I.

An overview of the theory of Zeta functions and L-series

K. Consani – Johns Hopkins University

Vanderbilt University, May 2006

(a) **Arithmetic L-functions**

(a1) **Riemann zeta function**: $\zeta(s)$, $s \in \mathbb{C}$

(a2) Dirichlet L-series: $L(\chi, s)$

$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

(a3) **Dedekind zeta funct**: $\zeta_K(s)$, $[K : \mathbb{Q}] \leq \infty$

(a4) Hecke L-series: $L_K(\chi, s)$

(a5) **Artin L-function**: $L(\rho, s)$

$\rho : Gal(K/\mathbb{Q}) \to GL_n(\mathbb{C})$ Galois representation

(a6) **Motivic L-function**: $L(M, s)$

M pure or mixed motive

(b) **Automorphic L-functions**

(b1) Classical theory (before Tate's thesis 1950)

$L(f,s)$; $L(f,\chi,s)$ **modular L-function**

associated to a modular cusp form $f:\mathfrak{H}\to\mathbb{C}$

(b2) Modern adelic theory: $L(\pi,s)$

automorphic L-function

$\pi=\otimes'_v\pi_v$, $(\pi_v, V_{\pi_v})=$ irreducible (admissible) representation of $GL_n(\mathbb{Q}_v)$

(a1) **The Riemann zeta function**

$$s \in \mathbb{C}, \qquad \boxed{\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}}$$

Main Facts

- converges absolutely and uniformly on $Re(s) > 1$

$(Re(s) \geq 1 + \delta \ (\delta > 0), \quad \sum_{n=1}^{\infty} |\frac{1}{n^s}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}})$

$\Rightarrow \zeta(s)$ represents an analytic function in $Re(s) > 1$

- **Euler's identity**: $$\boxed{\zeta(s) = \prod_{\substack{p \\ \text{prime}}} (1 - p^{-s})^{-1}}$$

$$\left(|\prod_{p \leq N} (1 - p^{-s})^{-1} - \zeta(s)| \leq \sum_{n > N} \frac{1}{n^{1+\delta}}\right)$$

Number-theoretic significance of the zeta-function:

▶ Euler's identity expresses the law of unique prime factorization of natural numbers

$$\boxed{\Gamma(s) := \int_0^\infty e^{-y} y^s \ \frac{dy}{y}}$$

Gamma-function

$s \in \mathbb{C}$, $Re(s) > 0$; absolutely convergent

- $\Gamma(s)$ analytic, has meromorphic continuation to $\mathbb{C}$

- $\Gamma(s) \neq 0$, has <u>simple</u> poles at $s = -n$, $n \in \mathbb{Z}_{\geq 0}$

$$Res_{s=-n}\Gamma(s) = \tfrac{(-1)^n}{n!}$$

- functional equations

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(s)\Gamma(1-s) = \tfrac{\pi}{\sin(\pi s)}$$

- Legendre's duplication formula

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \tfrac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$$

- special values

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma(k+1) = k!, \quad k \in \mathbb{Z}_{\geq 0}$$

The connection between $\Gamma(s)$ and $\zeta(s)$

$$y \mapsto \pi n^2 y \quad \Rightarrow \quad \pi^{-s}\Gamma(s)\tfrac{1}{n^{2s}} = \int_0^\infty e^{-\pi n^2 y} y^s \frac{dy}{y}$$

sum over $n \in \mathbb{N}$

$$\boxed{\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y} y^s \frac{dy}{y}} \qquad g(y) := \sum_{n\geq 1} e^{-\pi n^2 y}$$

$$\Theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z} \qquad \textbf{Jacobi's theta}$$

$$g(y) = \tfrac{1}{2}(\Theta(iy) - 1), \qquad \boxed{Z(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s)}$$

Main Facts

(1) $Z(s)$ admits the integral representation

$$Z(s) = \tfrac{1}{2}\int_0^\infty (\Theta(iy) - 1) y^{s/2} \frac{dy}{y} \qquad \overset{\text{Mellin Principle}}{\Rightarrow}$$

(2) $Z(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$, has simple poles at $s = 0$, $s = 1$

$$Res_{s=0} Z(s) = -1, \quad Res_{s=1} Z(s) = 1$$

(3) functional eq $Z(s) = Z(1-s)$

Implications for the Riemann zeta $\zeta(s)$

(4) $\zeta(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$

has simple pole at $s = 1$, $Res_{s=1}\zeta(s) = 1$

(5) (functional eq) $\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s)$

Moreover, from $Z(s) = Z(1-s)$ $\Rightarrow$

- the only zeroes of $\zeta(s)$ in $Re(s) < 0$ are the poles of $\Gamma(\frac{s}{2})$ ($s \in 2\mathbb{Z}_{<0}$, "trivial zeroes")
- other zeroes of $\zeta(s)$ (*i.e.* on $Re(s) > 0$) must lie on the **critical strip**: $0 \leq Re(s) \leq 1$

Riemann Hypothesis The "non-trivial" zeroes of $\zeta(s)$ lie on the line $Re(s) = \frac{1}{2}$

(a2) **Dirichlet L-series**

$$m \in \mathbb{N}, \quad \chi : (\mathbb{Z}/m\mathbb{Z})^* \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Dirichlet character mod.m

$$\chi : \mathbb{Z} \to \mathbb{C}, \quad \chi(n) = \begin{cases} \chi(n \text{ mod } m) & (n,m) = 1 \\ 0 & (n,m) \neq 1 \end{cases}$$

$$s \in \mathbb{C}, \quad \boxed{L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s}} \qquad Re(s) > 1$$

for $\chi = 1$ (principal character): $L(1, s) = \zeta(s)$

Main Facts

(1) Euler's identity: $\boxed{L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}}$

(2) $L(\chi, s)$ converges absolutely and unif. on $Re(s) > 1$ (represents an analytic function)

$$\chi(-1) = (-1)^p \chi(1), \quad p \in \{0,1\} \text{ **exponent**}$$

$$\chi : \{(n) \subset \mathbb{Z} \mid (n,m) = 1\} \to S^1$$

$$\chi((n)) := \chi(n)\left(\tfrac{n}{|n|}\right)^p$$

Grössencharacter mod.m (multiplicative fct)

$$\Gamma(\chi, s) := \Gamma\left(\tfrac{s+p}{2}\right) = \int_0^\infty e^{-y} y^{(s+p)/2} \frac{dy}{y}$$ **Gamma integral**

$$y \mapsto \pi n^2 y/m, \quad \theta(\chi, iy) = \textstyle\sum_n \chi(n) n^p e^{-\pi n^2 y/m} \quad \ldots \Rightarrow$$

$$\boxed{L_\infty(\chi, s) := \left(\tfrac{m}{\pi}\right)^{\frac{s}{2}} \Gamma(\chi, s)}$$ **archimedean Euler factor**

$$\Lambda(\chi, s) := L_\infty(\chi, s) L(\chi, s), \qquad Re(s) > 1$$

completed L-series of the character χ

$\Lambda(\chi, s)$ has integral representation $\overset{\text{Mellin principle}}{\Rightarrow}$

- Functional eq.: If $\chi \neq 1$ is a primitive character,

$\Lambda(\chi, s)$ admits an analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi)\Lambda(\bar{\chi}, 1-s), \quad |W(\chi)| = 1$$

($\bar{\chi}$ = complex conjugate character)

(a3) Dedekind zeta function

$K/\mathbb{Q}$ number field, $[K:\mathbb{Q}] = n$

$s \in \mathbb{C}$
$$\boxed{\zeta_K(s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}}$$

$\mathfrak{a}$ = integral ideal of K, $N(\mathfrak{a})$ = absolute norm

Main Facts

(1) $\zeta_K(s)$ converges absolutely and unif. on $Re(s) > 1$

(2) (Euler's identity)

$$\boxed{\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}} \quad Re(s) > 1$$

$Cl_K = J/P$ **ideal class group** of K

$$\zeta_K(s) = \sum_{[\mathfrak{b}] \in Cl_K} \zeta(\mathfrak{b}, s), \quad \zeta(\mathfrak{b}, s) := \sum_{\substack{\mathfrak{a} \in [\mathfrak{b}] \\ \text{integral}}} \frac{1}{N(\mathfrak{a})^s}$$

$\zeta(\mathfrak{b}, s)$ **partial zeta functions**

$$L_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$$

$$L_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$$

$r_1 :=$ number of real embeddings $\quad v = \bar{v} : K \to \mathbb{C}$

$r_2 :=$ number of pairs of complex embeddings $\{v, \bar{v}\} : K \to \mathbb{C}$

$d_K =$ discriminant of K

$$\boxed{Z_\infty(s) := |d_K|^{s/2} L_{\mathbb{R}}(s)^{r_1} L_{\mathbb{C}}(s)^{r_2}}$$

