Incomplete Generalized Jacobsthal and Jacobsthal-Lucas Numbers

G. B. DJORDJEVIĆ
Department of Mathematics
Faculty of Technology
University of Niš
YU-16000 Leskovac, Serbia and Montenegro, Yugoslavia
ganedj@eunet.yu

H. M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsri@math.uvic.ca

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Abstract—In this paper, we present a systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal-Lucas numbers. The main results, which we derive here, involve the generating functions of these incomplete numbers. @ 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND DEFINITIONS

Recently, Djordjević [1,2] considered four interesting classes of polynomials: the generalized Jacobsthal polynomials $J_{n,m}(x)$, the generalized Jacobsthal-Lucas polynomials $j_{n,m}(x)$, and their associated polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$. These polynomials are defined by the following recurrence relations (cf., [1-3]):

1. $J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x)$,
2. $j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x)$,
3. $F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3$,
4. $f_{n,m}(x) = f_{n-1,m}(x) + 2xf_{n-m,m}(x) + 3$.

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\[ f_{n,m}(x) = f_{n-1,m}(x) + 2xf_{n-m,m}(x) + 5 \]
\[ (n \geq m; m, n \in \mathbb{N}; f_{0,m}(x) = 0; f_{n,m}(x) = 1, \text{ when } n = 1, \ldots, m - 1), \]

\(\mathbb{N}\) being the set of natural numbers and

\[ \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}. \]

Explicit representations for these four classes of polynomials are given by

\[ J_{n,m}(x) = \sum_{r=0}^{\left\lfloor \frac{n-1}{m} \right\rfloor} \binom{n-1-(m-1)r}{r} (2x)^r, \]

\[ J_{n,m}(x) = \sum_{r=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k, \]

\[ F_{n,m}(x) = J_{n,m}(x) + 3 \sum_{r=0}^{\left\lfloor \frac{n-m+1}{m} \right\rfloor} \binom{n-m+1-(m-1)r}{r+1} (2x)^r, \]

and

\[ f_{n,m}(x) = J_{n,m}(x) + 5 \sum_{r=0}^{\left\lfloor \frac{n-m+1}{m} \right\rfloor} \binom{n-m+1-(m-1)r}{r+1} (2x)^r, \]

respectively. Tables for \(J_{n,m}(x)\) and \(j_{n,m}(x)\) are provided in [2].

By setting \(x = 1\) in definitions (1.1)–(1.4), we obtain the \textit{generalized Jacobsthal numbers}

\[ J_{n,m} := J_{n,m}(1) = \sum_{r=0}^{\left\lfloor \frac{n-1}{m} \right\rfloor} (n-1-(m-1)r) 2^r, \]

and the \textit{generalized Jacobsthal-Lucas numbers}

\[ j_{n,m} := j_{n,m}(1) = \sum_{r=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r, \]

and their associated numbers

\[ F_{n,m} := F_{n,m}(1) = J_{n,m}(1) + 3 \sum_{r=0}^{\left\lfloor \frac{n-m+1}{m} \right\rfloor} \binom{n-m+1-(m-1)r}{r+1} 2^r \]

and

\[ f_{n,m} := f_{n,m}(1) = J_{n,m}(1) + 5 \sum_{r=0}^{\left\lfloor \frac{n-m+1}{m} \right\rfloor} \binom{n-m+1-(m-1)r}{r+1} 2^r. \]

Particular cases of these numbers are the so-called \textit{Jacobsthal numbers} \(J_n\) and the \textit{Jacobsthal-Lucas numbers} \(j_n\), which were investigated earlier by Horadam [4]. (See also a systematic investigation by Raina and Srivastava [5], dealing with an interesting class of numbers associated with the familiar Lucas numbers.)

Motivated essentially by the recent works by Filipponi [6], Pintéř and Srivastava [7], and Chu and Vicenti [8], we aim here at introducing (and investigating the generating functions of) the analogously \textit{incomplete} version of each of these four classes of numbers.
2. GENERATING FUNCTIONS OF THE INCOMPLETE GENERALIZED JACOBSTHAL AND JACOBSTHAL-LUCAS NUMBERS

We begin by defining the incomplete generalized Jacobsthal numbers $J_{n,m}^k$ by

$$J_{n,m}^k := \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \quad \left(0 \leq k \leq \left\lfloor \frac{n-1}{m} \right\rfloor; m, n \in \mathbb{N}\right), \quad (2.1)$$

so that, obviously,

$$J_{n,m}^{(n-1)/m(n-1)/m} = J_{n,m}, \quad (2.2)$$

$$J_{n,m}^k = 0 \quad \left(0 \leq n < mk + 1\right), \quad (2.3)$$

and

$$J_{mk+l,m}^k = J_{mk+l-1,m} \quad (l = 1, \ldots, m). \quad (2.4)$$

The following known result (due essentially to Pintőr and Srivastava [7]) will be required in our investigation of the generating functions of such incomplete numbers as the incomplete generalized Jacobsthal numbers $J_{n,m}^k$ defined by (2.1). For the theory and applications of the various methods and techniques for deriving generating functions of special functions and polynomials, we may refer the interested reader to a recent treatise on the subject of generating functions by Srivastava and Manocha [9].

