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Explicit Formulas and Combinatorial Identities for Generalized Stirling Numbers

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Abstract. In this paper, a modified approach to the multiparameter non-central Stirling numbers via differential operators, introduced by El-Desouky, and new explicit formulae of both kinds of these numbers are given. Also, some relations between these numbers and the generalized Hermite and Truesdel polynomials are obtained. Moreover, we investigate some new results for the generalized Stirling-type pair of Hsu and Shiue. Furthermore some interesting special cases, new combinatorial identities and a matrix representation are deduced.

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1. Introduction and preliminaries

Through this article we use the following notations. The falling and rising factorials are defined, respectively by

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (x)_0 = 1,$$

and

$$\langle x \rangle_n = x(x+1)\cdots(x+n-1), \quad \langle x \rangle_0 = 1.$$

The generalized falling and rising factorials $\langle x; \alpha \rangle_n$ and $\langle x; \alpha \rangle_n$, associated with parameter $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ where $\alpha_j, j = 0, 1, \ldots, n - 1$, is a sequence

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of real or complex numbers, are defined by
\[
(x; \alpha)_n = \prod_{j=0}^{n-1} (x - \alpha_j) \quad \text{and} \quad \langle x; \alpha \rangle_n = \prod_{j=0}^{n-1} (x + \alpha_j),
\]
respectively. Note that if \( \alpha_i = i\alpha, \ i = 0, 1, \ldots, n-1 \), then \( (x; \alpha)_n \) reduces to
\[
(x|\alpha)_n = i^n (x - (n - 1)\alpha).
\]
Hsu and Shiue [14] defined generalized Stirling-type pair
\[
\{S^1(n,k), S^2(n,k)\} = \{S(n,k; \alpha, \beta, r), S(n,k; \beta, \alpha, -r)\}
\]
by the inverse relations
\[
(x|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(x-r|\beta)_k \quad \text{(1.1)}
\]
and
\[
(x|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(x+r|\alpha)_k, \quad \text{(1.2)}
\]
where \( n \) is a nonnegative integer and the parameters \( \alpha, \beta \) and \( r \) are real or complex numbers, with \( (\alpha, \beta, r) \neq (0,0,0) \).

For the numbers \( S(n,k; \alpha, \beta, r) \), we have the following recurrence relation
\[
S(n+1,k; \alpha, \beta, r) = S(n,k-1; \alpha, \beta, r) + (k\beta - n\alpha + r)S(n,k; \alpha, \beta, r), \quad \text{(1.3)}
\]
where \( n \geq k \geq 1 \).

These numbers have the vertical generating functions (with \( \alpha \beta \neq 0 \))
\[
k! \sum_{n=0}^{\infty} S(n,k; \alpha, \beta, r) \frac{x^n}{n!} = (1 + \alpha x)^r/\alpha \left( \frac{(1 + \alpha x)^{\beta/\alpha} - 1}{\beta} \right)^k \quad \text{(1.4)}
\]
and
\[
k! \sum_{n=0}^{\infty} S(n,k; \beta, \alpha, -r) \frac{x^n}{n!} = (1 + \beta x)^{-r/\beta} \left( \frac{(1 + \beta x)^{\alpha/\beta} - 1}{\alpha} \right)^k.
\]

The multiparameter non-central Stirling numbers of the first and second kind, respectively, were introduced by El-Desouky [11] with
\[
(x)_n = \sum_{k=0}^{n} s(n,k; \alpha)(x; \alpha)_k \quad \text{(1.5)}
\]
and
\[
(x; \alpha)_n = \sum_{k=0}^{n} S(n,k; \alpha)(x)_k. \quad \text{(1.6)}
\]
These numbers \( s(n,k; \alpha) \) and \( S(n,k; \alpha) \) satisfy the recurrence relations
\[
s(n+1,k; \alpha) = s(n,k-1; \alpha) + (\alpha_k - n)s(n,k; \alpha) \quad \text{(1.7)}
\]
and
\[
S(n+1,k; \alpha) = S(n,k-1; \alpha) + (k - \alpha_k)S(n,k; \alpha), \quad \text{(1.8)}
\]
respectively.
The multiparameter non-central Stirling numbers of the first kind have the vertical exponential generating function
\[
\sum_{n \geq 0} s(n, k; \alpha) \frac{x^n}{n!} = \sum_{j=0}^{k} \frac{(1 + x)^{a_j}}{(\alpha_j)_k},
\]
(1.9)
where \((\alpha_j)_k = \prod_{j=0, j \neq k}^{\ell} (\alpha_k - \alpha_j), \ k \leq \ell\).

Also, more generalizations and extensions of Stirling numbers are given in [4] and [20]–[22].

Let \(\delta = xD, D = d/dx\). Then we have some well-known and useful relations of this differentiation operator (see [2, 10, 16] and [19]):

(i) \(\delta^n(x^\alpha) = \alpha^n x^\alpha\),

(ii) \(\delta^n(u \cdot v) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k} u \delta^k v\),

(iii) \(F(\delta)(x^\alpha f(x)) = x^\alpha F(\delta + \alpha)f(x)\),

(iv) \(F(\delta)(e^{g(x)} f(x)) = e^{g(x)} F(\delta + x g'(x)) f(x)\).