Euler's factor at infinity of $\zeta(\mathfrak{b}, s)$

► $Z(\mathfrak{b}, s) := Z_\infty(s)\zeta(\mathfrak{b}, s), \quad Re(s) > 1$

admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$ and satisfies a functional equation

$$Z_K(s) := \sum_{\mathfrak{b}} Z(\mathfrak{b}, s) = Z_\infty(s)\zeta_K(s)$$

From the corresponding properties of $Z(\mathfrak{b}, s)$ one deduces

Main Facts

(1) $Z_K(s) = Z_\infty(s)\zeta_K(s)$ has analytic continuation to

$$\mathbb{C} \setminus \{0, 1\}$$

(2) (functional eq) $\quad Z_K(s) = Z_K(1-s)$

$Z_K(s)$ has simple poles at $s = 0,\ 1$

$$Res_{s=0} Z_K(s) = -\frac{2^r hR}{w}, \quad Res_{s=1} Z_K(s) = \frac{2^r hR}{w}$$

$r = r_1 + 2r_2$, $\quad h =$ class nb. of K, $\quad R =$ regulator of K
$w =$ number of roots of 1 in K

[Hecke] Subsequent results for $\zeta_K(s)$

(3) $\zeta_K(s)$ has analytic continuation to $\mathbb{C} \setminus \{1\}$

with a simple pole at $s = 1$

(4) Class number formula

$$Res_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_K|}} \frac{hR}{w}$$

(5) (functional eq.) $\zeta_K(1-s) = A(s)\zeta_K(s)$

$$A(s) := |d_K|^{s-1/2}(\cos\tfrac{\pi s}{2})^{r_1+r_2}(\sin\tfrac{\pi s}{2})^{r_2}L_{\mathbb{C}}(s)^n$$

(6) $\zeta_K(s) \neq 0$ for $Re(s) > 1$ $\Rightarrow$

$$m \in \mathbb{Z}_{\geq 0}$$

$$ord_{s=-m}\zeta_K(s) = \begin{cases} r_1 + r_2 - 1 = rk(\mathcal{O}_K^*) & \text{if } m = 0 \\ r_1 + r_2 & \text{if } m > 0 \text{ even} \\ r_2 & \text{if } m > 0 \text{ odd} \end{cases}$$

The class number formula reads now as

$$\zeta_K^*(0) := \lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{hR}{w}$$

(a5) **Artin L-functions**

$L/K =$ Galois extension of nb field K, $\quad G := Gal(L/K)$

Artin L-functions generalize the classical L-series

in the following way

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}, \ Re(s) > 1$$

$$\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*, \quad G := Gal(\mathbb{Q}(\mu_m)/\mathbb{Q}) \overset{\simeq}{\leftarrow} (\mathbb{Z}/m\mathbb{Z})^*$$

p mod $m \mapsto \varphi_p$, $\quad \varphi_p(\zeta_m) = \zeta_m^p$ **Frobenius**

$\chi : G \to GL_1(\mathbb{C})$ 1-dim Galois representation

$$\blacktriangleright\blacktriangleright \quad L(\chi, s) = \prod_{p \nmid m} (1 - \chi(\varphi_p)p^{-s})^{-1}$$

this is a description of the Dirichlet L-series in a purely Galois-theoretic fashion

More in general:

$V =$ finite dim $\mathbb{C}$-vector space

$$\rho : G = Gal(L/K) \to GL(V) = Aut_{\mathbb{C}}(V)$$

$\mathfrak{p}$ prime ideal in K, $\mathfrak{q}/\mathfrak{p}$ prime ideal of L above $\mathfrak{p}$

$$D_{\mathfrak{q}}/I_{\mathfrak{q}} \xrightarrow{\simeq} Gal(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})), \quad D_{\mathfrak{q}}/I_{\mathfrak{q}} = < \varphi_{\mathfrak{q}} >$$

$$\varphi_{\mathfrak{q}} \mapsto (x \mapsto x^q) \quad q = N(\mathfrak{p})$$

$\varphi_{\mathfrak{q}} \in End(V^{I_{\mathfrak{q}}})$ finite order endomorphism

$\boxed{P_{\mathfrak{p}}(T) := \det(1 - \varphi_{\mathfrak{q}} T;\ V^{I_{\mathfrak{q}}})}$, **characteristic pol**

only depends on $\mathfrak{p}$ (not on $\mathfrak{q}/\mathfrak{p}$)

$$\boxed{\zeta_{L/K}(\rho, s) := \prod_{\substack{\mathfrak{p}\text{prime} \\ \text{in} K}} \det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{p})^{-s};\ V^{I_{\mathfrak{q}}})^{-1}}$$

Artin L-series

$$\det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{p})^{-s};\ \ V^{I_{\mathfrak{q}}}) = \prod_{i=1}^{d} (1 - \epsilon_i N(\mathfrak{p})^{-s})$$

$\epsilon_i =$ roots of 1: $\varphi_{\mathfrak{q}}$ has finite order

- $\zeta_{L/K}(\rho, s)$ converges absolutely and unif on $Re(s) > 1$

If $(\rho, \mathbb{C})$ is the trivial representation, then

$$\zeta_{L/K}(\rho, s) = \zeta_K(s) \quad \text{Dedekind zeta function}$$

- An additive expression analogous to $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$ does not exist for general Artin L-series.

- Artin L-series exhibit nice functorial behavior under change of extensions L/K and representations ρ

Character of (ρ, V) $\quad \chi_\rho : Gal(L/K) \to \mathbb{C}$

$$\chi_\rho(\sigma) = tr(\rho(\sigma)), \quad \chi_\rho(1) = \dim V = \deg(\rho)$$

$$(\rho, V) \sim (\rho', V') \quad \Longleftrightarrow \quad \chi_\rho = \chi_{\rho'}$$

$\zeta_{L/K}(\rho, s) = \zeta_{L/K}(\chi_\rho, s)$; functorial behavior $\Rightarrow$

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\chi \neq 1 \\ \chi \text{ irred of } G(L/K)}} \zeta_{L/K}(\chi, s)^{\chi(1)}$$

Artin conjecture $\forall \chi \neq 1$ irreducible, $\zeta_{L/K}(\chi, s)$ defines an entire function *i.e.* holom. function on $\mathbb{C}$

the conjecture has been proved for abelian extensions

For every <u>infinite</u> (archimedean) place $\mathfrak{p}$ of K

$$\zeta_{L/K,\mathfrak{p}}(\chi,s) = \begin{cases} L_{\mathbb{C}}(s)^{\chi(1)} & \mathfrak{p} \text{ complex} \\ L_{\mathbb{R}}(s)^{n_+} L_{\mathbb{R}}(s+1)^{n_-} & \mathfrak{p} \text{ real} \end{cases}$$

$$n_+ = \tfrac{\chi(1)+\chi(\varphi_{\mathfrak{q}})}{2}, \quad n_- = \tfrac{\chi(1)-\chi(\varphi_{\mathfrak{q}})}{2}; \quad \varphi_{\mathfrak{q}} \in Gal(L_{\mathfrak{q}}/K_{\mathfrak{p}})$$

$\zeta_{L/K,\mathfrak{p}}(\chi,s)$ has also nice <u>functorial behavior</u>

<u>For $\mathfrak{p}$ real</u>, $\varphi_{\mathfrak{q}}$ induces decomp $\quad V = V^+ \oplus V^-$

$$V^+ = \{x \in V : \varphi_{\mathfrak{q}} x = x\}, \ V^- = \{x \in V : \varphi_{\mathfrak{q}} x = -x\}$$

$$n_+ = \dim V^+, \quad n_- = \dim V^-$$

$$\boxed{\zeta_{L/K,\infty}(\chi,s) := \prod_{\mathfrak{p}|\infty} \zeta_{L/K,\mathfrak{p}}(\chi,s)}$$

$$\Lambda_{L/K}(\chi,s) := c(L/K,\chi)^{\frac{s}{2}} \zeta_{L/K}(\chi,s) \zeta_{L/K,\infty}(\chi,s)$$

completed Artin series

$$c(L/K,\chi) := |d_K|^{\chi(1)} N(\mathfrak{f}(L/K,\chi)) \in \mathbb{N}$$

$\mathfrak{f}(L/K,\chi) = \prod_{\mathfrak{p}\nmid\infty} \mathfrak{f}_{\mathfrak{p}}(\chi)$ Artin conductor of χ

$\mathfrak{f}_{\mathfrak{p}}(\chi) = \mathfrak{p}^{f(\chi)}$ local Artin conductor $\quad (f(\chi) \in \mathbb{Z})$

Main Facts

- $\Lambda_{L/K}(\chi, s)$ admits a meromorphic continuation to $\mathbb{C}$

- (Functional eq.) $\Lambda_{L/K}(\chi, s) = W(\chi)\Lambda_{L/K}(\bar{\chi}, 1-s)$

$$W(\chi) \in \mathbb{C}, \quad |W(\chi)| = 1$$

▶ the proof of the functional equation uses the fact that the Euler factors $\zeta_{L/K,\mathfrak{p}}(\chi, s)$ at the infinite places $\mathfrak{p}$ behave, under change of fields and characters, in exactly the same way as the Euler factors at the finite places:

$$\mathfrak{p} < \infty \qquad \zeta_{L/K,\mathfrak{p}}(\chi, s) := \det(1 - \varphi_{\mathfrak{q}} N(\mathfrak{q})^{-s};\ V^{I_{\mathfrak{q}}})^{-1}$$

this uniform behavior that might seem at first in striking contrast with the definition of the archimedean Euler factors has been motivated by a unified interpretation of the Euler's factors (archimedean and non)

▶▶ [Deninger 1991-92, Consani 1996]

$$\zeta_{L/K,\mathfrak{p}}(\chi, s) = \det{}_\infty\left(\tfrac{\log N(\mathfrak{p})}{2\pi i}(sid - \Theta_{\mathfrak{p}});\ H(X(\mathfrak{p})/\mathbb{L}_{\mathfrak{p}})\right)^{-1}$$

this result reaches far beyond Artin L-series and suggests a complete analogy with the theory of L-series of algebraic varieties over finite fields.

