Stirling matrix via Pascal matrix

Gi-Sang Cheon a,*, Jin-Soo Kim b

a Department of Mathematics, Daejin University, Pocheon 487-711, South Korea
b Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea

Received 11 May 2000; accepted 17 November 2000

Submitted by R.A. Brualdi

Abstract

The Pascal-type matrices obtained from the Stirling numbers of the first kind \( s(n, k) \) and of the second kind \( S(n, k) \) are studied, respectively. It is shown that these matrices can be factorized by the Pascal matrices. Also the LDU-factorization of a Vandermonde matrix of the form \( V_n(x, x+1, \ldots, x+n-1) \) for any real number \( x \) is obtained. Furthermore, some well-known combinatorial identities are obtained from the matrix representation of the Stirling numbers, and these matrices are generalized in one or two variables. © 2001 Elsevier Science Inc. All rights reserved.

AMS classification: 05A19; 05A10

Keywords: Pascal matrix; Stirling number; Stirling matrix

1. Introduction

For integers \( n \) and \( k \) with \( n \geq k \geq 0 \), the Stirling numbers of the first kind \( s(n, k) \) and of the second kind \( S(n, k) \) can be defined as the coefficients in the following expansion of a variable \( x \) (see [3, pp. 271–279]):

\[
[x]_n = \sum_{k=0}^{n} (-1)^{n-k} s(n, k)x^k
\]

and

---

* Corresponding author.

E-mail addresses: gscheon@road.daejin.ac.kr (G.-S. Cheon), lion@math.sskku.ac.kr (J.-S. Kim).

0024-3795/01/$ - see front matter © 2001 Elsevier Science Inc. All rights reserved.

PII: S 0 0 2 4 - 3 7 9 5 ( 0 1 ) 0 0 2 3 4 - 8
\[ x^n = \sum_{k=0}^{n} S(n, k)[x]_k, \tag{1.1} \]

where
\[ [x]_n = \begin{cases} x(x-1) \cdots (x-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{1.2} \]

It is known that for an \( n, k \geq 0 \), the \( s(n, k) \), \( S(n, k) \) and \( [n]_k \) satisfy the following Pascal-type recurrence relations:
\[
\begin{align*}
  s(n, k) &= s(n-1, k-1) + (n-1)s(n-1, k), \\
  S(n, k) &= S(n-1, k-1) + kS(n-1, k), \\
  [n]_k &= [n-1]_k + k[n-1]_{k-1},
\end{align*} \tag{1.3}
\]

where \( s(n, 0) = s(0, k) = S(n, 0) = S(0, k) = [0]_k = 0 \) and \( s(0, 0) = S(0, 0) = 1 \), and moreover the \( S(n, k) \) satisfies the following formula known as ‘vertical’ recurrence relation:
\[
S(n, k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l, k-1). \tag{1.4}
\]

As we did for the Pascal triangle, we can define the Pascal-type matrices from the Stirling numbers of the first kind and of the second kind, respectively. A matrix representation of the Pascal triangle has catalyzed several investigations (see [1, 2, 4, 6, 7]).

The \( n \times n \) Pascal matrix [4] (also see [2]), \( P_n \), is defined by
\[
(P_n)_{ij} = \begin{cases} 
  \binom{i-1}{j-1} & \text{if } i \geq j, \\
  0 & \text{otherwise}. 
\end{cases}
\]

More generally, for a nonzero real variable \( x \), the Pascal matrix was generalized in \( P_n[x] \) and \( Q_n[x] \), respectively which are defined in [6] (also see [1]), and these generalized Pascal matrices were also extended in \( \Phi_n[x, y] \) (see [7]) for any two nonzero real variables \( x \) and \( y \) where
\[
(\Phi_n[x, y])_{ij} = \begin{cases} 
  x^{i-j} y^{i+j-2} \binom{i-1}{j-1} & \text{if } i \geq j, \\
  0 & \text{otherwise}. 
\end{cases} \tag{1.5}
\]

By the definition, we see that
\[
P_n[x] = \Phi_n[x, 1], \quad Q_n[y] = \Phi_n[1, y],
\]
\[
P_n = P_n[1] = Q_n[1] = \Phi_n[1, 1]. \tag{1.6}
\]

Moreover, it is known that
\[
P_n^{-1}[x] = P_n[-x] = \left[ (-1)^{i-j} \binom{i-1}{j-1} x^{i-j} \right], \tag{1.7}
\]
and in particular, \( P_n^{-1} = P_n^{-1} \) [1].