The paper is organized as follows. In Section 2 a modified approach via differential operator to multiparameter non-central Stirling numbers is given. Also, some relations between these numbers and the generalized Hermite and Truesdell polynomials are obtained. Moreover, we show that some of Hsu and Shiue [14] results are investigated using the results obtained by El-Desouky [11] and consequently of this paper. In Section 3, new explicit formulae for those numbers, some special cases and new combinatorial identities are derived. Finally, in Section 4, a computer program is written using Maple and executed for calculating the multiparameter non-central Stirling matrix and some special cases and matrix representation of these numbers is given.

2. A modified approach to multiparameter non-central Stirling numbers

Let the differential operator \(\mathcal{D}\) be defined by \(\mathcal{D} = x^\alpha \delta x^{-\alpha}\), where \(\delta = xD\). Using (iii) we get \(\mathcal{D} f(x) = x^\alpha \delta x^{-\alpha} f(x) = (\delta - \alpha) f(x)\), hence \(\mathcal{D} = \delta - \alpha\) and by induction we get \(\mathcal{D}^n = (\delta - \alpha)^n = x^\alpha \delta^n x^{-\alpha}\). Generally, we can take the following.

**Definition 2.1.** Let the differential operator \(\mathcal{D}_n\) be defined by
\[
\mathcal{D}_n = (x^{a_{n-1}} \delta x^{-a_{n-1}}) \cdots (x^{a_0} \delta x^{-a_0}) = \prod_{i=0}^{n-1} x^{a_i} \delta x^{-a_i}, \ n \geq 1
\]
and \(\mathcal{D}_0 = I\), the identity operator.

Then we have \(\mathcal{D}_n f(x) = (\delta - a_{n-1}) \cdots (\delta - a_0) f(x)\), i.e.,
\[
\mathcal{D}_n = (\delta - a_{n-1}) \cdots (\delta - a_0) = \prod_{i=0}^{n-1} (\delta - a_i) = (\delta; \alpha)_n,
\]
and so we have the operational formula
\[ \prod_{j=0}^{n-1} x^\alpha \delta x^{-\alpha} = \prod_{j=0}^{n-1} (\delta - \alpha_j). \]

This leads us to define the multiparameter non-central Stirling numbers via the
differential operator \( D_n \). Therefore, equations (1.5) and (1.6) can be represented, in
terms of operational formulae, by
\[
(\delta)_n = x^n D^n = \sum_{k=0}^{n} s(n, k; \alpha)(\delta; \alpha)_k = \sum_{k=0}^{n} s(n, k; \alpha) D_k
\]
and
\[
D_n = (\delta; \alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha)(\delta)_k = \sum_{k=0}^{n} S(n, k; \alpha)x^k D^k,
\]
respectively.

Similarly, Comtet numbers (see [4] and [20]) can be defined by
\[
(\delta; \alpha)_n = \sum_{k=0}^{n} s\alpha(n, k)\delta^k
\]
and
\[
\delta^n = \sum_{k=0}^{n} S\alpha(n, k)(\delta; \alpha)_k.
\]

Setting \( \alpha_i = \alpha \), then (2.1) and (2.2), respectively, yield
\[
(\delta)_n = x^n D^n = \sum_{k=0}^{n} s(n, k; \alpha)(\delta - \alpha)_k = \sum_{k=0}^{n} s(n, k; \alpha)x^\alpha \delta^k x^{-\alpha}.
\]
and
\[
x^\alpha \delta^n x^{-\alpha} = (\delta - \alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha)(\delta)_k = \sum_{k=0}^{n} S(n, k; \alpha)x^k D^k,
\]
where \( s(n, k; \alpha) \) and \( S(n, k; \alpha) \) are the non-central Stirling numbers of the first and
second kind [11], respectively.

Acting with Eq. (2.3) on \( e^{\alpha x^r} \), then multiplying by \( e^{\alpha x^r} \) we obtain
\[
e^{\alpha x^r} x^n D^n e^{-\alpha x^r} = \sum_{k=0}^{n} s(n, k; \alpha)e^{\alpha x^r} x^\alpha \delta^k x^{-\alpha} e^{-\alpha x^r},
\]
hence we have,
\[
x^n H^n(x, 0, p) = (-1)^n \sum_{k=0}^{n} s(n, k; \alpha)T^{-\alpha}_k(x, r, p),
\]
where \( H^n(x, \alpha, p) \) and \( T^{-\alpha}_k(x, r, p) \) are the generalized Hermite and Truesdell polynomials, respectively (see [12] and [16]-[18]).

Setting \( r = p = 1 \) in (2.5), or (2.6), we get
\[
(-1)^n x^n = \sum_{k=0}^{n} s(n, k; \alpha)T^{-\alpha}_k(x).
\]
where \( T^{-\alpha}_n(x) \) are Truesdell polynomials, see [17].
Similarly, acting with Eq. (2.4) on $e^{-\alpha x}$, we obtain the inverse relations

$$T_n^{-\alpha}(x, r, p) = (-1)^n \sum_{k=0}^{n} S(n, k; \alpha)x^k \left(-1\right)^k H_k^r(x, 0, p)$$

and

$$T_k^{-\alpha}(x) = \sum_{k=0}^{n} s(n, k; \alpha)(-1)^k x^k.$$ 

Next, we derive some new results for Hsu and Shiue numbers [14].