(b) **Automorphic L-functions**

(b1) Classical theory (before Tate's thesis)

$f : \mathfrak{H} \to \mathbb{C}$ modular form of weight k for $\Gamma \subset SL_2(\mathbb{Z})$

- f holomorphic, $\mathfrak{H} =$ upper-half complex plane
- $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$
- f is regular at the cusps z of Γ, $|SL_2(\mathbb{Z}) : \Gamma| < \infty$ ($z \in \mathbb{Q} \cup \{i\infty\}$ fixed pts of parabolic elements of Γ)

Examples

- $\theta_q(z)$ theta series attached to a quadratic form $q(\underline{x})$

$$\theta_q(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}, \quad a(n) = \text{Card}\{\underline{v} : q(\underline{v}) = n\}$$

- $\Delta(z)$ discriminant function from the theory of elliptic modular functions

$$\Delta(z) = 2^{-4}(2\pi)^{12} \sum_{n=1}^{\infty} \tau(nz) e^{2\pi i n z}$$

For simplicity will assume: $\Gamma = SL_2(\mathbb{Z})$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \quad \Rightarrow \quad f(z+1) = f(z)$$

$$\boxed{f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}}$$ **Fourier expansion**

f is a **cusp form** if $a_0 = 0$ *i.e.*

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

the Fourier coefficients a_n often carry interesting arithmetical information:

- $f(z) = \theta_q(z)$, a_n counts the number of times n is represented by the quadratic form $q(\underline{x})$

- $f(z) = \Delta(z)$, $a_n = \tau(n)$ Ramanujan's τ-function

[Hecke 1936] Attached to each cusp form there is a complex analytic invariant function: its L-function

$$\boxed{L(f, s) = \sum_{n \geq 1} \frac{a_n}{n^s}}$$ **Dirichlet series**

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

This L-function is connected to f by an **integral representation**: its <u>Mellin transform</u>

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^\infty f(iy)y^s d^\times y$$

through this integral representation one gets

[Hecke] $L(f,s)$ is <u>entire</u> and satisfies the functional equation

$$\Lambda(f,s) = i^k \Lambda(f,k-s)$$

the functional equation is a consequence of having a modular transformation law under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sending $z \mapsto -\frac{1}{z}$

Since the Mellin transform has an <u>inverse integral transform</u>, one gets

Converse theorem [Hecke] If

$$D(s) = \sum_n \frac{a_n}{n^s}$$

has a "nice" behavior and satisfies the correct functional equation (as above) then

$$f(z) = \sum_n a_n e^{2\pi i n z}$$

is a cusp form (of weight k) for $SL_2(\mathbb{Z})$ and

$$D(s) = L(f, s)$$

in particular: the modularity of $f(z)$ is a consequence of the Fourier expansion and the functional equation

[Weil 1967] the Converse theorem for $\Gamma_0(N)$ holds, by using the functional equation not just for $L(f, s)$ but also for

$$L(f, \chi, s) = \sum_{n \geq 1} \frac{\chi(n) a_n}{n^s}$$

$\chi =$ Dirichlet character of conductor prime to the level $N \in \mathbb{N}$

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ mod \ N \}$$

[Hecke 1936]   An algebra of operators

$\mathcal{H} = \{T_n\}$ **Hecke operators**

acts on modular forms.

If $f(z)$ is a simultaneous eigen-function for the operators in $\mathcal{H}$, then $L(f, s)$ has an **Euler's product**

$$\boxed{L(f,s) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1}}$$

Conclusion

Arithmetic L-functions

are described by Euler's products, analytic properties are conjectural, arithmetic meaning is clear

Automorphic L-functions

are defined by Dirichlet series, characterized by analytic properties, Euler's product and arithmetic meaning are more mysterious...

(b2) Modern adelic theory

The modular form $f(z)$ for $SL_2(\mathbb{Z})$ (or congruence subgroup) is replaced by an automorphic representation of $GL_2(\mathbb{A})$

(in general by an automorphic representation of $GL_n(\mathbb{A})$)

this construction is a generalization of Tate's thesis for $GL_1(\mathbb{A})$

$$\boxed{\mathbb{A} := \prod_p{}' \mathbb{Q}_p \times \mathbb{R}}$$ ring of **adèles of** $\mathbb{Q}$

locally compact topological ring ($\prod'$ = restricted product)

$\mathbb{Q} \subset \mathbb{A}$ diagonal discrete embedding, $\mathbb{A}/\mathbb{Q}$ compact

$$GL_n(\mathbb{A}) = \prod_p{}' GL_n(\mathbb{Q}_p) \times GL_n(\mathbb{R})$$

$GL_n(\mathbb{Q}) \subset GL_n(\mathbb{A})$ diagonal discrete embedding

$Z(\mathbb{A})GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})$ finite volume

$GL_n(\mathbb{A})$ acts on the space

$\mathcal{A}_0(Z(\mathbb{A})GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ **cusp automorphic forms**

producing a decomposition:

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})) = \bigoplus_\pi m(\pi)V_\pi$$

the (infinite dimensional) factors are the **cuspidal automorphic representations**

The decomposition of $GL_n(\mathbb{A})$ as a <u>restricted</u> product corresponds to a decomposition of the representations

$$\pi \simeq \otimes'_v \pi_v = (\otimes'_p \pi_p) \otimes \pi_\infty$$

(π_v, V_{π_v}) = irreducible (admissible) representations of $GL_n(\mathbb{Q}_v)$

<u>Main relations</u>

$$\pi_\infty \longrightarrow L(\pi_\infty, s) \longleftrightarrow \Gamma(s)$$

$$\pi_p \longrightarrow L(\pi_p, s) = Q_p(p^{-s})^{-1}$$

$$\pi \longrightarrow \Lambda(\pi, s) = \prod_p L(\pi_p, s)L(\pi_\infty, s) = L(\pi, s)L(\pi_\infty, s)$$

for $Re(s) >> 0$

[Jacquet,P-S,Shalika] $L(\pi, s) := \prod_p L(\pi_p, s)$ is entire and satisfies a functional equation

$$\Lambda(\pi, s) = \epsilon(\pi, s)\Lambda(\tilde{\pi}, 1 - s)$$

[Cogdell, P-S] A Converse theorem holds

$$\Rightarrow$$

- "nice" degree n automorphic L-functions are modular, *i.e.* they are associated to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$

The theory or Artin L-functions $L(\rho, s)$ associated to degree n representations ρ of $G_{\mathbb{Q}} := Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ (and their conjectural theory) has suggested

Langlands' Conjecture (1967)

$$\{\rho : G_{\mathbb{Q}} \to GL_n(\mathbb{C})\} \subset \{\pi | \text{autom.rep. of } GL_n(\mathbb{A})\}$$

$$\text{s.t.} \quad L(\rho, s) = L(\pi, s)$$

modularity of Galois representations

there is a local version of this conjecture

In fact the local version (now a theorem!) can be stated very precisely, modulo replacing the local Galois group $G_{\mathbb{Q}_v}$ by the (local) Weil and Deligne groups

$$G_{\mathbb{Q}_v} \rightsquigarrow W_{\mathbb{Q}_v},\ W'_{\mathbb{Q}_v}$$

[Harris-Taylor, Henniart 1996-98] there is a 1-1 correspondence satisfying certain natural compatibilities (e.g. compatibility with local functional equations and preservation of L and epsilon factors of pairs)

$$\{\rho_v : W'_{\mathbb{Q}_v} \to GL_n(\mathbb{C}) : \text{admissible}\} \leftrightarrow$$

$$\leftrightarrow \{\pi \ : \ \text{irred.admiss rep of } GL_n(\mathbb{Q}_v)\}$$

Conclusion: local Galois representations are modular!

Global Modularity?

There is a global version of the Weil group $W_{\mathbb{Q}}$ but there is no definition for a global Weil-Deligne group (the conjectural Langlands group)
At the moment there is a conjectural re-interpretation of it: an "avatar" of this global modularity:

Global (local) functoriality...

