see [FS09, p. 385]. □

Remark: $\Gamma(\alpha)$ denotes the Eulerian Gamma function defined by

$$\Gamma(\alpha) := \int_0^{\infty} e^{-t} t^{\alpha-1} \; dt \tag{4.79}$$

In the book there are several special cases pointed out explicitly namely the cases where $\alpha = \frac{1}{2}$ and $\beta = -1$. Also the case where α is a nonnegative integer is interesting, because in this case the Gamma function evaluates to a simple factorial by the well known interpolation property

$$\forall n \in \mathbb{N} : \Gamma(n+1) = n!$$

that is easily proved by integration by parts.

Curriculum Vitae

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