In [6] and [7], the factorizations of \( P_n[x] \), \( Q_n[x] \), and \( \Phi_n[x, y] \) are obtained, respectively.

In Section 2, we study the Pascal-type matrices which will be called the Stirling matrices obtained from the Stirling numbers of the first kind \( s(n, k) \) and second kind \( S(n, k) \). As a consequence it is shown that such matrices can be factorized by the Pascal matrices. Also the LDU-factorization of a Vandermonde matrix of the form \( V_n(x, x+1, \ldots, x+n-1) \) for any real number \( x \) is obtained.

In Section 3, some well-known combinatorial identities are obtained from the matrix representation of the Stirling numbers.

Finally in Section 4, these matrices are generalized in one or two variables.

2. Stirling matrices of the second kind

For the Stirling numbers \( s(i, j) \) and \( S(i, j) \) of the first kind and of the second kind respectively, define \( s_n \) and \( S_n \) to be the \( n \times n \) matrices by

\[
(s_n)_{ij} = \begin{cases} s(i, j) & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}
\]

and

\[
(S_n)_{ij} = \begin{cases} S(i, j) & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}
\]

We call the matrices \( s_n \) and \( S_n \) Stirling matrix of the first kind and of the second kind, respectively (see [5, p. 144]).

For example,

\[
s_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{bmatrix} \quad \text{and} \quad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}.
\]

From now on, we will use the notation \( \oplus \) for the direct sum of two matrices.

Using the definition of \( S_n \), we can derive the following matrix representation from (1.1):

\[
X_n = ([1] \oplus S_{n-1}) F_n,
\]

where \( X_n = [1 x \ldots x^{n-1}]^T \) and \( F_n = [[x]_0[x]_1 \ldots [x]_{n-1}]^T \).

In this section, we mainly study Stirling matrix \( S_n \) of the second kind since

\[
S_n^{-1} = [(-1)^{i-j} s(i, j)] \quad \text{or} \quad S_n^{-1} = [(-1)^{i-j} S(i, j)].
\]

(2.2)

First, we will discuss for a factorization of \( S_n \).

For the \( k \times k \) Pascal matrix \( P_k \), we define the \( n \times n \) matrix \( \tilde{P}_k \) by

\[
\tilde{P}_k = \begin{bmatrix} I_{n-k} & O \\ O & P_k \end{bmatrix}.
\]
Thus, $\bar{P}_n = P_n$ and $\bar{P}_1$ is the identity matrix of order $n$.

**Lemma 2.1.** For the $n \times n$ Pascal matrix $P_n$,

$S_n = P_n([1] \oplus S_{n-1}).$

**Proof.** For each $i$ and $j$ with $i \geq j \geq 1$, since the $(i, j)$-entry of $[1] \oplus S_{n-1}$ is $S(i - 1, j - 1)$, from the definition of the matrix product and (1.4), we get

$$
(P_n([1] \oplus S_{n-1}))_{ij} = \sum_{l=j-1}^{i-1} p_{l+1} S(l, j-1) = \sum_{l=j-1}^{i-1} \binom{i-1}{l} S(l, j-1) = S(i, j) = (S_n)_{ij}.
$$

□

For example,

$S_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.$

The following theorem is an immediate consequence of Lemma 2.1.

**Theorem 2.2.** The Stirling matrix $S_n$ of the second kind can be factorized by the Pascal matrices $\bar{P}_k$'s:

$S_n = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1.$

For example,

$S_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.$

We now turn our attention to the special matrices which can be expressed by the Stirling matrices.

It is easy to see that Lemma 2.1 and (2.1) lead to

$$(x + 1)^n = \sum_{k=0}^{n} S(n + 1, k + 1)[x]_k \tag{2.3}$$

for each $n = 0, 1, \ldots$. Thus (2.1) and (2.3) suggest how the Vandermonde matrix which is defined by the following way can be factorized.

