Difference Equations for Generalized Meixner Polynomials

HERMAN BAVINCK AND HENK VAN HAERINGEN

Faculty of Technical Mathematics and Informatics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

Submitted by Bruce C. Berndt

Received February 24, 1993

DEDICATED TO RICHARD ASKEY ON HIS 60TH BIRTHDAY

In this paper is introduced a system of polynomials orthogonal with respect to the classical discrete weight function for Meixner polynomials with an extra point mass added at \( x = 0 \). A difference operator of infinite order is constructed for which these new polynomials are eigenfunctions and a second-order difference equation is given with polynomial coefficients, \( n \)-dependent and of at most degree 2, which these polynomials satisfy. © 1994 Academic Press, Inc.

1. INTRODUCTION

For the polynomials \( \{ L_n^{\alpha,N}(x) \}_{n=0}^{\infty} \) that are orthogonal on \([0, \infty)\) with respect to the weight function

\[
\frac{1}{f(x+1)} x^\alpha e^{-x} + N \delta(x), \quad \alpha > -1, \; N \geq 0
\]

(see Koornwinder [7]), Koekoek and Koekoek [6] found a differential equation of the form

\[
N \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + xy''(x) + (x+1-x) y'(x) + ny(x) = 0,
\]

where the coefficients \( a_i(x), \; i \in \{1, 2, 3, \ldots\} \), are independent of \( n \) and \( a_0(x) = a_0(n, x) \) depends on \( n \) but is independent of \( x \). For a more constructive approach to this differential equation see [2].

At a conference held in Erice (May 1990), Askey [1] posed the problem of finding difference equations of a similar form for generalizations of the discrete orthogonal polynomials that are orthogonal with respect to a
classical weight function at which a point mass at the point \( x = 0 \) is added. In [3] a solution to this problem for Charlier polynomials is given and in the present paper we deal with the more complicated case of Meixner polynomials. Moreover, we construct a second-order difference equation with polynomial coefficients, \( n \)-dependent and of degree at most 2, which the generalized Meixner polynomials satisfy.

2. MEIXNER POLYNOMIALS

Taking a normalization slightly different from the one used in [4], we define the classical Meixner polynomials \( M_n(x; \beta, c) \) by the generating function

\[
\sum_{n=0}^{\infty} M_n(x; \beta, c) t^n = \left( 1 - \frac{t}{c} \right)^x (1 - t)^{-\beta},
\]

from which it easily follows that

\[
M_n(x; \beta, c) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{-x - \beta}{n - k} c^{-k}
\]

\[
= \frac{(\beta)_n}{n!} {}_2F_1 \left( \begin{array}{c} -n, -x \\ \beta \end{array} \right| 1 - \frac{1}{c} \right), \quad n = 0, 1, 2, \ldots.
\]  

(2.2)

Note that (2.2) can be used for all values of \( \beta \) and \( c \) except \( c = 0 \). Formula (2.3) is not defined for \( \beta = 0, -1, -2, \ldots \), and for \( c = 0 \). Obviously

\[
M_n(0; \beta, c) = \frac{(\beta)_n}{n!}, \quad n = 0, 1, 2, \ldots.
\]

(2.4)

Meixner polynomials are closely related to Jacobi polynomials. In fact, we have

\[
M_n(x; \beta, c) = (-c)^{-n} P_n^{(x - \beta - 1, -\beta - 1)}(1 - 2c), \quad n = 0, 1, 2, \ldots.
\]

(2.5)

For \( \beta > 0 \) and \( 0 < c < 1 \) the Meixner polynomials satisfy the orthogonality relation

\[
(1 - c)^{\beta} \sum_{x=0}^{\infty} M_n(x; \beta, c) M_p(x; \beta, c) \frac{c^x (\beta)_x}{x!} = \frac{c^{-n}}{n!} (\beta)_n \delta_{np}, \quad n, p = 0, 1, 2, \ldots
\]

(2.6)
and the second-order difference equation
\[ x \Delta y(x) + [\beta c - x(1 - c)] D y(x) + n(1 - c) y(x) = 0, \] (2.7)
where \( \Delta y(x) = y(x + 1) - y(x) \) and \( \nabla y(x) = y(x) - y(x - 1) \).