BIBLIOGRAPHY

- J. Neukirch Algebraic Number Theory, Springer

- Manin, Panchishkin Introduction to Modern Number Theory, Springer

II.

Zeta functions of schemes and motivic L-functions

K. Consani – Johns Hopkins University

Vanderbilt University, May 2006

(1) Zeta functions of schemes over finite fields

(2) Zeta functions of arithmetic schemes

(3) Motivic L-functions

(1) **Zeta functions of schemes over finite fields**

$X_{/k}$ scheme of finite type, $\quad k = \mathbb{F}_q$ finite field

Main Example

$$X = \{\underline{a} = (a_1, \ldots, a_n) \in k^n : f_i(\underline{a}) = 0,\ i = 1, \ldots r\}$$

$$f_i(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$$

affine variety $\quad$ (e.g. $X = \mathbb{A}^n_k$)

$\underline{a} = (a_1, \ldots, a_n) \in k^n,\ f_i(\underline{a}) = 0\ \forall i \quad$ k-**rational point**

$$X(k) := \{x \in X : \ x = (a_1, \ldots, a_n) \in k^n\}, \quad (|\mathbb{A}^n_k(k)| = q^n)$$

More in general: $X \to Spec(k), \quad d \in \mathbb{N}$

$$\boxed{X(\mathbb{F}_{q^d}) := Mor_k(Spec(\mathbb{F}_{q^d}), X)} \quad \mathbb{F}_{q^d}\text{-}\textbf{rational point}$$

Fact: $k_d := \mathbb{F}_{q^d}, \quad N_d := |X(\mathbb{F}_{q^d})| < \infty$

$$\bar{X} := \{x \in X : \kappa(x)/k \text{ finite}\}, \quad \kappa(x) = \text{residue field}$$

$$N(x) := \#\kappa(x) = q^{\deg(x)}, \quad \deg(x) := [\kappa(x) : k]$$

$$n_l := \#\{x \in \bar{X} : \deg(x) = l\} < \infty, \quad N_d = \sum_{l|d} l n_l$$

$N_d = |X(k_d)|$ Diophantine invariant of $X_{/k}$

$$\boxed{Z(X_{/k}, T) := \exp(\sum_{d \geq 1} N_d \frac{T^d}{d}) \in \mathbb{Q}[[T]]}$$

Zeta-function of $X_{/k}$

$s \in \mathbb{C}$, $\boxed{\zeta_X(s) := Z(X_{/k}, q^{-s})}$ **Hasse-Weil zeta**

carries the "complete package" of the Diophantine information associated to the set $\{N_d : d \in \mathbb{N}\}$

Examples

1) $\mathbb{P}^1_{/\mathbb{F}_q}$, $\quad N_d = q^d + 1$

$$Z(\mathbb{P}^1, T) = \exp(\textstyle\sum_{d\geq 1}(q^d+1)\frac{T^d}{d}) = \frac{1}{(1-qT)(1-T)} \in \mathbb{Q}(T)$$

$$\zeta_{\mathbb{P}^1}(s) = (1-q^{-s})^{-1}(1-q^{-(s-1)})^{-1}$$

2) $\mathbb{P}^m_{/\mathbb{F}_q}$, $\quad N_d = \dfrac{q^{d(m+1)}-1}{q^d-1} = q^{md} + \ldots + q^{2d} + q^d + 1$

$$Z(\mathbb{P}^m, T) = \frac{1}{(1-q^mT)\cdots(1-qT)(1-T)} \in \mathbb{Q}(T)$$

$$\zeta_{\mathbb{P}^m}(s) = \prod_{n=0}^{m}(1-q^{-(s-n)})^{-1}$$

3) $\mathbb{A}^m_{/\mathbb{F}_q}$, $\quad N_d = q^{md}$

$$Z(\mathbb{A}^m, T) = \exp(\textstyle\sum_{d\geq 1} q^{md}\ \frac{T^d}{d}) = (1-q^mT)^{-1} \in \mathbb{Q}(T)$$

$$\zeta_{\mathbb{A}^m}(s) = (1-q^{-(s-m)})^{-1}$$

Main Facts

$$(1) \quad Z(X_{/k}, T) = \prod_{x \in \bar{X}} (1 - T^{deg(x)})^{-1}$$

absolutely convergent in $Re(s) > \dim X$

(2) **Theorem** [Dwork, Grothendieck 1959-64] The zeta function of a scheme of finite type over a finite field **is rational**

$$Z(X, T) = \frac{\prod_i (1 - \alpha_i T)}{\prod_j (1 - \beta_j T)} \in \mathbb{Q}(T), \qquad \alpha_i, \beta_j \in \mathbb{C}$$

$$F : X(\bar{k}) \to X(\bar{k}), \quad F(\underline{a}) = \underline{a}^q \quad \underline{a} = (a_i), \ a_i \in \bar{k}$$

Frobenius morphism

$N_d = \sharp\{x \in X(\bar{k}) : \ F^d(\underline{a}) = \underline{a}\}$ fixed points of F^d

$\underline{a} =$ description in local coordinates of x

Theorem [Grothendieck 1964] $X_{/k}$ scheme of finite type, smooth and proper over $k = \mathbb{F}_q$

$$N_d = \sum_{i=0}^{2\dim X} (-1)^i Tr((F^d)^*; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)) \qquad \Rightarrow$$

$$\boxed{Z(X_{/k}, q^{-s}) = \prod_{i=0}^{2\dim X} \det(1 - F^* q^{-s}; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}}$$

in $\mathbb{Q}_\ell[[q^{-s}]]$, $X_{\bar{k}} := X \times_k \bar{k}$, $(\ell, q) = 1$, $\ell =$ prime

in 1964 it was <u>not known</u> in general (although expected) that

$$\det(1 - F^* q^{-s}; H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Q}[q^{-s}]$$

independently of the auxiliary choice of the prime ℓ

Theorem [Deligne 1974] Assume $X_{/k}$ is <u>smooth</u>, and <u>proper</u> ($\dim X = m$)

$$(1) \quad Z(X_{/k}, T) = \frac{P_1(T) \cdots P_{2m-1}(T)}{P_0(T) \cdots P_{2m}(T)} \quad \text{in} \quad \mathbb{Q}(T)$$

$$P_i(T) := \det(1 - F^* T; H^i(X_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Q}[T]$$

<u>In particular</u>

$$P_0(T) = 1 - T, \quad P_{2m}(T) = 1 - q^m T$$

(**2**) (functional equations)

$$P_{2m-i}(T) = (-1)^{B_i} \frac{q^{mB_i} T^{B_i}}{\det(F^*; H^i_{et})} P_i(\frac{1}{q^m T})$$

$$B_i := \dim H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$$

$$Z(\frac{1}{q^m T}) = \pm q^{mE/2} T^E Z(T), \qquad E := \sum (-1)^i B_i$$

(**3**) <u>**Riemann Hypothesis**</u>

$$P_i(T) = \prod_j (1 - \alpha_{i_j} T) \in \mathbb{Z}[T], \quad \alpha_{i_j} \in \bar{\mathbb{Q}}, \quad |\alpha_{i_j}| = q^{i/2}$$

<u>Example</u>

$E_{/k}$ smooth, proper elliptic curve

$$Z(E, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}, \quad \text{in } \mathbb{Q}(T)$$

$$1 - aT + qT^2 = (1 - \alpha_{1_1} T)(1 - \alpha_{1_2} T), \quad |\alpha_{1_i}| = q^{1/2}$$

$$a = \alpha_{1_1} + \alpha_{1_2} = Tr(F^*; H^1_{et}(E_{\bar{k}}, \mathbb{Q}_\ell)) \in \mathbb{Z}$$

(2) **Zeta functions of arithmetic schemes**

$X \to Spec(\mathbb{Z})$ scheme separated and of finite type

$\bar{X}(=|X|) = \{x \in X : \kappa(x) \text{ finite}\}, \quad N(x) = |\kappa(x)|$

$s \in \mathbb{C}$, $$\boxed{\zeta_X(s) := \prod_{x \in \bar{X}} (1 - N(x)^{-s})^{-1}}$$

Hasse-Weil Zeta function of X

Examples

1) $X = Spec(\mathbb{Z})$, $\zeta_X(s) = \prod_p (1 - p^{-s})^{-1} = \zeta(s)$

2) $X = Spec(\mathbb{Z}[T_1, \ldots, T_n]) = \mathbb{A}^n_{\mathbb{Z}}$

$$\zeta_X(s) = \prod_p (1 - p^{-(s-n)})^{-1} = \zeta(s-n)$$

3) $X = \mathbb{P}^n_{\mathbb{Z}}$

$$\zeta_X(s) = \prod_p \prod_{m=0}^{n} (1 - p^{-(s-m)})^{-1} = \prod_{m=0}^{n} \zeta(s-m)$$

4) $X = Spec(\mathcal{O}_K)$, $\mathcal{O}_K$ = ring of integers of $K/\mathbb{Q}$ number field

$$\zeta_X(s) = \zeta_K(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} (1 - N(\mathfrak{p})^{-s})^{-1} \text{ Dedekind zeta}$$