**Theorem 2.2.** For the special case $\alpha_i = -(r - i\alpha)/\beta, i = 0, 1, \ldots, n - 1$, we have

$$s(n, k; \bar{\alpha}) = \beta^{k-n} S^2(n, k)$$

(2.7)

**Proof.** From equation (1.2) we get

$$\beta^n(t/\beta) \cdot (t/\beta - 1) \cdot \cdots \cdot (t/\beta - (n - 1)) = \sum_{k=0}^{n} S^2(n, k)(t + r|\alpha)_k.$$ 

Setting $t/\beta = x$, we have

$$\beta^n x(x - 1) \cdot \cdots \cdot (x - (n - 1)) = \sum_{k=0}^{n} S^2(n, k)(\beta x + r)(\beta x + r - \alpha) \cdot \cdots \cdot (\beta x + r - (k - 1)\alpha),$$

hence

$$(x)_n = \sum_{k=0}^{n} \beta^{k-n} S^2(n, k) (x + r/\beta)(x + (r - \alpha)/\beta) \cdot \cdots \cdot (x + (r - (k - 1)\alpha)/\beta).$$

Comparing this equation with (1.5) yields (2.7). \[\square\]

Also, it can be shown that $S(n, k; \bar{\alpha}) = \beta^{k-n} S^1(n, k)$ if $\alpha_i = (i\alpha - r)/\beta$.

Furthermore, we show that the generating function (1.4), see [14], of Hsu-Shiue numbers can be investigated from the generating function (1.9), see [11], where $\alpha_i = (r + i\beta)/\alpha, i = 0, 1, \ldots, n - 1$.

**Theorem 2.3.** For the special case $\alpha_i = (r + i\beta)/\alpha, i = 0, 1, \ldots, n - 1$, the generating function (1.9) is reduced to the generating function (1.4) and

$$S^1(n, k) = \alpha^{n-k} s(n, k; \bar{\alpha}).$$

**Proof.** If we start from (1.9)

$$\sum_{n=0}^{\infty} s(n, k; \bar{\alpha}) x^n/n! = \sum_{j=0}^{k} \frac{(1 + z)^{\alpha_j}}{(\alpha_j)_k}$$
and setting \( \alpha_j = r/\alpha + j\beta/\alpha, \) \( j = 0, 1, \ldots, k, \) we get

\[
RHS = \sum_{j=0}^{k} \frac{(1+z)^{r+j\beta}}{\prod_{j=0, i \neq j}^{k} \left( \frac{r}{\alpha} + j\beta \right)} = \sum_{j=0}^{k} \frac{(1+z)^{r+j\beta}}{\prod_{i=0, i \neq j}^{k} (j-i)}
\]

\[
= \frac{(1+z)^{\tilde{\beta}}}{\tilde{\beta}^k} \sum_{j=0}^{k} \frac{(1+z)^j}{\prod_{i=0, i \neq j}^{k} (j-i)} = \frac{(1+z)^{\tilde{\beta}}}{\tilde{\beta}^k} \sum_{j=0}^{k} \frac{(1+z)^j}{(-1)^{k-j}(k-j)!} = \frac{(1+z)^{\tilde{\beta}}}{\tilde{\beta}^k} \left( (1+z)^{\frac{\beta}{\alpha} - 1} \right)^k,
\]

where we used the identity,

\[
\prod_{i=0, i \neq j}^{k} (j-i) = (-1)^{k-j}(k-j)!.
\]

Thus, putting \( z = \alpha r \) in (1.9) we get

\[
\sum_{n=0}^{\infty} s(n,k; \alpha) \frac{\alpha^n}{n!} = \frac{1}{k!} \alpha^k (1+\alpha r)^{\beta/\alpha} \left( \frac{(1+\alpha r)^{\beta/\alpha} - 1}{\beta} \right)^k,
\]

hence by virtue of (1.4) we have

\[
\sum_{n=0}^{\infty} S^1(n,k) \frac{\mu^n}{n!} = \sum_{n=0}^{\infty} \alpha^{n-k}s(n,k; \alpha) \frac{\mu^n}{n!} = \frac{1}{k!} (1+\alpha r)^{\beta/\alpha} \left( \frac{(1+\alpha r)^{\beta/\alpha} - 1}{\beta} \right)^k,
\]

where \( \alpha \equiv (i\beta + r)/\alpha. \)

In fact, this equation shows that

\[
S^1(n,k) = \alpha^{n-k}s(n,k; \alpha),
\]

where \( \alpha_i = (i\beta + r)/\alpha, \) \( i = 0, 1, \ldots, n-1. \) This completes the proof. \( \square \)

Now, setting \( \alpha_i = (i\alpha - r)/\beta, \) \( i = 0, 1, \ldots, n-1 \) in (2.2) we get the following new operational formula for Hsu-Shiu'e numbers:

\[
\sum_{k=0}^{n} \beta^{k-n} S^1(n,k)(\delta)_k = \sum_{k=0}^{n} \beta^{k-n} S^1(n,k)(\delta)_k.
\]  

(2.8)

\textbf{Remark 2.4.} Note that equations (2.2) and (2.8) for \( \alpha_i = 0, i = 0, 1, \ldots, n-1 \) and \( \alpha = 0, \beta = 1 \) and \( r = 0, \) respectively, are reduced to the well known operational representation of the ordinary Stirling numbers of the second kind

\[
\delta^n = \sum_{k=0}^{n} S(n,k)x^k D^k.
\]
3. New explicit formulae and combinatorial identities

We derive new explicit formulae for both kinds of the multiparameter non-central Stirling numbers.

