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On a generating function of Laguerre polynomials.


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On a generating function of Laguerre polynomials

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Summary. - A generating function for the Laguerre polynomials is derived from their recursion formulae.

Introduction: The following generating function of Laguerre polynomials $L_n^{(a)}(x)$:

$$\sum_{n=0}^{\infty} \frac{L_n^{(a)}(x)t^n}{\Gamma(n + a + 1)} = e^{t(x t - \frac{1}{2} a)} J_a[2(x t)^{\frac{1}{2}}]; \quad (a > -1)$$

is due to Doetsch and follows from

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)t^n = (1 - t)^{-a-1} \exp\left(\frac{xt}{t-1}\right); \quad (|t| < 1)$$

by means of the Laplace transformation. Recently Tschauner [1] proved the known generating function for Gegenbauer polynomials $C_n^{p+\frac{1}{2}}(x)$, $n = 0, 1, 2, \ldots; p > -\frac{1}{2}$:

$$\sum_{n=0}^{\infty} \frac{C_n^{p+\frac{1}{2}}(x)}{\Gamma(1 + n + 2p)} t^n = \frac{\Gamma(1 + p)}{\Gamma(1 + 2p)} e^{xt} \left(\frac{yt}{2}\right)^{-p} J_p(yt),$$

where $y = (1 - x^2)^{\frac{1}{2}}$

from the recursion formula. Following the method adopted by Tschauner we like to prove (1) from the recursion formula of Laguerre polynomials:

$$(n + 2)u_{n+2} - (2n + 3 + a - x)u_{n+1} + (n + a + 1)u_n = 0$$

where $u_n = u_n(x, a) = L_n^{(a)}(x)$, $u_0 = 1$, $u_1 = a + 1 - x$.

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Now using

\[ v_n(x, \omega) = \frac{u_n(x, \omega)}{\Gamma(n + \alpha + 1)} = \frac{L_n^{(\alpha)}(x)}{\Gamma(n + \alpha + 1)} \]

we derive from (4) the recursion formula for \( v_n(x, \omega) \):

\[ (n + 2)(n + \alpha + 2)v_{n+2} - (2n + 3 + \alpha - x)v_{n+1} + v_n = 0 \]

with

\[ v_0 = \frac{1}{\Gamma(\alpha + 1)}, \quad v_1 = \frac{\alpha + 1 - x}{(\alpha + 1)\Gamma(\alpha + 1)}, \]

so that \( (\alpha + 1)v_1 - (\alpha + 1 - x)v_0 = 0 \).

Next multiplying both members of (5) by \( t^n \) and summing from \( n = 0 \) to \( \infty \) we obtain the following homogeneous differential equation of order two:

\[ t \frac{d^2V}{dt^2} + (\alpha + 1 - 2t)\frac{dV}{dt} - (\alpha + 1 - x - t)V = (\alpha + 1)v_1 - (\alpha + 1 - x)v_0 = 0, \]

where

\[ V(t) = \sum_{n=0}^{\infty} v_n t^n. \]

Using the substitution

\[ V(t) = e^t (t - \frac{1}{2})^\frac{\alpha}{2} W(t) \]

we derive from (6) the following differential equation

\[ 4t^2 \frac{d^2W}{dt^2} + 4t \frac{dW}{dt} + (4tx - x^2)W = 0 \]

Finally using the substitution

\[ \sqrt{t} = z \text{ and } 2\sqrt{x} = y \]

we obtain from (7) the Bessel differential equation

\[ z^2 \frac{d^2W}{dz^2} + z \frac{dW}{dz} + (y^2z^2 - x^3)W = 0, \]

which leads to the particular solutions

\[ W = AJ_\alpha(yz), \quad W = BY_\alpha(yz). \]
Thus the particular solutions of (6) are

\[ V(t) = Ae^t \left( -\frac{1}{2} \alpha \right) J_\alpha(2\sqrt{x}t), \quad V(t) = Be^t \left( -\frac{1}{2} \alpha \right) Y_\alpha(2\sqrt{x}t). \]

Now since \( V(0) = v_0 \) is finite, we take the particular solution

\[ V(t) = Ae^t \left( -\frac{1}{2} \alpha \right) J_\alpha(2\sqrt{x}t). \]

To determine the normalising factor \( A \), we notice

\[ V(0) = v_0 = \frac{1}{1 + \alpha} = A \cdot \frac{(2\sqrt{x})^\alpha}{2^\alpha \Gamma(\alpha + 1)}, \]

whence \( A = x^{-\frac{1}{2} \alpha}. \)

Thus we obtain the generating function for the Laguerre polynomials

\[ V(t) = \sum_{n=0}^{\infty} v_n(x, \alpha) t^n = \sum_{n=0}^{\infty} \frac{J_n^{(\alpha)}(x)t^n}{\Gamma(n + \alpha + 1)}, \quad (\alpha > -1). \]

In particular, when \( \alpha = 0 \), we derive from (10)

\[ \sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!} = e^{tx} J_\alpha(2\sqrt{x}t). \]

Next using \( \alpha = -\frac{1}{2}, \cdot x = x^2 \) and noticing the relation

\[ H_{2n}(x) = (-)^n 2^{2n} n! L_n \left( -\frac{1}{2} x \right), \]

we derive ultimately

\[ \sum_{n=0}^{\infty} \frac{(-)^n H_{2n}(x) z^{2n}}{2^{2n}(2n)!} = e^{z^2} \cos (2xz), \]

where \( z = 2\sqrt{t}. \)

Again using \( \alpha = \frac{1}{2}, x = x^2 \) and noticing the relation

\[ H_{2n+1}(x) = (-)^n 2^{2n+1} (n+1)! \alpha L_n \left( \frac{1}{2} x \right), \]
we similarly derive

\[
\sum_{n=0}^{\infty} \frac{(-)^n H_{2n+1}(x) z^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{z^2}{4} \sin(2x).
\]

Lastly from (12) and (13) we obtain on taking \(z = 2\):

\[
\sum_{n=0}^{\infty} \frac{(-)^n H_{2n}(x)}{(2n)!} = e \cos(2x)
\]

\[
\sum_{n=0}^{\infty} \frac{(-)^n H_{2n+1}(x)}{(2n+1)!} = e \sin(2x);
\]

both of which may be compared with the formulae in [2].

REFERENCES