Define $V_n(x)$ to be the $n \times n$ Vandermonde matrix by

$V_n(x) := V_n(x, x + 1, \ldots, x + n - 1)$
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x & x + 1 & \cdots & x + n - 1 \\
x^2 & (x + 1)^2 & \cdots & (x + n - 1)^2 \\
\vdots & \vdots & \ddots & \vdots \\
x^{n-1} & (x + 1)^{n-1} & \cdots & (x + n - 1)^{n-1}
\end{bmatrix},
\]

and use the definition of \([x]_n\) in (1.2) to define the \(n \times n\) matrix \(L_n\) by

\[
(L_n)_{ij} = \begin{cases} 
[i-1]_{j-1} & \text{if } i \geq j, \\
0 & \text{otherwise.}
\end{cases}
\]

For example,

\[
L_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 2 & 0 \\
1 & 3 & 6 & 6
\end{bmatrix}.
\]

By a simple computation we obtain

\[
L_n = P_n D_n, \quad (2.4)
\]

where \(D_n = \text{diag}(1, 1, 2!, \ldots, (n - 1)!).\) Thus, we have

\[
L_n^{-1} = D_n^{-1} P_n^{-1} = \begin{bmatrix} (-1)^{i-j} \frac{1}{(i-1)!} \binom{i-1}{j-1} \end{bmatrix}.
\]

Applying the binomial theorem, it is easy to show that for any real number \(x\) and for the Pascal matrix \(P_n\),

\[
P_n V_n(x) = V_n(x + 1).
\]

Thus, we have

\[
\det V_n(x) = \det V_n(x + 1).
\]

Lemma 2.3. For the \(n \times n\) Stirling matrix \(S_n\) of the second kind,

\[
V_n(1) = S_n L_n^T.
\]

Proof. Applying (1.1) and (1.3) for each \(i, j = 1, 2, \ldots, n\), we have

\[
(S_n L_n^T)_{ij} = \sum_{k=1}^{i} S(i, k) [j - 1]_{k-1}
\]

\[
= \sum_{k=1}^{i} [S(i - 1, k - 1) + k S(i - 1, k)] [j - 1]_{k-1}
\]

\[
= \sum_{k=1}^{i} [S(i - 1, k - 1)[j - 1]_{k-1} + S(i - 1, k) k [j - 1]_{k-1}]
\]
\[
\sum_{k=1}^{i} [S(i - 1, k - 1)[j - 1]_{k-1} + S(i - 1, k)\{[j]_{k} - [j - 1]_{k}\}]
\]

\[
\sum_{k=1}^{i-1} S(i - 1, k)[j]_{k}
\]

\[
j^{i-1} = (V_n(1))_{ij},
\]

since

\[
\sum_{k=1}^{i} S(i - 1, k - 1)[j - 1]_{k-1} = \sum_{k=1}^{i} S(i - 1, k)[j - 1]_{k},
\]

which completes the proof. □

In the following theorem, we obtain the LDU factorization of \(V_n(x)\) for any real number \(x\).

**Theorem 2.4.** For any real number \(x\) and the generalized Pascal matrix \(P_n[x]\) in (1.6),

\[
V_n(x) = (P_n[x - 1]S_n)D_nP_n^T.
\]

**Proof.** From (2.4) and Lemma 2.3, for any real number \(x\) we get

\[
((P_n[x - 1]S_n)L_n^T)_{ij} = (P_n[x - 1]V_n(1))_{ij}
\]

\[
= \sum_{k=0}^{i-1} \binom{i - 1}{k} (x - 1)^{i-1-k} j^k = (x + j - 1)^{i-1}
\]

\[
= (V_n(x))_{ij}. \quad \Box
\]