Direct consequences of (2.1) are the following formulae which are valid for all real \( x, \beta, c \) (except \( c = 0 \)), and \( v \), and for all \( n \in \{0, 1, 2, \ldots\} \):
\[ M_n(x + v; \beta - v, c) = \sum_{k=0}^{n} \binom{v}{k} \left( -\frac{1}{c} \right)^k M_{n-k}(x; \beta, c), \] (2.8)
\[ M_n(x; \beta - v, c) = \sum_{k=0}^{n} \binom{v}{k} (-1)^k M_{n-k}(x; \beta, c), \] (2.9)
\[ \Delta M_n(x; \beta, c) = \left( \frac{c - 1}{c} \right) M_{n-1}(x; \beta + 1, c). \] (2.10)

In the sequel we always take \( \beta > 0, \quad 0 < c < 1 \), and since \( c \) is kept fixed during the whole paper we simplify the notation putting \( M_n(x; \beta) \) instead of \( M_n(x; \beta, c) \).

3. GENERALIZED MEIXNER POLYNOMIALS

Let \( P \) denote the space of all polynomials with real coefficients. We consider the inner product
\[ \langle f(x), g(x) \rangle = (1 - c)^{\beta} \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} f(x) g(x) \]
\[ + N f(0) g(0), \quad N \geq 0, f, g \in P. \] (3.1)

We show that coefficients \( A_n \) and \( B_n \) can be chosen in such a way that the polynomials \( M_n^N(x; \beta) = M_n^N(x; \beta, c) \) that are orthogonal with respect to the inner product (3.1) can be written as
\[ M_n^N(x; \beta) = A_n M_n(x; \beta) + B_n M_{n-1}(x - 1; \beta + 1). \]

Suppose that \( n \geq 2 \) and \( p(x) = xq(x) \) with degree \( \lceil q(x) \rceil \leq n - 2 \). Then we obtain using (2.6)
\[ \langle p(x), M_n^N(x; \beta) \rangle \]
\[ = B_n (1 - c)^{\beta} \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} xq(x) M_{n-1}(x - 1; \beta + 1) \]
\[ = B_n (1 - c)^{\beta} c \beta \sum_{x=1}^{\infty} \frac{c^{x-1}(\beta + 1)_{x-1}}{(x-1)!} q(x) M_{n-1}(x - 1; \beta + 1) = 0. \]
Hence for \( n \geq 1 \) the coefficients \( A_n \) and \( B_n \) have to fulfill only the following condition:

\[
0 = \langle 1, M_n^N(x; \beta) \rangle \\
= B_n(1-c)\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)}{x!} M_{n-1}(x-1; \beta+1) \\
+ NA_nM_n(0; \beta) + NB_nM_{n-1}(-1; \beta+1).
\]

By using (2.8) with \( v = -1 \) and (2.6) it follows that

\[
(1-c)\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)}{x!} M_{n-1}(x-1; \beta+1) = c^{-n+1}.
\]

So a possible choice for \( A_n \) and \( B_n \) is

\[
A_n = Nc^{n-1}M_{n-1}(-1; \beta+1) + 1 \quad \text{and} \quad \\
B_n = -Nc^{n-1}M_n(0; \beta),
\]

and we put

\[
M_n^N(x; \beta) = [Nc^{n-1}M_{n-1}(-1; \beta+1) + 1] M_n(x; \beta) \\
- Nc^{n-1}M_n(0; \beta) M_{n-1}(x-1; \beta+1), \quad n = 0, 1, 2, \ldots.
\]

(3.2)

Here and in the sequel we use \( M_{-k}(x; \beta, c) = 0 \), \( k = 1, 2, \ldots \). Note that \( M_n^N(x; \beta) = M_n(x; \beta) \) and that \( M_n^0(0; \beta) = M_n(0; \beta) \).

4. The Difference Equation

We are looking for a difference equation of the form

\[
N \sum_{i=0}^{\infty} a_i(x) \Delta^i y(x) + x \Delta \nabla y(x) + [\beta c - x(1-c)] \\
\times \Delta y(x) + n(1-c) \cdot y(x) = 0
\]

(4.1)

for the polynomials \( \{M_n^N(x; \beta)\}_{n=0}^{\infty} \) given by (3.2), where the coefficients \( \{a_i(x)\}_{i=1}^{\infty} = \{a_i(x, \beta, c)\}_{i=1}^{\infty} \) are functions of \( x, \beta, \) and \( c \), but are independent of the degree \( n \). Moreover, since we want the polynomials \( \{M_n^N(x; \beta)\}_{n=0}^{\infty} \) to be eigenfunctions of a difference operator, we assume \( a_0(x) = a_0(n, \beta, c) \) to be independent of \( x \) but dependent on \( n \). So we insert

