1. Put

\[ A(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1} \cdot \]

where it is understood that

\[ \binom{n}{-1} = \binom{n}{n+1} = 0 \quad (n > 0). \]

Consideration of this sum was suggested by the following problem proposed by H. W. Gould [1]. Let

\[ A_p(n) = \sum_{0 \leq 2k \leq n} (-1)^k \left\{ \binom{n}{k} \binom{n}{k-1} \right\}^p. \]

Then

\[ A_1(2m + 1) = (2m + 1)A_1(2m + 1). \]

It is noted that this result does not hold for even \( n \).

Since

\[ A(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{k} \binom{n}{k-1} \right\}^3 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{k} \binom{n}{k-1} \right\}^3, \]

so that

\[ A(n) = (-1)^n A(n), \]

therefore

\[ A(2m + 1) = 0. \]

However (1.2) gives no information about \( A(2m) \). By (1.1) we have

\[ A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 \]

\[ - \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}^3 = 2 \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2. \]

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Thus if we put
\[ S_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2, \quad S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1}, \]
\[ S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2, \]
it is clear that
\[ (1.4) \quad A(n) = 2S_0(n) - 3S_1(n) + 3S_2(n). \]
In the next place, we have
\[ S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 \]
\[ = \sum_{k=0}^{n+1} (-1)^n \binom{n}{n-k+1} \binom{n}{n-k}^2 \]
so that
\[ (1.5) \quad S_2(n) = (-1)^{n+1} S_1(n) \]
and (1.4) becomes
\[ (1.6) \quad A(n) = 2S_0(n) - 3 \left\{ 1 + (-1)^n \right\} S_1(n). \]
In particular we have
\[ (1.7) \quad \begin{cases} A(2m) = 2S_0(2m) - 6S_1(2m) \\ A(2m + 1) = 2S_2(2m + 1). \end{cases} \]
It is well known (see for example [2, p. 13], [3, p. 243]) that \( S_0(2m + 1) = 0 \), while
\[ (1.8) \quad S_0(2m) = (-1)^m \frac{(3m)!}{(m!)^3}. \]
However \( S_1(n) \) does not seem to be known.

2. In order to evaluate \( S_1(2m) \) we proceed as follows. We have
\[ S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \left\{ \binom{n+1}{k} - \binom{n}{k} \right\} \]
\[ = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \left( \binom{n+1}{k} - \binom{n}{k} \right) - S_0(n) \]
so that
\[ (2.1) \quad S_1(n) = T_0(n) - T_1(n) - S_0(n), \]
where
\[ T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2, \quad T_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n}{k-1}. \]
Now
\[
T_1(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1} \binom{n+1}{n-k} \binom{n}{n}
\]

\[
= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} \binom{n+1}{n} \binom{n}{k} ,
\]

that is,

\[
(2.2) \quad T_1(n) = (-1)^{n+1} T_1(n).
\]

Therefore \(T_1(2m) = 0\) and (2.1) yields

\[
(2.3) \quad S_n(2m) = T_6(2m) - S_8(2m).
\]

In the next place

\[
T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \binom{n+1}{n-k}^2
\]

\[
= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1}^2 = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k+1} \right\}
\]

\[
= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n+2}{k} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k+1} \right\}
\]

\[
= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2
\]

\[
- (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+1}{k} + \binom{n+1}{k+1} \right\}
\]

\[
= -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} - (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1}^2
\]

\[
+ (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2 ,
\]

so that

\[
(2.4) \quad \{1 + (-1)^n\} T_0(n) = -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}^2.
\]

For \(n = 2m + 1\), (2.4) gives no information about \(T_6(2m+1)\); indeed each sum on the right vanishes. For \(n = 2m\), however, (2.4) becomes
It is known [3, p. 243] that
\[
\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m}{k+1} (2m+1) = -2 \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{2m+2}{k+1}^2.
\]

Substituting from (2.6) and (2.7) in (2.5), we get
\[
T_0(2m) = (-1)^m \frac{(3m+1)!}{m!m!(m+1)!}.
\]
Therefore by (2.3) and (1.8)
\[
S_{1}(2m) = (-1)^m \frac{(3m)!}{m!(m-1)!(2m+1)}.
\]
Finally, by (1.6) and (2.9),
\[
A(2m) = -2(-1)^m \frac{(3m)!(m-1)}{(m!)^2(2m+1)}.
\]
This completes the evaluation of the sum \( A(2m) \). Note that we have not evaluated \( S_{2}(2m) \).