**Theorem 3.1.** The numbers $S(n, k; \alpha)$, have the following new explicit formula

$$S(n, k; \alpha) = \sum_{i_{n-1} = n-k}^{\sum_{i_{n-2} = (n-1)-(k-1)}} \left( -\alpha_0 \right) \left( \begin{array}{c} 1 - \alpha_1 - i_0 \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n - 1 - \alpha_{n-1} - i_0 \cdots - i_{n-2} \\ i_{n-1} \end{array} \right),$$

where $i_j \in \{0, 1\}$, $j \in \{0, 1, \ldots, n-1\}$, and $l_{n-1} := i_0 + i_1 + \cdots + i_{n-1}$.

**Proof.** The statement for $k = 0$ gives that

$$S(n, 0; \alpha) = (-1)^n \alpha_0 \alpha_1 \cdots \alpha_{n-1},$$

which agrees with the definition of $S(n, k; \alpha)$ (see [11]).

Also, if $i_{n-1} \in \{0, 1\}$, we have that

$$S(n, k; \alpha) = \sum_{l_{n-2} = (n-1)-(k-1)}^{\sum_{l_{n-3} = (n-2)-(k-2)}} \left( -\alpha_0 \right) \left( \begin{array}{c} 1 - \alpha_1 - i_0 \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n - 2 - \alpha_{n-2} - i_0 \cdots - i_{n-3} \\ i_{n-2} \end{array} \right) 
+ \sum_{l_{n-2} = (n-1)-(k)} \left[ n - 1 - \alpha_{n-1} - (n-k-1) \right] \left( -\alpha_0 \right) \left( \begin{array}{c} 1 - \alpha_1 - i_0 \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n - 2 - \alpha_{n-2} - i_0 \cdots - i_{n-3} \\ i_{n-2} \end{array} \right),$$

i.e.,

$$S(n, k; \alpha) = S(n, 1, k - 1; \alpha) + (k - \alpha_{n-1}) S(n, 1, k; \alpha).$$

Therefore, by (1.8) and induction we get the desired result.

**Remark 3.2.** We used $\alpha_0 = 0 \iff i_0 = 0$.

**Theorem 3.3.** The multiparameter non-central Stirling numbers of the first kind have the following new explicit expression

$$s(n, k; \alpha) = \sum_{i_1 + \cdots + i_k = k; i_j \in \{0, 1\}} \left( i_1 + \alpha_1 \\ 1 - i_1 \right) \left( i_2 + \alpha_1 + i_2 - 1 \\ 1 - i_2 \right) \cdots \left( i_n + \alpha_1 + \cdots + i_n - n + 1 \\ 1 - i_n \right).$$

(3.1)

**Proof.** For $k = 0$ we have

$$s(n, 0; \alpha) = \alpha_0 (\alpha_0 - 1) \cdots (\alpha_0 - n + 1) = (\alpha_0)_n.$$
and if \( i_n \in \{0, 1\} \), we have

\[
\begin{align*}
    s(n, k; \alpha) &= \sum_{i_1 + \ldots + i_{n-1} = k} \left( \begin{array}{c} i_1 + \alpha_1 \\ 1 - i_1 \end{array} \right) \left( \begin{array}{c} i_{n-1} + \alpha_{i_{n-1}} - n + 2 \\ 1 - i_{n-1} \end{array} \right) \\
    &+ \sum_{i_1 + \ldots + i_{n-1} = k} \left( \alpha_{i_1 + \ldots + i_{n-1} - n + 1} \right) \left( \begin{array}{c} i_1 + \alpha_1 \\ 1 - i_1 \end{array} \right) \left( \begin{array}{c} i_{n-1} + \alpha_{i_{n-1}} - n + 2 \\ 1 - i_{n-1} \end{array} \right) \\
    &= s(n - 1, k - 1; \alpha) + (\alpha_k - n + 1)s(n - 1, k; \alpha),
\end{align*}
\]

where \( i_1 + \ldots + i_{n-1} = k \). By virtue of (1.7), this completes the proof. \( \square \)

It is worth noting that setting \( \alpha_i = (r + i\beta)/\alpha, \ i \in \{0, 1, \ldots n - 1\} \) in the recurrence relation (1.7) for multiparameter non-central Stirling numbers of the first kind, we can get the recurrence relation (1.3), and hence

\[
s(n, k; \alpha) = \alpha^{k-n} S^1(n, k).
\]

Furthermore, Corcino, Hsu and Tan [9] mentioned that the multiparameter non-central Stirling numbers are related with the generalized Stirling-type pair of Hsu-Shiue (see also [23]) by \( S(n, k; 1, \alpha, 0) = s(n, k; \alpha) \) and \( S(n, k; \alpha, 1, 0) = S(n, k; \alpha) \) for special case \( \alpha = i\alpha, \ i = 0, 1, \ldots, n - 1 \).