For example,

\[
V_4(x) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
x & x+1 & x+2 & x+3 \\
x^2 & (x+1)^2 & (x+2)^2 & (x+3)^2 \\
x^3 & (x+1)^3 & (x+2)^3 & (x+3)^3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
(x-1) & 1 & 0 & 0 \\
(x-1)^2 & 2(x-1) & 1 & 0 \\
(x-1)^3 & 3(x-1)^2 & 3(x-1) & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{bmatrix}
\]
Corollary 2.5. For any real number $x$,

$$
\det V_n(x) = \prod_{k=0}^{n-1} k!
$$

For the inverse of $V_n(x)$, from (1.7) and Theorem 2.4 we see that

$$
V_n(x)^{-1} = P_n^T [-1] D_n^{-1} S_n^{-1} P_n [1 - x].
$$

3. Some combinatorial identities

In this section, we obtain some well-known identities for a Stirling number from its matrix representation.

Applying Theorem 2.4 for $x = 1$, we obtain

$$
S_n = V_n(1) \left( P_n^{-1} \right)^T D_n^{-1}.
$$

(3.1)

Computing the matrix product in (3.1) and comparing with the last row of $S_n$, we can obtain the following representation for $S(n, k)$, known as Stirling formula:

$$
S(n, k) = \frac{1}{(k - 1)!} \sum_{t=1}^{k} (-1)^{k-t} \binom{k-1}{t-1} t^{n-1} \quad (k = 1, 2, \ldots, n).
$$

Again applying (1.7) and (2.2) to Lemma 2.1, since

$$
s_n = ([1] \oplus s_{n-1}) P_n,
$$

(3.2)

by a simple matrix product, it is easy to see that the Stirling number $s(n, k)$ of the first kind satisfies the following ‘horizontal’ recurrence relation which gives other explicit formula for the $s(n, k)$ (see [5, p. 215])
\[ s(n, k) = \sum_{l=k-1}^{n-1} \binom{l}{k-1} s(n - 1, l). \]

Moreover, from Lemma 2.1 and (3.2), since
\[ P_n = S_n([1] \oplus s_{n-1}) \quad \text{or} \quad P_n = ([1] \oplus S_{n-1})s_n, \]
a binomial coefficient \( \binom{n}{k} \) can be expressed by the Stirling numbers of the first kind and of the second kind as follows:
\[
\binom{n}{k} = \sum_{t=k}^{n} (-1)^{t-k} S(n + 1, t + 1)s(t, k)
\]
or
\[
\binom{n}{k} = \sum_{t=k}^{n} (-1)^{n-t} S(n, t)s(t + 1, k + 1).
\]

Finally, note that the Bell number \( \omega(n) \) is defined by
\[
\omega(n) = \sum_{k=1}^{n} S(n, k), \quad n \geq 1.
\]

By virtue of the matrix, the \( i \)th Bell number \( \omega(i) \) is just the sum of the entries in the \( i \)th row of the Stirling matrix \( S_n \) of the second kind. Thus, from Lemma 2.1 we get
\[
P_n \begin{bmatrix} \omega(0) & \omega(1) & \cdots & \omega(n - 1) \\ \omega(1) & \omega(2) & \cdots & \omega(n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(n - 1) & \omega(n) & \cdots & \omega(2n - 2) \end{bmatrix}^T = \begin{bmatrix} \omega(1) & \omega(2) & \cdots & \omega(n) \end{bmatrix}^T.
\]

More generally, if we note that for each \( n = 0, 1, \ldots \)
\[
\Delta^m \omega(n) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \omega(n + k) \quad (m = 0, 1, \ldots)
\]
where \( \omega(0) := 1 \) and \( \Delta \) is the difference operator which is defined by
\[
\Delta \omega(n) = \omega(n + 1) - \omega(n) \quad \text{and} \quad \Delta^m \omega = \Delta (\Delta^{m-1} \omega) \quad (m = 2, 3, \ldots),
\]
by a simple matrix computation we get
\[
P_n \begin{bmatrix} \omega(0) & \omega(1) & \cdots & \omega(n - 1) \\ \omega(1) & \omega(2) & \cdots & \omega(n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(n - 1) & \omega(n) & \cdots & \omega(2n - 2) \end{bmatrix} = \begin{bmatrix} \omega(1) & \Delta \omega(1) & \cdots & \Delta^{n-1} \omega(1) \\ \omega(2) & \Delta \omega(2) & \cdots & \Delta^{n-1} \omega(2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \Delta \omega(n) & \cdots & \Delta^{n-1} \omega(n) \end{bmatrix}.
\]  \quad (3.3)
From (3.3), for each \( n=1,2, \ldots \) it is easy to establish the following identity:

\[
\Delta^n \omega(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \omega(m+k) \quad (m = 0, 1, \ldots, n - 1),
\]

(3.4)

where \( \Delta^0 \omega(n) := \omega(n) \).