Question on the asymptotic distribution of closed points on X (*i.e.* $x \in \bar{X}$) can be translated into analytic questions about $\zeta_X(s)$

<u>Fact</u> $\zeta_X(s)$ is <u>absolutely convergent</u> (holomorphic) in $Re(s) > \dim X$

<u>Expected</u>: $\zeta_X(s)$ has a meromorphic continuation to $\mathbb{C}$ and a functional equation (once suitably completed)

More in general, consider

$$X \xrightarrow{\pi} Spec(\mathcal{O}_K), \quad \pi = \text{proper}$$

irreducible, arithmetic scheme, K = number field

$$|X| = \coprod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p}\text{prime}}} |X_\mathfrak{p}|, \qquad X_\mathfrak{p} := X \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{p})$$

$$\zeta_X(s) = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p}\text{prime}}} \zeta_{X_\mathfrak{p}}(s), \quad Re(s) > \dim X$$

<u>Assume</u>: $X_K := X \times_{Spec(\mathcal{O}_K)} Spec(K)$ (generic fiber)

is smooth and proper (dim $X_K = m$)

Known: $X_\mathfrak{p}$ is smooth and proper for almost all $\mathfrak{p}$ (*i.e.* all $\mathfrak{p}$ except a finite number)

$$\zeta_X(s) = \prod_{i=0}^{2m} L_i(X,s)^{(-1)^{i+1}}$$

$$\boxed{L_i(X,s) := \prod_{\substack{\mathfrak{p} \\ X_\mathfrak{p} \text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} \times L_i^{\text{(bad)}}(X,s)}$$

<u>FACT</u>: $\prod_{\substack{\mathfrak{p} \\ X_\mathfrak{p} \text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1}$ depends <u>only</u> on X_K

$L_i^{\text{(bad)}}(X,s)$ depends also on X (the "geometric model" of X_K)

$$\boxed{P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s}) := \det(1 - F_\mathfrak{p}^* N(\mathfrak{p})^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell))}$$

$X_{\bar{\mathfrak{p}}} := X_\mathfrak{p} \times_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})}$, $q = N(\mathfrak{p})$, $F_\mathfrak{p}^{-1} \in Gal(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$

▶ $P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s}) = \det(1 - F_\mathfrak{p}^* N(\mathfrak{p})^{-s}; H^i(X_{\bar{\mathfrak{p}}}, \mathbb{Q}_\ell))$

because of the base-change theorem in étale cohomology

▶ [Deligne] $\prod_{\substack{\mathfrak{p} \\ X_\mathfrak{p} \text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} = L(\rho_{X,i}, s)$

$$\rho_{X,i} : G_K \to Aut(H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)) \quad G_K = Gal(\bar{K}/K)$$

$$L(\rho_{X,i}, s) := \prod_{v \notin S} P_{v,\rho}((Nv)^{-s})^{-1}$$ **Artin L-series**

$$P_{v,\rho}((Nv)^{-s}) := \det(1 - F^*_{v,\rho} N(v)^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell))$$

$$F^{-1}_{v,\rho} \in G_{k(v)} \cong D_w/I_w, \quad w|v, \quad \mathfrak{p} = \mathfrak{p}_v$$

$v \in \Sigma_K$ classes of normalized valutations of K

$S \subset \Sigma_K,\ S = \{v : X_{\mathfrak{p}_v} \text{ not smooth}\} \cup \{v : \text{archim}\} \cup \{w|\ell\}$

$\rho_{X,i}$ factors through $G_{k(v)} =< F_{\mathfrak{p}_v} >$

▶ [Deligne] The conjugacy classes $\{F_{v,\rho}\}$ describe a system of (local) Galois reprentations which defines $\rho_{X,i}$

Because the infinite product

$$\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_K \\ X_\mathfrak{p}\text{smooth}}} P_{i,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1}$$

is known to have in some cases (e.g. abelian varieties with CM) meromorphic continuation to $\mathbb{C}$ and functional equation, if completed at the bad and at the archimedean primes

▶ One is led to study $L_i(X,s)$ "per se" as a function associated to $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$: the ℓ-adic realization of the (pure) motive $h^i(X_K)$

▶ The definition of the Euler's factors at the places $\mathfrak{p}$ of bad reduction for X (*i.e.* where $X_\mathfrak{p}$ is not smooth) is deduced by analogy with the case of a scheme defined over a global field of positive characteristic

Main Point (Analogy with the function field case)

$Y_{/\mathbb{F}_q}$ smooth, projective curve, $\quad K(Y) = K$

$$X \xrightarrow{\pi} Spec(K), \quad Spec(K) \overset{j}{\hookrightarrow} Y$$

$$\mathcal{F} := j_* R^i \pi_* \mathbb{Q}_\ell = j_* H^i(X_{\bar{K}}, \mathbb{Q}_\ell), \quad (\ell, q) = 1$$

$$y \in |Y|, \quad \mathcal{F}_{\bar{y}} = H^i(X_{\bar{K}}, \mathbb{Q}_\ell)^{I_y} \cong H^i(X_{\bar{K}_y}, \mathbb{Q}_\ell)^{I_y}$$

$\bar{K}_y =$ completion of K at y, $\quad I_y \subset G_{K_y}$ inertia group

$$L_i(X,s) = \prod_{y\in|Y|} \det(1 - F_y^* N(y)^{-s}; H^i(X_{\bar{K}_y}, \mathbb{Q}_\ell)^{I_y})^{-1}$$

$$\zeta_Y(\mathcal{F},s) = \prod_{i=0}^{2} \det(1 - F_y^* N(y)^{-s}; H^i(Y_{\bar{\mathbb{F}}_q}, \mathcal{F}))^{(-1)^{i+1}}$$

has functional equation (as $Y_{/\mathbb{F}_q}$ is smooth and proper)

This result suggests to define in the number-field case $L_i^{(bad)}(X,s)$ as a product of local factors such as

$$\boxed{P_{i,\mathfrak{p}}^{(bad)}(X, N(\mathfrak{p})^{-s}) := \det(1 - F_{\mathfrak{p}} N(\mathfrak{p})^{-s}; H^i(X_{\bar{K}}, \mathbb{Q}_\ell)^{I_{\mathfrak{p}}})^{-1}}$$

and assuming that the coefficients belong to $\mathbb{Q}$ and are independent of ℓ

Example $X_{/K}$ algebraic curve, $K =$ number-field, $g(X) = g$

$$H^1_{et}(X_{\bar{K}}, \mathbb{Q}_\ell) \simeq T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell =: V_\ell(J) \simeq \mathbb{Q}_\ell^{2g}$$

Tate's module of the Jacobian $J = Jac(X)$ of X

$$T_\ell(X) := \lim_m Ker(J \xrightarrow{\ell^m} J) \simeq \mathbb{Z}_\ell^{2g}$$

$$L_1(X,s) = \prod_{\mathfrak{p}} P_{1,\mathfrak{p}}(X, N(\mathfrak{p})^{-s})^{-1} \quad \text{L-function of } X$$

$$L_0(X,s) = \zeta_K(s), \quad L_2(X,s) = \zeta_K(s-1)$$

Cohomology classes are represented by cocycles (cells for CW complexes)

Grothendieck conjectured that an analogue of the CW-decomposition should exist for any algebraic scheme.

The factorization of the zeta-function

$$\zeta_X(s) = \prod_{i=0}^{2m} L_i(X,s)^{(-1)^{i+1}}$$

should then be interpreted as an arithmetic manifestation of a decomposition, holding at the level of the geometric space, into more general types of "cells":

the motives $h^i(X)$

$h^i(X)$ are no longer algebraic schemes but elements of a suitable **abelian category** constructed by enlarging the category of smooth, projective schemes over K

(3) **Motivic L-functions**

$K,\ E =$ number fields

$\mathcal{M}_K(E) =$ category of (pure, mixed) motives over K with coefficients in E, endowed with **realization functors**

$$H^*_{\mathcal{H}} : \mathcal{M}_K(E) \to Vect_E$$

these functors describe the realizations of a motive M in a (Weil) cohomology theory with coefficients in E: $H^*_{\mathcal{H}}(M, E)$

Example

$H^*_{et,\ell}(M) = H^*_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$, $X_{/K} =$ smooth, projective K-scheme

ℓ-adic realization, ℓ prime number

$\mathfrak{p}|p$ prime ideal in K, $[K_\mathfrak{p} : \mathbb{Q}_p] < \infty$

$$I_\mathfrak{p} \subset G_{K_\mathfrak{p}}, \quad \varphi_\mathfrak{p} \in G_{K_\mathfrak{p}}/I_\mathfrak{p}, \quad \varphi_\mathfrak{p}(x) = x^{N(\mathfrak{p})},\ F_\mathfrak{p} = \varphi_\mathfrak{p}^{-1}$$

$\underline{\text{Fix}}\quad \ell \neq p, \quad \iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$