**Corollary 3.4.** A new explicit expression for generalized Stirling numbers (Hsu-Shiue numbers type (1.3)) is given by

\[
S(n, k; \alpha, \beta, r) = \beta^{n-k} \sum_{i_{n-1} = n-k \atop i_j \in \{0, 1\}} \left( \begin{array}{c} r/\beta \\ i_0 \end{array} \right) \left( \begin{array}{c} 1 + (r - \alpha)/\beta - i_0 \\ i_1 \end{array} \right) \ldots \left( \begin{array}{c} n - 1 + (r - (n - 1)\alpha)/\beta - i_0 - \ldots - i_{n-2} \\ i_{n-1} \end{array} \right)
\]

or in the modified form

\[
S(n, k; \alpha, \beta, r) = \sum_{i_{n-1} = n-k \atop i_j \in \{0, 1\}} \left( \begin{array}{c} r/\beta \\ i_0 \end{array} \right) \left( \begin{array}{c} r + \beta - \alpha - \beta i_0 \\ i_1 \end{array} \right) \left( \begin{array}{c} r + 2(\beta - \alpha) - \beta(i_0 + i_1) \\ i_2 \end{array} \right) \ldots \left( \begin{array}{c} r + (n - 1)(\beta - \alpha) - \beta(i_0 + i_1 + \ldots + i_{n-2}) \\ i_{n-1} \end{array} \right),
\]

where we use \( i_0 = 0 \iff r = 0 \).

**Proof.** The proof of this result follows directly from Theorem 3.1. Namely, by setting \( \alpha_i = (r + i\beta)/\beta, \ i = 0, 1, \ldots, n - 1 \), we get the statement. \( \square \)

The previous corollary implies new explicit expressions for the usual Stirling numbers of the second and first kind

\[
S(n, k; 0, 1, 0) = S(n, k) = \sum_{i_1 + \ldots + i_{n-1} = n-k \atop i_j \in \{0, 1\}} \left( \begin{array}{c} 1 \\ i_1 \end{array} \right) \left( \begin{array}{c} 2 - i_1 \\ i_2 \end{array} \right) \ldots \left( \begin{array}{c} n - 1 - i_1 - \ldots - i_{n-2} \\ i_{n-1} \end{array} \right)
\]
and
\[ S(n, k; 1, 0, 0) = s(n, k) = \sum_{i_1 + \cdots + i_{n-1} = n-k \atop i_j \in \{0,1\}} \left( \begin{array}{c} -1 \\ i_1 \\ 2 \\ i_2 \\ \vdots \\ (n-1) \\ i_{n-1} \end{array} \right) \]
\[ = (-1)^{n-k} \sum_{i_1 + \cdots + i_{n-1} = n-k \atop i_j \in \{0,1\}} \left( \begin{array}{c} i_1 \\ i_1 \\ 1 + i_2 \\ i_2 \\ \vdots \\ n - 2 + i_{n-1} \end{array} \right), \]
respectively.

**Remark 3.5.** From the above corollary we obtain the explicit formula for the numbers \( F_{\alpha, \gamma}(n, k) \) (see [7]), defined by the triangular recurrence relation
\[ F_{\alpha, \gamma}(n + 1, k) = F_{\alpha, \gamma}(n, k - 1) + (\gamma - n\alpha)F_{\alpha, \gamma}(n, k), \]
where \( F_{\alpha, \gamma}(0, 0) = 1 \) and \( F_{\alpha, \gamma}(n, k) = 0 \) when \( n < 0, k < 0 \) and \( n < k \). Namely, for \( \beta = 0 \) and \( r = \gamma \), the modified form of \( S(n, k; \alpha, \beta, r) \) in Corollary 3.4 reduces to
\[ F_{\alpha, \gamma}(n, k) = S(n, k; \alpha, 0, \gamma) = \sum_{i_1 + \cdots + i_{n-1} = n-k \atop i_j \in \{0,1\}} \left( \begin{array}{c} \gamma \\ i_0 \\ \gamma - \alpha \\ i_1 \\ \gamma - (n-1)\alpha \\ i_{n-1} \end{array} \right). \]

Note that some properties and identities on \( S(n, k; \alpha, \beta, r) \) cannot be obtained for \( \beta = 0 \), which is a reason why the explicit formula, i.e., the modified form (from Corollary 3.4), is very convenient to use.

Corte et al. [8] defined the \((r, \beta)\)-Stirling numbers, denoted by \( \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{r, \beta} \), using the following recurrence relation
\[ \left\langle \begin{array}{c} n+1 \\ k \end{array} \right\rangle_{r, \beta} = \left\langle \begin{array}{c} n \\ k-1 \end{array} \right\rangle_{r, \beta} + (r\beta + r)\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{r, \beta}. \]

It is easy to conclude that \( \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{r, \beta} = S(n, k; 0, \beta, r) \). Thus, we can obtain an explicit formula for \((r, \beta)\)-Stirling numbers from Corollary 3.4
\[ \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{r, \beta} = S(n, k; 0, \beta, r) \]
\[ = \sum_{i_1 + \cdots + i_{n-1} = n-k \atop i_j \in \{0,1\}} \left( \begin{array}{c} r/\beta \\ i_0 \\ 1 + r/\beta - i_0 \\ i_1 \\ \vdots \\ n - 1 + r/\beta - i_0 - \cdots - i_{n-2} \end{array} \right). \]