In particular, from (3.4) we get the following well-known identities (see [5, pp. 210–211]):

\[
\omega(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \omega(k) \quad (n \geq 1)
\]

and

\[
\omega(n) = \Delta^n \omega(1).
\]

4. Generalizations of the Stirling matrices

For any nonzero real number \( x \), the \( n \times n \) generalized Stirling matrix of the first kind \( s_n[x] \) and of the second kind \( S_n[x] \) are defined by

\[
(s_n[x])_{ij} = \begin{cases} 
   x^{i-j}s(i, j) & \text{if } i \geq j, \\
   0 & \text{otherwise}
\end{cases}
\]

and

\[
(S_n[x])_{ij} = \begin{cases} 
   x^{i-j}S(i, j) & \text{if } i \geq j, \\
   0 & \text{otherwise}.
\end{cases}
\]

By the definition, we see that \( s_n[1] = s_n \) and \( S_n[1] = S_n \).

Also, for the \( k \times k \) generalized Pascal matrix \( P_k[x] \) we define the \( n \times n \) matrix \( \tilde{P}_k[x] \) by

\[
\tilde{P}_k[x] = \begin{bmatrix} 
   I_{n-k} & 0 \\
   0 & P_k[x]
\end{bmatrix}.
\]

Since \( s_n[1] = s_n \) and \( S_n[1] = S_n \), it is easy to prove the following lemma.

**Lemma 4.1.** Let \( x \) be a nonzero real number. Then
(a) \( s^{-1}_n[x] = S_n[-x] \),
(b) \( S^{-1}_n[x] = s_n[-x] \).

The following theorem follows from Lemmas 2.1 and 4.1.

**Theorem 4.2.** Let \( x \) be a nonzero real number. Then
(a) \( S_n[x] = P_n[x]([1] \oplus S_{n-1}[x]) \),
(b) \( S_n[x] = \tilde{P}_n[x] \tilde{P}_{n-1}[x] \cdots \tilde{P}_2[x] \tilde{P}_1[x] \),
(c) \( S^{-1}_n[x] = \tilde{P}_1[-x] \tilde{P}_2[-x] \cdots \tilde{P}_{n-1}[-x] \tilde{P}_n[-x] \).
For example,

\[
S_4[x] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
x^2 & 3x & 1 & 0 \\
x^3 & 7x^2 & 6x & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
x^2 & 2x & 1 & 0 \\
x^3 & 3x^2 & 3x & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & x & 1 & 0 \\
0 & x^2 & 3x & 1
\end{bmatrix}.
\]

Again, if we define the \(n \times n\) matrices \(t_n[x]\) and \(T_n[x]\) by

\[
(t_n[x])_{ij} = \begin{cases} 
  x^{i+j-2}s(i, j) & \text{if } i \geq j, \\
  0 & \text{otherwise},
\end{cases}
\]

and

\[
(T_n[x])_{ij} = \begin{cases} 
  x^{i+j-2}S(i, j) & \text{if } i \geq j, \\
  0 & \text{otherwise},
\end{cases}
\]

it is easy to see that the following theorem holds by the similar arguments for \(s_n[x]\) and \(S_n[x]\).