$E \otimes \mathbb{C} \simeq \mathbb{C}^{Hom(E,\mathbb{C})}$, consider the functor

$$\mathcal{M}_{K_\mathfrak{p}}(E) \to F_\mathfrak{p}Mod_{E\otimes\mathbb{C}}$$

$F_\mathfrak{p}Mod_{E\otimes\mathbb{C}}$ = category of $(E\otimes\mathbb{C})[F_\mathfrak{p}]$-modules of finite rank over $E\otimes\mathbb{C}$

$$M \mapsto M^I_{\ell,\iota} := (M^{I_\mathfrak{p}}_{\ell,\iota,\sigma})_{\sigma\in Hom(E,\mathbb{C})}$$

$$M^{I_\mathfrak{p}}_{\ell,\iota,\sigma} = M^{I_\mathfrak{p}}_{\ell,\iota} \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C}, \;\; M^{I_\mathfrak{p}}_{\ell,\iota} = H^*_{et}(M_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)^{I_\mathfrak{p}} \otimes_{\mathbb{Q}_{\ell,\iota}} \mathbb{C}$$

Expected These functors are isomorphic for different choices of ℓ and ι

- This is in fact the case if $M = h(X_{K_\mathfrak{p}})$, and $X_{K_\mathfrak{p}}$ is smooth, projective with good reduction (at $\mathfrak{p}$):

$$H^*_{et}(M_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell)^{I_\mathfrak{p}} = H^*_{et}(X_{\bar{K}_\mathfrak{p}}, \mathbb{Q}_\ell) \qquad E = \mathbb{Q}$$

In general

$$L_{\mathfrak{p}}(M,s) := (\text{det}_{\mathbb{C}}(1 - F_{\mathfrak{p}} N(\mathfrak{p})^{-s}; M_{\ell,\iota,\sigma}^{I_{\mathfrak{p}}})^{-1})_{\sigma \in Hom(E,\mathbb{C})}$$

$$L_{\mathfrak{p}}(M,s) = (L_{\mathfrak{p}}(M,\sigma,s))_{\sigma \in Hom(E,\mathbb{C})}$$

Expected to be independent of ℓ and ι

If K is a number field, M_K a motive over K (with coefficients in E)

$M_{K_{\mathfrak{p}}} := M \otimes_K K_{\mathfrak{p}}$ is a motive over the local field $K_{\mathfrak{p}}$

$$L(M,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M_{\mathfrak{p}},s)$$

expected to be independent of ℓ, ι

To state the convergency properties of the motivic L-function

consider the integer $w_m :=$ largest weight of M

Example

$w_m = 2n$, $X_{/K}$ = smooth projective algebraic variety, $\dim X = n$, $M = h(X)$

$$\prod_{\mathfrak{p}} L_{\mathfrak{p}}(M_{\mathfrak{p}}, s)$$

FACT: this function converges absolutely in $Re(s) > \frac{w_m}{2} + 1 = n + 1$

Expected $L(M, s)$ has meromorphic continuation to $\mathbb{C}$ with functional equation holding for the complete L-function

$$\hat{L}(M, s) := L(M, s) \cdot L_{\infty}(M, s)$$

The Archimedean factors $L_\infty(M,s)$

$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

$L_\infty(M,s)$ depends on the isomorphic class of the Betti realization

$$H^m_B(M)\otimes\mathbb{C}$$

of the motive, endowed with the Hodge decomposition and an involution F_∞

Conjecture the completed motivic L-function $\hat{L}(M,s)$ has a meromorphic continuation to $\mathbb{C}$ and

(functional eq) $\hat{L}(M,s) = \epsilon(M,s)\hat{L}(M^*,1-s)$

$M^*=$ dual motive, $\epsilon(M,s)=$ epsilon factor

▶ In all cases where the conjecture has been verified, the proof runs through the identification of $\widehat{L}(M,s)$ with an automorphic L-series!

If M is a pure, geometric motive of weight i, then $M^* \simeq M(i)$ and the (expected) functional equation is

$$\widehat{L}(M,s) = \epsilon(M,s)\widehat{L}(M,i+1-s)$$

Main Conjecture the zeroes of $\widehat{L}(M,s)$ lie on the line

$$Re(s) = \frac{i+1}{2}$$

BIBLIOGRAPHY

• Manin, Panchishkin Introduction to Modern Number Theory, Springer

• Deninger L-functions of mixed motives, in Motives I, Proceedings of Symposia in Pure Math. 55

• Deligne, Valeurs de fonctions L et périodes d'intégrales, Proc. Symposia Pure Math 33 (2)

III.

Archimedean factors of L-functions of geometric motives Lefschetz trace formulas

K. Consani – Johns Hopkins University

Vanderbilt University, May 2006

$K =$ number field, $M =$ geometric pure motive over K

with realizations: $H^m_{\mathcal{H}}(M)$

$\Sigma_K =$ set of places of K; $v \in \Sigma_K^{ar}$ $v : K_v \to \mathbb{C}$

Examples

- $H^m_B(M_v) = H^m(X(\mathbb{C}), \mathbb{Q})$, $v \in \Sigma_K^{ar}$

- $M^m_{et}(M_v) = H^m_{et}(X_{\bar{K}_v}, \mathbb{Q}_\ell)$, $v \in \Sigma_K^{nar}$

▶ the motive M is realized by the family of all (Weil) cohomological theories associated to a scheme X of finite type over K

$$H^m_B(M_v) \otimes \mathbb{C} = \bigoplus_{\substack{p+q=m \\ p,q \geq 0}} H^{p,q}, \quad h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$$

$$H^{p,p} = H^{p,+} \oplus H^{p,-}$$

$$H^{p,+} := \{\mathrm{v} \in H^{p,p} : F_\infty(\mathrm{v}) = (-1)^p \mathrm{v}\}$$

$F_\infty = \mathbb{C}$-linear involution induced by the complex conjugation on $X(\mathbb{C})$

$$h^{p\pm} := \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$$

$u \in \mathbb{C}, \quad L_{\mathbb{C}}(u) = 2(2\pi)^{-u}\Gamma(u), \quad L_{\mathbb{R}}(u) = \pi^{-\frac{u}{2}}\Gamma(\frac{u}{2})$

(Legendre's formula) $\quad L_{\mathbb{R}}(u)L_{\mathbb{R}}(u+1) = L_{\mathbb{C}}(u)$

$$L_\infty(M,u) := \prod_{v|\infty} L_v(M,u), \qquad L_v(M,u) =$$

$$\begin{cases} L_{\mathbb{C}}(M_v,u) = \prod_{p+q=m} L_{\mathbb{C}}(u - \text{min(p,q)})^{h^{p,q}}; & \text{v complex} \\ L_{\mathbb{R}}(M_v,u) = \prod_p L_{\mathbb{R}}(u-p)^{h^{p,+}} L_{\mathbb{R}}(u-p+1)^{h^{p,-}} \prod_{p<q} L_{\mathbb{C}}(u-p)^{h^{p,q}} \end{cases}$$

Archimedean factor attached to M

Assume:

(1) $\quad L(M,u) := \prod_v L_v(M,u) =$

$\prod_{v<\infty} \det(1 - F_v N(v)^{-u}; H^m(X_{\bar{K}_v}, \mathbb{Q}_\ell)^{I_v})^{-1} \times L_\infty(M,u)$

converges absolutely in $Re(u) > \frac{m}{2} + 1$

(2) $\quad L(M,u)$ has meromorphic continuation to $\mathbb{C}$

$\widehat{L}(M,u) := L(M,u) \cdot L_\infty(M,u)$ satisfies functional eq

(3) $\quad \widehat{L}(M,u) = \epsilon(M,u) \cdot \widehat{L}(M, m+1-u)$

Then

▶ The location and the multiplicity of the zeroes of

$L(M,u)$ in $Re(u) < \frac{m}{2}$

are determined by the poles of $L_\infty(M,s)$

(thanks to the functional equation)

The Γ-function has simple poles at $u = -n$ ($n \in \mathbb{Z}_{\geq 0}$)
$\Rightarrow$

▶ the multiplicities of the zeroes of $L(M,u)$ in $Re(u) < \frac{m}{2}$ must depend on the Hodge structure of M.