Moreover, El-Desouky [11, Theorem 2.1] derived the following explicit formula for \( s(n, k; \alpha) \)
\[ s(n, k; \alpha) = \sum_{r=k}^{n} (-1)^{n-r} n! \sum_{i_1 + \cdots + i_r = n} \frac{1}{i_1 \cdots i_r} \sum_{j=0}^{k} \frac{\alpha^j}{(\alpha)_j}, \quad (3.2) \]
where, in the second sum, the summation extends over all ordered \( n \)-tuples of integers \((i_1, i_2, \ldots, i_r)\) satisfying the condition \( i_1 + i_2 + \cdots + i_r = n \) and \( i_j \geq 0, j = 1, 2, \ldots, r \).
From (3.1) and (3.2), we obtain the following new combinatorial identity

\[
\sum_{i_1+\ldots+i_n=n-k} \left( \frac{i_1}{1-i_1} \right) \left( \frac{i_2 + \alpha_{i_1+i_2}}{1-i_2} \right) \ldots \left( \frac{i_n + \alpha_{i_1+i_2+\ldots+i_n-n+1}}{1-i_n} \right) = \sum_{r=k}^{n} (-1)^{n-r} \frac{n!}{r!} \sum_{i_1}^{\ldots} \sum_{i_r}^{\ldots} \frac{1}{(\alpha_{i_j})_k}.
\]

**Remark 3.6.** Now, it is worth noting that all special cases, (i)–(xi), derived in [14], are special cases of the multiparameter non-central Stirling numbers.

Corcino, Hsu and Tan [9] proved that

\[
S(n,k) = \frac{n!\alpha^n}{k!\beta^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{(\beta/\alpha) j + (r/\alpha)}{n}.
\] (3.3)

From (3.3) and Corollary 3.4, we have the combinatorial identity

\[
\frac{n!\alpha^n}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{(\beta/\alpha) j + (r/\alpha)}{n} = \beta^n \sum_{i_{n-1}}^{\ldots} \sum_{i_{i_0}}^{\ldots} \binom{r/\beta}{i_0} \times
\]

\[
\left(1+(r-\alpha)/\beta-i_0\right) \ldots \left(n-1+(r-(n-1)\alpha)/\beta-i_0-\ldots-i_{n-2}\right).
\]

Hongquan Yu [23, Theorem 4 and Corollary 5] proved that

\[
S(p+l,k;\beta,\alpha,0) \equiv 0 \pmod{p},
\] (3.4)

where \(p\) is a prime number, \(k\) and \(l\) are integers such that \(l+1 < k < p\) and

\[
S(p,k;\alpha,\beta,r) \equiv 0 \pmod{p},
\] (3.5)

where \(\alpha, \beta\) and \(r\) are integers and \(p\) is a prime, \(1 < k < p\).

By virtue of (3.4), (3.5) and Corollary 3.4, we obtain the combinatorial identities

\[
\alpha^{p+l-k} \sum_{i_1+\ldots+i_{p+l-1}=p+l-k} \left( \frac{1-\beta/\alpha}{i_1} \right) \left( \frac{2(1-\beta/\alpha)-i_1}{i_2} \right) \ldots
\]

\[
\times \left( \frac{(p+l-1)(1-\beta/\alpha)-i_1-\ldots-i_{p+l-2}}{i_{p+l-1}} \right) \equiv 0 \pmod{p}, \ l+1 < k < p,
\]

and

\[
\beta^{p-k} \sum_{i_{p-k}+\ldots+i_0=p-k} \left( \frac{r/\beta}{i_0} \right) \left( \frac{1+(r-\alpha)/\beta-i_0}{i_1} \right) \ldots
\]

\[
\times \left( \frac{p-1+(r-(p-1)\alpha)/\beta-i_0-\ldots-i_{p-2}}{i_{p-1}} \right) \equiv 0 \pmod{p}, \ 1 < k < p.
\]

The degenerate weighted Stirling numbers, denoted by \(S(n,k,\lambda|\theta)\), are the pair \((\theta, 1, \lambda)\) (see [13]).
Howard [13] derived the following explicit formula

\[ S(n,k,\lambda|\theta) = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k+l} \binom{k}{l} (\lambda + l|\theta)_n. \quad (3.6) \]

Next, we find new explicit formulae for the following special cases of Stirling numbers.

**Corollary 3.7.** The degenerate weighted Stirling numbers \( S(n,k,\lambda|\theta) \) have the following explicit formula

\[ S(n,k,\lambda|\theta) = \sum_{\substack{i_0, \ldots, i_{n-1} = n-k \atop i_j \in \{0, 1\}}} \left( \begin{array}{c} \lambda \\ i_0 \\ \end{array} \right) \left( \begin{array}{c} \lambda - \theta + 1 - i_0 \\ i_1 \\ \end{array} \right) \left( \begin{array}{c} \lambda + 2(1 - \theta) - i_0 - i_1 \\ i_2 \\ \end{array} \right) \cdots \]

\[ \times \left( \begin{array}{c} \lambda + (n-1)(1 - \theta) - (i_0 + i_1 + \cdots + i_{n-2}) \\ i_{n-1} \\ \end{array} \right). \quad (3.7) \]

**Proof.** The proof follows by setting \( \alpha = \theta, \beta = 1 \) and \( r = \lambda \) in the modified form of Corollary 3.4. \( \Box \)