**Lemma 1.** (See [7, p. 593].) Let \( \{s_n\}_{n=0}^{\infty} \) be a complex sequence satisfying the following nonhomogeneous recurrence relation:

$$s_n = s_{n-1} + 2s_{n-m} + r_n \quad (n \geq m; m, n \in \mathbb{N}), \quad (2.5)$$

where \( \{r_n\} \) is a given complex sequence. Then the generating function \( S(t) \) of the sequence \( \{s_n\} \) is

$$S(t) = s_0 - r_0 + \sum_{l=1}^{m-1} t^l (s_l - s_{l-1} - r_l) + G(t) \frac{1}{(1 - t - 2t^m)^{-1}}, \quad (2.6)$$

where \( G(t) \) is the generating function of the sequence \( \{r_n\} \).

Our first result on generating functions is contained in Theorem 1 below.

**Theorem 1.** The generating function of the incomplete generalized Jacobsthal numbers $J_{n,m}^k$ ($k \in \mathbb{N}_0$) is given by

$$R_m^k(t) = \sum_{r=0}^{\infty} J_{k,m}^r t^r$$

$$= t^{mk+1} \left[ J_{mk,m} + \sum_{l=1}^{m-1} t^l \left( J_{mk+l,m} - J_{mk+l-1,m} \right) \right] \frac{1}{(1 - t)^{k+1} - 2^{k+1} t^m} \quad (2.7)$$

$$\cdot \frac{1}{(1 - t - 2t^m)^{-1}}.$$
PROOF. From (1.1) (with $x = 1$) and (2.1), we get

$$J_{n,m}^k - J_{n-1,m}^k - 2J_{n-m,m}^k = \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^{r+1} = \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r$$

$$= \sum_{r=0}^{k+1} \binom{n-2-(m-1)r}{r-1} 2^r - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1}$$

$$= -\sum_{r=1}^{k} \left[ \binom{n-2-(m-1)r}{r} + \binom{n-2-(m-1)r}{r-1} \right] 2^r - 1 - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} + \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r$$

$$= \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r + 1 - \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1}$$

$$= -\binom{n-1-m-(m-1)k}{k} 2^{k+1}$$

$$= -\binom{n-1-m-(m-1)k}{n-1-m-mk} 2^{k+1} \quad (n \geq m + 1 + mk; \ k \in \mathbb{N}_0).$$

Next, in view of (2.3) and (2.4), we set

$$s_0 = J_{mk+1,m}^k, s_1 = J_{mk+2,m}^k, \ldots, s_{m-1} = J_{mk+m,m}^k$$

and

$$s_n = J_{mk+n+1,m}^k.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = 2^{k+1} \binom{n-m+k}{n-m}.$$

Then, for the generating function $G(t)$ of the sequence $\{r_n\}$, we can show that

$$G(t) = \frac{2^{k+1}t^m}{(1-t)^{k+1}}.$$

Thus, in view of the above lemma, the generating function $S_n^k(t)$ of the sequence $\{s_n\}$ satisfies the following relationship:

$$S_n^k(t)(1-t-2t^m) + \frac{2^{k+1}t^m}{(1-t)^{k+1}} = J_{mk,m}(k) + \sum_{l=1}^{m-1} t^l (J_{mk+1,m} - J_{mk+l-1,m}) + \frac{2^{k+1}t^m}{(1-t)^{k+1}}.$$
Hence, we conclude that
\[ R_m^k(t) = t^{mk+1} s_m^k(t). \]
This completes the proof of Theorem 1.

**Corollary 1.** The incomplete Jacobsthal numbers \( J_n^k \) \((k \in \mathbb{N}_0)\) are defined by
\[ J_n^k := J_n^{k,2} = \sum_{r=0}^{k} \binom{n-1-r}{r} 2^r \quad \left(0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor; \ n \in \mathbb{N} \setminus \{1\} \right) \]
and the corresponding generating function is given by (2.7) when \( m = 2 \), that is, by
\[ R_2^k(t) = t^{2k+1} \left[J_{2k} + t (J_{2k+1} - J_{2k}) (1-t)^{k+1} - 2^{k+1} t^2 \right] \cdot \left[(1-t)(1-t^2) (1-t)^{k+1} \right]^{-1}. \] (2.9)

**3. INCOMPLETE GENERALIZED JACOBSTHAL-LUCAS NUMBERS**

For the incomplete generalized Jacobsthal-Lucas numbers \( j_{n,m}^k \) defined by [cf. equation (1.10)]
\[ j_{n,m}^k := \sum_{r=0}^{m-1} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r \quad \left(0 \leq k \leq \left\lfloor \frac{n}{m} \right\rfloor; \ m, n \in \mathbb{N} \right), \] (3.1)
we now prove the following generating function.