Assume(for simplicity): $K = \mathbb{Q}$

Fact: The poles of $L_\infty(M,u)$ at $Re(u) = n \leq \frac{m}{2}$.

have multiplicities

$$\nu_{m,n} := \begin{cases} \sum_{p<q} h^{p,q} & \text{m odd} \\ \sum_{n \leq p < q} h^{p,q} + h^{\frac{m}{2},(-1)^{n-\frac{m}{2}}} & \text{m even} \end{cases}$$

are described by the difference

$$\nu_{m,n} = \dim_{\mathbb{C}} H^m(X(\mathbb{C}), \mathbb{R}(m-n))^{(-1)^{m-n}} - \dim_{\mathbb{C}} F^{m+1-n} H^m_{dR}(X_{/\mathbb{R}})$$

Main Facts ($K = \mathbb{Q}$)

$$\boxed{\nu_{m,n} = \dim_{\mathbb{R}} H^{m+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(m+1-n))}$$

$$0 \to F^{m+1-n} H^m_{dR}(X_{/\mathbb{R}}) \xrightarrow{\alpha} H^m(X(\mathbb{C}), \mathbb{R}(m-n))^{(-1)^{m-n}} \to$$
$$\to H^{m+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(m+1-n)) \to 0$$

$$\Rightarrow$$

$$\boxed{H^{m+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(m+1-n)) = Coker(\alpha)} \qquad !!$$

$$H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) := H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))^{DR}, \qquad i \geq 0$$

$DR =$ deRham conjugation *i.e.* $\mathbb{R}$-linear involution induced by the complex conjugation on $(X(\mathbb{C}), \Omega^{\cdot})$

$$H^i_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) := \mathbb{H}^i(\mathbb{R}(p)_{\mathcal{D}} : \mathbb{R}(p) \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0)$$

$$0 \to \Omega^{\cdot}_{<p}[-1] \to \mathbb{R}(p)_{\mathcal{D}} \to \mathbb{R}(p) \to 0$$

Desirable to have a description of the formulae of the local factors so that the archimedean and the non-archimedean cases are treated on equal footing: *i.e.* similar definition

keep in mind the similarity in functorial behavior of the Euler factors of the Artin L-functions

2 approaches to this problem

1) [Deninger 1991, Consani 1996] the archimedean local factor is interpreted using the definition of an infinite determinant for the action of a (logarithm of) suitable archimedean Frobenius operator on an infinite-dimensional $\mathbb{R}$-vector space

[Deninger] $v|\infty$ $L_v(M,u) = det_\infty(\frac{u}{2\pi} - \frac{\theta}{2\pi}; H^m_{ar}(M_v))^{-1}$

$$H^m_{ar}(M_v) := \begin{cases} Fil^0(H^m_B(M_v) \otimes_{\mathbb{C}} B_{ar})^{c=id} & v \text{ complex} \\ Fil^0(H^m_B(M_v) \otimes_{\mathbb{C}} B_{ar})^{c=id, F_\infty=1} & v \text{ real} \end{cases}$$

$B_{ar} \cong \mathbb{C}[T, T^{-1}]$, $c(H^{p,q}) = H^{q,p}$ conjugate linear inv

c induced by complex conj on $\mathbb{C}$, $F_\infty = \mathbb{C}$-linear inv

For example: if $H^m_B(M_v) = H^{p,p}$

$$L_v(M,u)^{-1} = [\amalg_{\nu=0}^{\infty}(\frac{u}{2\pi} - \frac{p-2\nu}{2\pi})]^{h^{p,+}}[\amalg_{\nu=0}^{\infty}(\frac{u}{2\pi} - \frac{p-1-2\nu}{2\pi})]^{h^{p,-}}$$

[Consani] $v|\infty, \quad L_v(M,u) =$

$$= \begin{cases} det_\infty(\frac{u}{2\pi} - \frac{\Phi}{2\pi}; H^m(\tilde{X}^*_{\bar{K}})^{N=0})^{-1}, & v \text{ complex} \\ \\ det_\infty(\frac{u}{2\pi} - \frac{\Phi}{2\pi}; H^m(\tilde{X}^*_{\bar{K}})^{N=0,\bar{F}_\infty=1})^{-1}, & v \text{ real} \end{cases}$$

$H^m(\tilde{X}^*_{\bar{K}})^{N=0}$ <u>archimedean inertia invariants</u>

$H^m(\tilde{X}^*_{\bar{K}})$ infinite dim. graded $\mathbb{R}$-vector space associated to the nearby-fiber in an infinitesimal neighborhood of the fiber over v

Φ = multiplication by the (pure) weight associated to each graded piece of $H^m(\tilde{X}^*_{\bar{K}})^{N=0}$

2) [Connes-Consani-Marcolli 2005] Reinterpret the archimedean local factors through a semi-local **trace formula** over a

(non-commutative) generalization of the motive M:

an "extension" of M by a <u>suitable modification</u> of the space of adeles $\mathbb{A}_K$, by replacing the local field K_v, with a division algebra, at each <u>real</u> archimedean place $v \in \Sigma_K$

Recall:

$F : V \to V$ endomorphism of a v. space V

$$\boxed{T\tfrac{d}{dT}\log(\det(1-FT;V)^{-1}) = \sum_{n\geq 0} Tr(F^n;V)T^n}$$

IF: $X_{/\mathbb{F}_q}$, $\quad Z(X,T) = \prod_{x\in|X|}(1-T^{deg(x)})^{-1}$

$$T\tfrac{d}{dT}\log Z(X,T) = \sum_n \sum_m (-1)^m Tr((F^*)^n; H^m_{et}(X,\mathbb{Q}_\ell))T^n$$

Seek for a similar formula at the archimedean places

$$H^m_B(M_v)\otimes\mathbb{C} = \bigoplus_{p+q=m} H^{p,q}(M_v), \quad c = \text{complex conj on } \mathbb{C}$$

$$(1\otimes c)(H^{p,q}(M_v)) = H^{q,p}(M_v)$$

$\bar{v} = c\circ v : K_v \to \mathbb{C}$, conjugate to $v : K_v \to \mathbb{C}$

by transport of structure $\exists\ \tau : H^m_B(M_v) \xrightarrow{\simeq} H^m_B(M_{\bar{v}})$

s.t. $(\tau\otimes c)$ preserves bigrading on $H^m_B\otimes\mathbb{C}$

$\Rightarrow$

$F_\infty := (\tau\otimes 1) : H^{p,q}(M_v) \xrightarrow{\simeq} H^{q,p}(M_{\bar{v}})$ $\mathbb{C}$-linear involution

(Local) Weil group action

1 case $\quad K_v$ complex (local) field

$v : K_v \overset{\simeq}{\to} \mathbb{C} \overset{\simeq}{\leftarrow} K_{\bar{v}} : \bar{v}$ isomorphisms

$W_{K_v} := \mathbb{C}^\times$ local Weil group

$$\boxed{\pi(H_B^m(M_v), u)\xi = u^{-p}\bar{u}^{-q}\xi}, \quad u \in \mathbb{C}^\times, \ \xi \in H^{p,q}(M_v)$$

$$\pi((\tau \otimes 1)(H^{p,q}(M_v)), u)\xi = (\tau \otimes 1) \circ \pi(H^{p,q}(M_v), u)\xi$$

i.e. $F_\infty = (\tau \otimes 1)$ is W_{K_v}-equivariant

$$\Rightarrow$$

$\pi(H_B^m(M_v)) \simeq \pi(H_B^m(M_{\bar{v}}))$ as representations of $\mathbb{C}^\times$

2 case $\quad K_v$ real (local) field

i.e. $\quad v = \bar{v} : K_v \to \mathbb{C}$

$$M_v = M_{\bar{v}}, \quad \pi(H_B^m(M_v)) = \pi(H_B^m(M_{\bar{v}}))$$

$F_\infty : M_v \overset{\simeq}{\to} M_{\bar{v}}$ involution (automorphism)

$$W_{K_v} := \mathbb{C}^\times \cup j\mathbb{C}^\times, \quad W_{K_v} = \text{normalizer of } \mathbb{C}^\times \text{ in } \mathbb{H}^*$$

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \quad \text{quaternion division algebra}$$

Rules $\quad j^2 = -1, \quad juj^{-1} = \bar{u}, \quad \forall u \in \mathbb{C}$

$$w = uj^e \in W_{K_v}, \quad u \in \mathbb{C}^\times, \quad e \in \{0, 1\}$$

$$\boxed{\pi(H_B^m(M_v), uj)\xi := i^{p+q} u^{-p} \bar{u}^{-q} F_\infty(\xi)}, \quad \xi \in H^{p,q}(M_v)$$

$$\pi(H_B^m(M_v), j)^2 = \pi(H_B^m(M_v), -1)$$

$$\pi(H^m), j)\pi(H^m, u) = \pi(H^m, \bar{u})\pi(H^m, j)$$

Trace formulas for the action of W_{K_v}

(with A. Connes & M. Marcolli)

Theorem 1 $K_v = \mathbb{C}$, $Re(z) = \frac{m+1}{2}$ (critical line)

$$\mathbb{C} \ni z = \tfrac{m+1}{2} + is, \quad s \in \mathbb{R}, \quad u \in \mathbb{C}^\times$$

$$\int'_{W_{K_v} = \mathbb{C}^\times} \frac{Tr(\pi(H^m(M_v), u)|u|_{\mathbb{C}}^z}{|1-u|_{\mathbb{C}}}\, d^\times u = -2 \frac{d}{ds} \Im \log L_{\mathbb{C}}(M_v, z)$$

Theorem 2 $K_v = \mathbb{R}$, $Re(z) = \frac{m+1}{2}$ (critical line)

$$z = \tfrac{m+1}{2} + is, \quad s \in \mathbb{R}, \quad w \in W_{K_v}$$

$$\int'_{W_{K_v}} \frac{Tr(\pi(H^m(M_v), w))|w|_{\mathbb{H}}^z}{|1-w|_{\mathbb{H}}}\, d^\times w = -2 \frac{d}{ds} \Im \log L_{\mathbb{R}}(M_v, z)$$