From (3.6) and (3.7) we have the new combinatorial identity

\[ \sum_{\substack{i_0, \ldots, i_{n-1} = n-k \atop i_j \in \{0, 1\}}} \left( \begin{array}{c} \lambda \\ i_0 \\ \end{array} \right) \left( \begin{array}{c} \lambda - \theta + 1 - i_0 \\ i_1 \\ \end{array} \right) \left( \begin{array}{c} \lambda + 2(1 - \theta) - i_0 - i_1 \\ i_2 \\ \end{array} \right) \cdots \]

\[ \times \left( \begin{array}{c} \lambda + (n-1)(1 - \theta) - (i_0 + i_1 + \cdots + i_{n-2}) \\ i_{n-1} \\ \end{array} \right) = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k+l} \binom{k}{l} (\lambda + l|\theta)_n. \]

Also, setting \( \lambda = 0 \) in (3.7) (see [13, Lemma 2.1]), we have a new explicit formula for degenerate Stirling numbers

\[ S(n,k|\theta) = \sum_{\substack{i_1, \ldots, i_{n-1} = n-k \atop i_j \in \{0, 1\}}} \left( \begin{array}{c} 1 - \theta \\ i_1 \\ \end{array} \right) \left( \begin{array}{c} 2(1 - \theta) - i_1 \\ i_2 \\ \end{array} \right) \cdots \]

\[ \times \left( \begin{array}{c} (n-1)(1 - \theta) - (i_1 + i_2 + \cdots + i_{n-2}) \\ i_{n-1} \\ \end{array} \right). \]

In the special case \( \theta = \lambda = 0 \) we obtain the identity

\[ \sum_{\substack{i_1, \ldots, i_{n-1} = n-k \atop i_j \in \{0, 1\}}} \left( \begin{array}{c} 1 \\ i_1 \\ \end{array} \right) \left( \begin{array}{c} 2 - i_1 \\ i_2 \\ \end{array} \right) \cdots \left( \begin{array}{c} (n-1) - (i_1 + i_2 + \cdots + i_{n-2}) \\ i_{n-1} \\ \end{array} \right) \]

\[ = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k+l} \binom{k}{l} p^n. \]

From (1.1) we get

\[ \alpha^n(t/\alpha)((t/\alpha) - 1) \cdots ((t/\alpha) - (n-1)) = \sum_{k=0}^{n} S^l(n,k)(t-r|\beta)_k. \]
Setting \( t/\alpha = x \),
\[
\alpha^n x(x-1) \cdots (x-(n-1)) = \sum_{k=0}^{n} S_1(n,k) (\alpha x - r)(\alpha x - r - \beta) \cdots (\alpha x - r - (k-1)\beta),
\]
and hence
\[
(\alpha)_n = \sum_{k=0}^{n} \alpha^{k-n} S_1(n,k) (x - (r/\alpha))(\beta/\alpha)_k. \tag{3.8}
\]

From (1.1) and (3.8), we obtain \( \alpha^{k-n} S_1(n,k) = S(n,k; 1, r/\alpha, \beta/\alpha) \), whence
\[
S(n,k; \alpha, \beta, r) = \alpha^{n-k} S(n,k; 1, \beta/\alpha, r/\alpha). \tag{3.9}
\]

Similarly, we can prove that
\[
S(n,k; \alpha, \beta, r) = \beta^{n-k} S(n,k; \alpha/\beta, 1, r/\beta). \tag{3.10}
\]

Indeed (3.9) and (3.10) (see [1]) can be investigated from (1.4).

Using (3.10), as well as the fact that the degenerate weighted Stirling numbers \( S(n,k, \lambda|\theta) \) are the pair \( \langle \theta, 1, \lambda \rangle \), then the numbers \( S(n,k; \alpha, \beta, r) \) can be represented in terms of \( S(n,k, \lambda|\theta) \) where
\[
S(n,k; \alpha, \beta, r) = \beta^{n-k} S(n,k, (r/\beta)(\alpha/\beta)). \tag{3.11}
\]

**Remark 3.8.** In fact, (3.11) agrees with Corollary 3.4.

Furthermore, Munagi [15] defined \( S_b(n,k) \), the \( B \)-Stirling numbers of the second kind, by
\[
S_b(n,k) = S_b(n-1,k-1) + (k+b-1)S_b(n-1,k), \quad n \geq k,
\]
\[
S_b(n,0) = \delta_{n,0}, S_b(n,1) = b^{n-1}, \quad \text{where } \delta_{i,j} \text{ is the Kronecker delta.}
\]

**Corollary 3.9.** The \( B \)-Stirling numbers of the second kind, \( S_b(n,k) \), have the following new explicit formula
\[
S_b(n+1,k+1) = S(n,k; 0,1,b)
\]
\[
= \sum_{i_0, i_1, \ldots, i_n \geq 0} \binom{b}{i_0} \binom{b + 1 - i_0}{i_1} \binom{b + 2 - (i_0 + i_1)}{i_2} \cdots \times \binom{b + n - 1 - (i_0 + i_1 + \cdots + i_{n-2})}{i_{n-1}}. \tag{3.12}
\]

where \( b \geq 0 \) and we use \( i_0 = 0 \iff b = 0 \).