**Theorem 4.3.** Let \(x\) be a nonzero real number. Then for the generalized Pascal matrices \(P_n[x]\) and \(Q_n[x]\) defined in (1.6), the following results hold:

(a) \(t_n^{-1}[x] = T_n[-\frac{1}{x}]\),

(b) \(T_n^{-1}[x] = t_n[-\frac{1}{x}]\),

(c) \(t_n[x] = ([1] \oplus s_{n-1}[x])Q_n[x]\),

(d) \(T_n[x] = Q_n[x]([1] \oplus S_{n-1}[-\frac{1}{x}])\),

(e) \(t_n[x] = \tilde{P}_1[x] \tilde{P}_2[x] \ldots \tilde{P}_{n-1}[x]Q_n[x]\),

(f) \(T_n[x] = Q_n[x] \tilde{P}_{n-1}[\frac{1}{x}] \ldots \tilde{P}_2[\frac{1}{x}] \tilde{P}_1[\frac{1}{x}]\),

(g) \(t_n^{-1}[x] = \tilde{Q}_n[-\frac{1}{x}] \tilde{P}_{n-1}[-x] \ldots \tilde{P}_2[-x] \tilde{P}_1[-x]\),

(h) \(T_n^{-1}[x] = \tilde{P}_1[-\frac{1}{x}] \tilde{P}_2[-\frac{1}{x}] \ldots \tilde{P}_{n-1}[-\frac{1}{x}] \tilde{Q}_n[-\frac{1}{x}]\).

Furthermore, for any two nonzero real numbers \(x\) and \(y\) we define the \(n \times n\) matrices \(\psi_n[x, y]\) and \(\Omega_n[x, y]\) by

\[
(\psi_n[x, y])_{ij} = \begin{cases} 
  x^{-j}y^{i+j-2}s(i, j) & \text{if } i \geq j, \\
  0 & \text{otherwise},
\end{cases}
\]

and
\[(\omega_n[x, y])_{ij} = \begin{cases} x^{i-j} y^{i+j-2} S(i, j) & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases}\]

By the definition, we see that
\[\Omega_{n}[x, 1] = S_{n}[x], \quad \Omega_{n}[1, y] = T_{n}[y],\]
\[\Psi_{n}[x, 1] = s_{n}[x], \quad \Psi_{n}[1, y] = t_{n}[y].\]

It is easy to see that the following theorems hold by the similar arguments for \(s_{n}[x]\) and \(S_{n}[x]\).

**Theorem 4.4.** Let \(x\) and \(y\) be any nonzero real numbers. Then for the extended generalized Pascal matrix \(\Phi_{n}[x, y]\) defined in (1.5), the following results hold:

(a) \(\Omega_{n}[-x, y] = \Omega_{n}[x, -y],\)
(b) \(\Psi_{n}[-x, y] = \Psi_{n}[x, -y],\)
(c) \(\Omega_{n}^{-1}[x, y] = \Psi_{n}[-x, \frac{1}{y}] = \Psi_{n}[x, -\frac{1}{y}],\)
(d) \(\Psi_{n}^{-1}[x, y] = \Omega_{n}[-x, \frac{1}{y}] = \Omega_{n}[x, -\frac{1}{y}],\)
(e) \(\Omega_{n}[x, y] = \Phi_{n}[x, y]([1] \oplus S_{n-1}[\frac{x}{y}]),\)
(f) \(\Psi_{n}[x, y] = ([1] \oplus s_{n-1}[xy]) \Phi_{n}[x, y],\)
(g) \(\Omega_{n}[x, y] = \Phi_{n}[x, y] \tilde{P}_{n-1}[\frac{x}{y}] \ldots \tilde{P}_{2}[\frac{x}{y}] \tilde{P}_{1}[\frac{x}{y}],\)
(h) \(\Psi_{n}[x, y] = \tilde{P}_{1}[xy] \tilde{P}_{2}[xy] \ldots \tilde{P}_{n-1}[xy] \Phi_{n}[x, y],\)
(i) \(\Psi_{n}^{-1}[x, y] = \Phi_{n}[x, -\frac{1}{y}] \tilde{P}_{n-1}[-xy] \ldots \tilde{P}_{2}[-xy] \tilde{P}_{1}[-xy],\)
(j) \(\Omega_{n}^{-1}[x, y] = \tilde{P}_{1}[-\frac{x}{y}] \tilde{P}_{2}[\frac{x}{y}] \ldots \tilde{P}_{n-1}[-\frac{x}{y}].\)

**References**