\[
y(x) = M_n^N(x; \beta) \\
= [Nc^{n-1}M_{n-1}(-1; \beta+1) + 1] M_n(x; \beta) \\
- Nc^{n-1}M_n(0; \beta) M_{n-1}(x-1; \beta+1)
\]
into (4.1). Using the difference equation (2.7) for the classical Meixner polynomials, (2.10), and (2.8) in the case \( v = 1 \), we find

\[
N\left[Nc^{n-1}M_{n-1}(-1; \beta + 1) + 1\right] \sum_{i=0}^{\infty} a_i(x) \Delta^i M_n(x; \beta) \\
- N^2c^{n-1}M_n(0; \beta) \sum_{i=0}^{\infty} a_i(x) \Delta^i M_{n-1}(x-1; \beta + 1) \\
- Nc^{n-1}M_n(0; \beta)(1-c) M_{n-1}(x-2; \beta + 2) = 0.
\]

This relation has to hold for all values of \( n, \beta, c, \) and \( N > 0 \). Its left-hand side is a polynomial in \( N \), so each coefficient must be zero. Thus we find

\[
M_{n-1}(-1; \beta + 1) \sum_{i=0}^{\infty} a_i(x) \Delta^i M_n(x; \beta) \\
- M_n(0; \beta) \sum_{i=0}^{\infty} a_i(x) \Delta^i M_{n-1}(x-1; \beta + 1) = 0 \tag{4.2}
\]

and

\[
\sum_{i=0}^{\infty} a_i(x) \Delta^i M_n(x; \beta) - c^{n-1}(1-c) M_n(0; \beta) M_{n-1}(x-2; \beta + 2) = 0.
\]

This leads to the following systems of equations:

\[
\sum_{i=0}^{\infty} a_i(x) \Delta^i M_n(x; \beta) = c^{n-1}(1-c) M_n(0; \beta) M_{n-1}(x-2; \beta + 2), \tag{4.3}
\]

\[
\sum_{i=0}^{\infty} a_i(x) \Delta^i M_{n-1}(x-1; \beta + 1) = c^{n-1}(1-c) \\
x M_{n-1}(-1; \beta + 1) M_{n-1}(x-2; \beta + 2). \tag{4.4}
\]

Formula (4.2) can be rewritten as

\[
\sum_{i=1}^{\infty} a_i(x) \left[ M_{n-1}(-1; \beta + 1) \Delta^i M_n(x; \beta) \\
- M_n(0; \beta) \Delta^i M_{n-1}(x-1; \beta + 1) \right] \\
= a_0(n, \beta, c) \left[ M_{n-1}(x-1; \beta + 1) M_n(0; \beta) \\
- M_n(x; \beta) M_{n-1}(-1; \beta + 1) \right].
\]
The right-hand side is 0 for $x = 0$ and since this holds for all values of $n$, \( \beta \), and \( c \), we conclude step by step that \( a_i(0) = a_i(0, \beta, c) = 0 \) for all \( i \in \{1, 2, \ldots\} \). Therefore, setting $x = 0$ in formula (4.3), we obtain

$$a_0(x) = a_0(0) = a_0(n, \beta, c) = c^{n-1}(1 - c) M_{n-1}(-2; \beta + 2), \quad (4.5)$$

and Eqs. (4.3) and (4.4) can be rewritten as

$$\sum_{i=1}^\infty a_i(x) \Delta^i M_n(x; \beta) = c^{n-1}(1 - c) \left[ M_n(0; \beta) M_{n-1}(x - 2; \beta + 2) - M_n(x; \beta) M_{n-1}(-2; \beta + 2) \right], \quad (4.6)$$

and

$$\sum_{i=1}^\infty a_i(x) \Delta^i M_{n-1}(x - 1; \beta + 1) = c^{n-1}(1 - c) \left[ M_{n-1}(0; \beta + 1) M_{n-1}(x - 2; \beta + 2) - M_{n-1}(x - 1; \beta + 1) M_{n-1}(-2; \beta + 2) \right]. \quad (4.7)$$