**Proof.** The proof follows by setting \( \alpha_i = -b, i = 0,1,\ldots,n-1 \), in Theorem 3.1 (or setting \( \alpha = 0, \beta = 1 \) and \( r = b \) in Corollary 3.4).

This show that the \( B \)-Stirling numbers of the second kind is a special case of the multiparameter non-central Stirling numbers and consequently of Hsu-Shiue numbers.
From (3.12) and [15, Corollary 6.2] we obtain the new combinatorial identity
\[
\sum_{l_{n-1}=n-k}^{n} \binom{b}{l_0} \binom{b+1-i_0}{l_1} \cdots \binom{b+n-1-(i_0+i_1+\cdots+i_{n-2})}{l_{n-1}} = \sum_{j=0}^{k} \frac{(-1)^{k-j}(b+j)^n}{j!(k-j)!}.
\]

4. The matrix representation

Let \(S, s(\overline{\alpha}), S(\overline{\alpha})\) and \(S^1, S^2\) be \((n+1) \times (n+1)\) lower triangular matrices, where \(s\) and \(S\) are the matrices whose entries are the Stirling numbers of the first and second kinds (i.e., \(s = [s_{ij}]_{i,j=0}^{n}\) and \(S = [S_{ij}]_{i,j=0}^{n}\)); \(s(\overline{\alpha})\) and \(S(\overline{\alpha})\) are the matrices whose entries are the multi-parameter non-central Stirling numbers of the first and second kinds (i.e., \(s(\overline{\alpha}) = [s(\overline{\alpha})_{ij}]_{i,j=0}^{n}\) and \(S(\overline{\alpha}) = [S(\overline{\alpha})_{ij}]_{i,j=0}^{n}\)); and \(S^1, S^2\) are the matrices whose entries are the generalized Stirling-type pair of Hsu and Shiue (i.e., \(S^1 = [S^1_{ij}]_{i,j=0}^{n}\) and \(S^2 = [S^2_{ij}]_{i,j=0}^{n}\)), respectively.

The multi-parameter non-central Stirling numbers of the first and second kinds, Eqs. (2.1) and (2.2), can be represented in a matrix form as
\[
\delta = s(\overline{\alpha}) \tilde{\delta} \quad \text{and} \quad \tilde{\delta} = S(\overline{\alpha}) \delta,
\]
respectively, where \(\tilde{\delta} = ((\delta)_0, (\delta)_1, \ldots, (\delta)_n)^T\) and \(\tilde{\delta} = ((\alpha_0, \alpha_1, \ldots, \alpha_n)^T\).

A computer program is written using Maple and executed for calculating the multi-parameter non-central Stirling matrix of the first kind \(s(\overline{\alpha})\) (and \(S(\overline{\alpha}) = s^{-1}(\overline{\alpha})\)) of the second kind) and the generalized Stirling-type pair of Hsu and Shiue as a special case when \(\alpha_i = (r+i\beta)/\alpha\).

For example if \(0 \leq n \leq 3\), then
\[
s(\overline{\alpha}) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\alpha_0 & 1 & 0 & 0 \\
\alpha_0(\alpha_0-1) & \alpha_0 + \alpha_1 - 1 & 1 & 0 \\
\alpha_0(\alpha_0-1)(\alpha_0-2) & \alpha_0\alpha_1 - 3\alpha_0 + \alpha_2^2 - 3\alpha_1 + \alpha_1^2 + 2 & \alpha_0 + \alpha_1 + \alpha_2 - 3 & 1
\end{bmatrix}
\]
and
\[
S^1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
r & 1 & 0 & 0 \\
r(r - \alpha) & 2r + \beta - \alpha & 1 & 0 \\
r(r - \alpha)(r - 2\alpha) & 3r^2 - 6r\alpha + 3\beta r + \beta^2 - 3r\alpha + 2\alpha^2 - 3\beta + 3\alpha - 3 & 1
\end{bmatrix}.
\]

Remark 4.1. Note that there are some missing terms in [23, Table 1].

Similarly, the degenerate weighted Stirling matrix \(S(\lambda, \theta)\), the degenerate Stirling matrix \(S(\theta)\) and the Stirling matrix of the second kind \(S\) are given by
\[
S(\lambda, \theta) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\lambda & 1 & 0 & 0 \\
\lambda(\lambda - \theta) & 2\lambda + 1 - \theta & 1 & 0 \\
\lambda(\lambda - \theta)(r - 2\theta) & 3\lambda^2 - 6\lambda \theta + 3\lambda + 1 - 3\theta + 2\theta^2 & 3\lambda + 3 - 3\theta & 1
\end{bmatrix},
\]
\[
S(\theta) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\theta(\theta - \theta) & 1 & 0 & 0 \\
\theta(\theta - \theta)(r - 2\theta) & 2\theta + 1 - \theta & 1 & 0 \\
\theta(\theta - \theta)(r - 2\theta)^2 & 3\theta^2 - 6\theta \theta + 3\theta + 1 - 3\theta + 2\theta^2 & 3\theta + 3 - 3\theta & 1
\end{bmatrix},
\]
\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]
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\[ S(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - \theta & 1 & 0 \\ 0 & (1 - \theta)(1 - 2\theta) & 3(1 - \theta) & 1 \end{bmatrix}, \]

and

\[ S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}, \]

respectively, as special cases.

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**References**


Explicit Formulas and Identities for Generalized Stirling Numbers


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