3. For completeness we give a simple proof of (1.8), (2.6) and (2.7). We assume Saalschütz’s theorem [2, p. 9]:
\[
\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!(c)_k (d)_k} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \quad (a)_0 = 1
\]
where
\[
(a)_k = a(a+1) \cdots (a+k-1),
\]
and
\[
c + d = -n + a + b + 1.
\]
We rewrite (3.1) in the following way:
\[
\sum_{j=0}^{\infty} \frac{(-j)_r (a+j)_r (b+c-a+1)_r}{r!(b+1)_r (c+1)_r} = \frac{(a-b)_j (a-c)_j}{(b+1)_j (c+1)_j};
\]
the condition (3.2) is automatically satisfied. Multiplying both sides of (3.3) by \((a)_j x^j/j!\) and summing over \(j\), it follows that
\[
\sum_{j=0}^{\infty} \frac{(-j)_r (a+j)_r (b+c-a+1)_r}{r!(b+1)_r (c+1)_r} x^j = \sum_{j=0}^{\infty} \frac{(-j)_r (a+j)_r (b+c-a+1)_r}{r!(b+1)_r (c+1)_r} x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_2r (b+c-a+1)_r}{r!(b+1)_r (c+1)_r} x^r (1-x)^{-2r}.
\]
Now take $a = -n$ and we get

\[
\sum_{j=0}^{\infty} \frac{(-n)j(-n-b)j(-n-c)j}{j!(b+1)j(c+1)j} x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(-n)2r(b+c+n-1)r}{r!(b+1)r(c+1)r} x^r(1-x)^{n-2r}.
\]

For $n = 2m$ and $x = 1$, (3.4) reduces to

\[
\sum_{j=0}^{\infty} \frac{(-2m)j(-2m-b)j(-2m-c)j}{j!(b+1)j(c+1)j} = (-1)^m \frac{(2m)!((b+c+2m+1)m}{m!(b+1)m(c+1)m}.
\]

Now let $b, c$ be non-negative integers. Then (3.5) yields

\[
\sum_{j=0}^{2m} (-1)^m \left( \begin{array}{c} 2m \\ j \end{array} \right) \left( \begin{array}{c} 2m+b+c \\ j+b \end{array} \right) \left( \begin{array}{c} 2m+b+c \\ j+c \end{array} \right)
\]

\[
= (-1)^m \frac{(2m)!(3m+b+c)(2m+b+c)!}{m!(m+b)!(m+c)!(2m+b)!(2m+c)!}.
\]

For $b = c = 0$ we get (1.8); for $b = 0, c = 1$ we get (2.6); for $b = c = 1$ we get (2.7).

**REFERENCES**


[Continued from Page 214.]

\[
\frac{1}{k} \log \frac{1 + \sqrt{5}}{2}
\]

as $n \to \infty$. Since this limiting value is an irrational number, the sequence $(u_n)$ is u.d. mod 1.

**REMARK.** Let $p$ and $q$ be non-negative integers. Then the sequence

\[
p, q, p + q, p + 2q, 2p + 3q, ...
\]

or $(H_n), n = 1, 2, ...$ with

\[
H_n = qF_{n-1} + pF_{n-2} \quad (n > 3), \quad H_1 = p, \quad H_2 = q
\]

possesses the property shown in Theorem 1. For if \(v_n = \log H_n^{1/k}\), we have

\[
v_{n+1} - v_n \to \frac{1}{k} \log \frac{1 + \sqrt{5}}{2}
\]

as $n \to \infty$.

**Theorem 2.** Let $p, q, p^*$ and $q^*$ be non-negative integers. Let $(H_n)$ be the sequence

\[
p, q, p + q, p + 2q, 2p + 3q, ...
\]

and $(H_n^*)$ the sequence

\[
p^*, q^*, p^* + q^*, p^* + 2q^*, 2p^* + 3q^*, ...
\]

[Continued on Page 276.]