**Theorem 2.** The generating function of the incomplete generalized Jacobsthal-Lucas numbers \( j_{n,m}^k \) \((k \in \mathbb{N}_0)\) is given by
\[ W_m^k(t) = \sum_{r=0}^{\infty} j_{n,m}^r t^r \]
\[ = t^{mk} \left[ j_{mk-1,m} + \sum_{l=1}^{m-1} t^l (j_{mk+l-1,m} - j_{mk+l-2,m}) (1-t)^{k+1} - 2^{k+1} t^m (2-t) \right] \cdot \left[(1-t-2t^m)(1-t)^{k+1} \right]^{-1}. \] (3.2)

**Proof.** First of all, it follows from definition (3.1) that
\[ j_{n,m}^{\lfloor n/m \rfloor} = j_{n,m}, \] (3.3)
\[ j_{n,m}^k = 0 \quad (0 \leq n < mk), \] (3.4)
and
\[ j_{mk+l,m}^1 = j_{mk+l-1,m} \quad (l = 1, \ldots, m). \] (3.5)
Thus, just as in our derivation of (2.8), we can apply (1.2) and (1.10) (with \( x = 1 \)) in order to obtain
\[ j_{n,m}^k - j_{n-1,m}^k - 2 j_{n,m,m}^k = - \frac{n-m+2k}{n-m+k} \left( n-m+k \right) 2^{k+1}. \] (3.6)
Let
\[ s_0 = j_{mk-1,m}, \quad s_1 = j_{mk,m}, \ldots, s_{m-1} = j_{mk+m,m}, \]
and
\[ s_n = j_{mk+n+1,m}. \]

Suppose also that
\[ r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = \frac{n - m + 2k}{n - m + k} \left( \frac{n - m + k}{n - m} \right) 2^{k+1}. \]

Then, the generating function \( G(t) \) of the sequence \( \{r_n\} \) is given by
\[ G(t) = \frac{2^{k+1} t^m (2 - t)}{(1 - t)^{k+1}}. \]

Hence, the generating function of the sequence \( \{s_n\} \) satisfies relation (3.2), which leads us to Theorem 2.

**Corollary 2.** For the incomplete Jacobsthal-Lucas numbers \( j^k_{n,2} \), the generating function is given by (3.2) when \( m = 2 \), that is, by
\[ W^k_2(t) = t^{2k} \left[ (j_{2k-1} + t(j_{2k} - j_{2k-1})) (1 - t)^{k+1} - 2^{k+1} t^2 (2 - t) \right] \cdot \left[ (1 - t - 2t^2) (1 - t)^{k+1} \right]^{-1}. \]

4. TWO FURTHER PAIRS OF INCOMPLETE NUMBERS

For a natural number \( k \), the incomplete numbers \( F_{n,m}^k \) corresponding to the numbers \( F_{n,m} \) in (1.11) are defined by
\[ F_{n,m}^k := j_{n,m} + 3 \sum_{r=0}^{k} \left( \frac{n - m + 1 - (m - 1)r}{r + 1} \right) 2^r \left( 0 \leq k \leq \left\lfloor \frac{n - 1}{m} \right\rfloor ; \ m, n \in \mathbb{N} \right), \quad (4.1) \]
where
\[ F_{n,m}^k = j_{n,m} = 0, \quad (n < m + mk). \]

**Theorem 3.** The generating function of the incomplete numbers \( F_{n,m}^k \) \((k \in \mathbb{N}_0)\) is given by \( t^{mk+1} S_m^k(t) \), where
\[ S_m^k(t) = \left[ F_{mk,m} + \sum_{l=1}^{m-1} t^l (F_{mk+l,m} - F_{mk+l-1,m}) \right] (1 - t - 2t^m)^{-1} + \frac{3t^m (1 - t)^{k+1} - 2^{k+1} t^m (1 - t + 3t^{m-1})}{(1 - t - 2t^m) (1 - t)^{k+2}}. \quad (4.2) \]