$|w|_{\mathbb{H}} = |w|_{W_{K_v}}$, $|1-w|_{\mathbb{H}}$ = reduced norm in $\mathbb{H}$

Proof of Theorem 1 follows from

Lemma 1 $\quad K_v = \mathbb{C},\ \mathbb{R}, \quad z = \frac{1}{2} + is,\ s \in \mathbb{R}$

$$\int'_{K_v^*} \frac{|u|^z}{|1-u|} d^*u = -2\frac{d}{ds}\Im \log \Gamma_{K_v}(z)$$

$\int' \cdots =$ principal value on K_v^* of the distribution

$$\frac{|u|^z}{|1-u|} \quad \text{on } K_v^*$$

$$-2\tfrac{d}{ds}\Im \log \Gamma_{K_v}(\tfrac{1}{2}+is) = -(\tfrac{\Gamma'_{K_v}}{\Gamma_{K_v}}(\tfrac{1}{2}+is) + \tfrac{\Gamma'_{K_v}}{\Gamma_{K_v}}(\tfrac{1}{2}-is)) =$$

$$= \begin{cases} 2\log(2\pi) - (\frac{\Gamma'}{\Gamma}(\frac{1}{2}+is) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}-is)), & K_v = \mathbb{C} \\ \log(\pi) - \frac{1}{2}(\frac{\Gamma'}{\Gamma}(\frac{1}{4}+i\frac{s}{2}) + \frac{\Gamma'}{\Gamma}(\frac{1}{4}-i\frac{s}{2})), & K_v = \mathbb{R} \end{cases}$$

$\Gamma_{K_v}(z)$ is a real function, *i.e.* $\Gamma_{K_v}(\bar{z}) = \overline{\Gamma_{K_v}(z)}$

Similar formula holds for

$$\frac{d}{ds}\Im \log \Gamma_{K_v}(\tfrac{1}{2} + is + \tfrac{|n|}{2}), \quad n \in \mathbb{Z} \quad !!$$

<u>Main Lemma 2</u> $K_v = \mathbb{C}$, $z = \frac{(1+m)}{2} + is$, $s \in \mathbb{R}$

$$m = p + q \geq 0$$

$$\int_{\mathbb{C}^\times}' \frac{u^{-p}\bar{u}^{-q}|u|_{\mathbb{C}}^z}{|1-u|_{\mathbb{C}}}\, d^*u = -2\frac{d}{ds}\Im \log L_{\mathbb{C}}(z - \min(p,q))$$

The shift by $\min(p,q)$ in the argument of $L_{\mathbb{C}}$ appears when one considers the principal value on $\mathbb{C}^\times$ of the distribution

$$\frac{u^{-p}\bar{u}^{-q}|u|_{\mathbb{C}}^z}{|1-u|_{\mathbb{C}}}$$

$$n := p - q, \quad \min(p,q) = \frac{m}{2} - \frac{|n|}{2} = \frac{p+q}{2} - \frac{|p-q|}{2}$$

$$|u|_{\mathbb{C}} := u\bar{u} \quad u^{-p}\bar{u}^{-q} = e^{-in\theta}|u|_{\mathbb{C}}^{-\frac{m}{2}}, \quad \theta = \arg(u)$$

The above equality can then be written in the following equivalent form

$$\int'_{C^\times} \frac{e^{-in\theta}|u|_{\mathbb{C}}^{\frac{1}{2}+is}}{|1-u|_{\mathbb{C}}}\, d^*u = -2\frac{d}{ds}\Im \log \Gamma_{\mathbb{C}}(\frac{1}{2}+is+\frac{|n|}{2})$$

$$-2\tfrac{d}{ds}\Im \log \Gamma_{\mathbb{C}}(\tfrac{1}{2}+is+\tfrac{|n|}{2}) =$$

$$2\log(2\pi) - (\tfrac{\Gamma'}{\Gamma}(\tfrac{1}{2}+is+\tfrac{|n|}{2}) + \tfrac{\Gamma'}{\Gamma}(\tfrac{1}{2}-is+\tfrac{|n|}{2}) - 2\Gamma'(1))$$

Proof of Theorem 2, when $m = p+q$ odd (resp. $m = 2p$ and $h^{p,+} = h^{p,-}$) is proven by using the same arguments as for Theorem 1 (resp. using duplication formula)

$$\Gamma_{\mathbb{R}}(z)\Gamma_{\mathbb{R}}(z+1) = \Gamma_{\mathbb{C}}(z)$$

When $m = 2p$, $h^{p,+} \neq h^{p,-}$ one refers instead to

Lemma 2 $K_v = \mathbb{R}$, $z = \frac{1}{2}+is$, $s \in \mathbb{R}$

$$\int_{\mathbb{R}_+^*} \frac{u^z}{1+u}\, d^*u = -2\frac{d}{ds}\Im \log \left(\frac{\Gamma_{\mathbb{R}}(z)}{\Gamma_{\mathbb{R}}(z+1)}\right)$$

The space on which the trace formula for $K_v = \mathbb{R}$

$$\int'_{W_{K_v}} \frac{Tr(\pi(H^m(M_v), w))|w|^z_{\mathbb{H}}}{|1-w|_{\mathbb{H}}}\ d^{\times}w = -2\frac{d}{ds}\Im \log L_{\mathbb{R}}(M, z)$$

is computed has as base $\quad \mathbf{B} = \mathbb{H} \quad$ the quaternions

thought of as a **complex manifold** (right action of $\mathbb{C}$) with a left-action by the Weil group

More precisely

for a single archimedean place the space on which the trace formula is computed is a

vector bundle E over

$$\mathbf{B} = \begin{cases} \mathbb{C} & v \text{ complex} \\ \mathbb{H} & v \text{ real} \end{cases}$$

with fiber a $\mathbb{Z}$-graded vector space

$$E = \oplus_m E^{(m)} = \oplus_m H^m_B(M_v) \ \ \& \ \ \text{repr. of } W_{K_v}$$

$$\pi_v : W_{K_v} \to Aut(E/B)$$

$$\pi_v(w)(z,\xi) = \begin{cases} (wz, w^{-p}\overline{w}^{-q}\xi) & \text{v complex} \\ (wz, i^m u^{-p}\overline{u}^{-q}F_{\infty}(\xi) & \text{v real} \end{cases}$$

$\mathcal{H} := L^2(B, E^{(m)})$ Hilbert sp of L^2-sections of $E^{(m)}$

$$\pi_v : W_{K_v} \to Aut(\mathcal{H})$$

Theorem $v =$ complex, $h \in S(\mathbb{R}_+^*)$ with compact support, view $h \in S(W_{K_v})$ by composition with the module

$$Tr(R_\Lambda \pi_v(h)) =$$

$$2h(1) B_m \log \Lambda + \int'_{W_{K_v}} \frac{h(|u|) Tr(\pi_v(H^m(M_v)))}{|1-u|_{\mathbb{C}}} \, d^*u + o(1)$$

as $\Lambda \to \infty$

$B_m =$ Betti number, $R_\Lambda = \widehat{P}_\Lambda P_\Lambda$

$P_\Lambda =$ orthogonal projection onto the subspace

$$\{\xi \in L^2(B, E^{(m)}) : \ \xi(b) = 0 \ \forall b \in B, \ |b|_{\mathbb{C}} > \Lambda\}$$

$\widehat{P}_\Lambda = F P_\Lambda F^{-1}$, $F =$ Fourier transform

Conjecture The above trace formula generalizes to the semi-local case

i.e. $v \in S \subset \Sigma_K$ finite set of archimedean places of K

$W =$ Weil group, $u \mapsto |u| \in \mathbb{R}_+^*$ module

$W_{K_v} \subset W$, $h \in S(\mathbb{R}_+^*)$, $h \in S(W)$ with compact support

$$Tr(R_\Lambda \pi(h)) =$$

$$2h(1)B_m \log \Lambda + \sum_{v \in S} \int'_{W_v^*} \frac{h(|w|) Tr(\pi_v(H^m(M_v)))}{|1-u|_{\mathbb{H}_v}} \, d^*w + o(1)$$

as $\Lambda \to \infty$

(Serre) B_m is independent of the place v

BIBLIOGRAPHY

• Serre, Facteurs locaux des fonctions zêta des variétés algébriques, Seminaire Delange-Pisot-Poitou (1969-70)

• P. Schneider, Introduction to the Beilinson Conjectures, in Beilinson's conjectures on special values of L-functions, Perspectives in mathematics

• Deninger, On the gamma-factors attached to motives, Inv. Math. 104 (1991)

• Consani, Double complexes and Euler L-factors, Compositio Math. 111 (1998)

• Connes, Consani, Marcolli, Noncommutative geometry and motives: the thermodynamics of endomotives, arXiv:math. QA/0512138