**Proof.** Our proof of Theorem 3 is much akin to those of Theorems 1 and 2 above. Here, we let
\[ s_0 = F_{mk+1,m}^k = F_{mk}, \]
\[ s_1 = F_{mk+2,m}^k = F_{mk-1,m}, \ldots, \]
\[ s_{m-1} = F_{mk+m,m}^k = F_{mk+m-1,m}, \]
and
\[ s_n = F_{mk+n+1,m}^k. \]
Suppose also that
\[ r_0 = r_1 = \cdots = r_{m-1} = 0 \]
and
\[ r_n = \left( \frac{n - m + k}{n - m} \right) 2^{k+1} + 3 \left( \frac{n - m + 2 + k}{n - m + k} \right) 2^{k+1}. \]
Then, by using the standard method based upon the above lemma, we can prove that
\[
G(t) = \sum_{n=0}^{\infty} r_n t^n = \frac{2^{k+1} t^m (1 - t + 3t^{m-1})}{(1 - t)^{k+2}}.
\]
Let \( S_m^k(t) \) be the generating function of \( F_{n,m}^k \). Then, it follows that
\[
\begin{align*}
S_m^k(t) &= s_0 + ts_1 + \cdots + s_n t^n + \cdots, \\
tS_m^k(t) &= ts_0 + t^2 s_1 + \cdots + t^n s_{n-1} + \cdots, \\
2t^m S_m^k(t) &= 2t^m s_0 + 2t^{m+1} s_1 + \cdots + 2t^n s_{n-m} + \cdots,
\end{align*}
\]
and
\[
G(t) = r_0 + r_1 t + \cdots + r_n t^n + \cdots.
\]
The generating function \( t^{m+1} S_m^k(t) \) asserted by Theorem 3 would now result easily.

**COROLLARY 3.** For the incomplete numbers \( F_{n,m}^k \) defined by (4.1) with \( m = 2 \), the generating function is given by
\[
S_m^k(t) = t^{2k+1} \left( \frac{F_{2k+1} - F_{2k}}{1 - t} \right)^k + 3t \left( \frac{1 - t + 3t^2}{1 - t} \right)^k.
\]
Finally, the incomplete numbers \( f_{n,m}^k \) (\( k \in \mathbb{N}_0 \)) corresponding to the numbers \( f_{n,m} \) in (1.12) are defined by
\[
f_{n,m}^k := J_{n,m}^k + 5 \sum_{r=0}^{k} \binom{n+1-m-(m-1)r}{r+1} 2^r \quad \left( 0 \leq k \leq \left\lfloor \frac{n-1}{m} \right\rfloor ; m, n \in \mathbb{N} \right).
\]

**THEOREM 4.** The incomplete numbers \( f_{n,m}^k \) (\( k \in \mathbb{N}_0 \)) have the following generating function:
\[
W_m^k(t) = t^{m+1} \left[ f_{m,k,m} + \sum_{i=1}^{m-1} t^i (f_{m,k+i,m} - f_{m,k+i-1,m}) \right] (1 - t - 2t^m)^{-1}
+ t^{m+1} \left( \frac{5t^m (1 - t)^{k+1} - 2^{k+1} t^m (1 - t + 5t^{m-1})}{(1 - t - 2t^m)^{k+2}} \right).
\]

**PROOF.** Here, we set
\[
\begin{align*}
s_0 &= f_{m,k+1,m} = f_{m,k,m}, \\
s_1 &= f_{m,k+2,m} = f_{m,k+1,m}, \\
&\vdots \\
s_{m-1,m} &= f_{m,k+m-1,m} = f_{m,k+m-1,m},
\end{align*}
\]
and

\[ s_n = f_{m+n+1,m}^k = f_{m+n,m}. \]

We also suppose that

\[ r_0 = r_1 = \ldots = r_{m-1} = 0 \]

and

\[ r_n = 2^{k+1} \left( \frac{n - m + k}{n - m} \right) + 5 \cdot 2^{k+1} \left( \frac{n - 2m + 2 + k}{n - 2m + 1} \right). \]

Then, by using the known method based upon the above lemma, we find that

\[ G(t) = \frac{2^{k+1} t^n (1 - t + 5 t^{m-1})}{(1 - t)^{k+2}} \]

is the generating function of the sequence \( \{r_n\} \). Theorem 4 now follows easily.

In its special case when \( m = 2 \), Theorem 4 yields the following generating function for the incomplete numbers investigated in [6,7].

**Corollary 4.** The generating function of the incomplete numbers \( f_{n,2}^k \) is given by (4.5) when \( m = 2 \), that is, by

\[ W_2^k(t) = t^{2k+1} \left( \frac{[f_{2k} + t(f_{2k+1} - f_{2k})]}{(1 - t)^{k+2}} + 5t^2 (1 - t)^{k+1} - 2^{k+1} t^2 (1 + 4t) \right) \]

**References**