We now show that any solution of the system (4.7) also satisfies (4.6). Since by (2.8) with \( v = 1 \) we have

$$\sum_{i=1}^\infty a_i(x) \Delta^i M_n(x; \beta) = \sum_{i=1}^\infty a_i(x) \Delta^i M_n(x - 1; \beta + 1) - \frac{1}{c} \sum_{i=1}^\infty a_i(x) \Delta^i M_{n-1}(x - 1; \beta + 1),$$

it remains to be proved that for \( n \geq 1 \),

$$c^{n-1}(1 - c) \left[ M_n(0; \beta) M_{n-1}(x - 2; \beta + 2) - M_n(x; \beta) M_{n-1}(-2; \beta + 2) \right] = c^n(1 - c) \left[ M_n(-1; \beta + 1) M_n(x - 2; \beta + 2) - M_n(x - 1; \beta + 1) \right] \times M_n(-2; \beta + 2) - c^{n-2}(1 - c) \left[ M_{n-1}(-1; \beta + 1) \times M_{n-1}(x - 2; \beta + 2) - M_{n-1}(x - 1; \beta + 1) M_{n-1}(-2; \beta + 2) \right].$$

This can be shown by combining terms and applying the formula (2.8) with \( v = 1 \) for different values of \( x \) and \( \beta \).

5. A Formula for the Coefficients \( a_i(x) \)

We now solve the system (4.7). Writing \( n \) instead of \( n - 1 \) and using (2.10), we get
\[
\sum_{i=1}^{\infty} a_i(x) \left( \frac{c-1}{c} \right)^i M_{\nu-i}(x-1; \beta + i + 1) \\
= c^n (1 - c) [ M_{\nu}(-1; \beta + 1) M_{\nu}(x-2; \beta + 2) \\
- M_{\nu}(x-1; \beta + 1) M_{\nu}(-2; \beta + 2) ], \quad n = 1, 2, \ldots \tag{5.1}
\]

If we consider \( a_i(x)(1 - 1/c)^i \) as unknown, the matrix \( T \) of the system (5.1) is triangular with entries \( t_{ij} \) for which we have

\[
t_{ij} = M_{i-j}(x-1; \beta + j + 1), \quad \text{for} \quad i, j = 1, 2, \ldots
\]

We show that the entries \( u_{ij} \) of the inverse matrix are

\[
u_{ij} = M_{i-j}(-x+1; -\beta - i), \quad \text{for} \quad i, j = 1, 2, \ldots \tag{5.2}
\]

In order to prove (5.2) we use the generating function (2.1) to find that

\[
\sum_{n=0}^{\infty} M_{n}(-x+1; -\beta - i) t^n \sum_{n=0}^{\infty} M_{n}(x-1; \beta + j + 1) t^n = (1 - t)^{i-j-1}.
\]

Equating the coefficients of \( t^{i-j} \) \((i \geq j)\) on both sides we obtain

\[
\sum_{k=j}^{i} M_{i-k}(-x+1; -\beta - i) M_{k-j}(x-1; \beta + j + 1) = \delta_{ij}.
\]

We conclude that the unique solution of the system (5.1) reads

\[
a_i(x) = \left( \frac{c-1}{c} \right)^i \sum_{k=1}^{i} M_{i-k}(-x+1; -\beta - i) \\
\times c^k (1 - c) [ M_k(-1; \beta + 1) M_k(x-2; \beta + 2) \\
- M_k(x-1; \beta + 1) M_k(-2; \beta + 2) ], \quad i = 1, 2, \ldots \tag{5.3}
\]

The difference equation (4.1), with \( a_0(x) \) given by (4.5) and \( a_i(x) \) by (5.3) for \( i = 1, 2, \ldots \), is of infinite order for all values of \( \beta \) and \( c \) \((0 < c < 1)\) and \( N > 0 \). We prove this by evaluating the coefficient \( k_i = k_i(\beta, c) \) of \( x^i \) in \( a_i(x) \), \( i \geq 1 \). From (2.3) we derive that

\[
M_n(x; \beta, c) = \frac{1}{n!} \left( \frac{c-1}{c} \right)^n x^n + \text{terms with lower powers of} \ x. \tag{5.4}
\]

Furthermore, in the case \( v = n \), formula (2.8) can be rewritten as

\[
(-c)^n M_n(x + n; \beta - n, c) = \sum_{k=0}^{n} \binom{n}{k} (-c)^k M_k(x; \beta, c). \tag{5.5}
\]
Hence we can write, using (5.3), (5.4), (5.5), and (2.8) with \( v = 1 \),

\[
k_i = \left( \frac{c}{c-1} \right)^i \sum_{k=1}^{i} \frac{1}{(i-k)!} \left( -\frac{c-1}{c} \right)^{i-k} c^k (1-c) \\
\times \frac{1}{k!} \left( \frac{c-1}{c} \right)^k [M_k(-1; \beta+1) - M_k(-2; \beta+2)]
\]

\[
= \frac{(-1)^i}{i!} (1-c) \sum_{k=0}^{i} \binom{i}{k} (-c)^k [M_k(-1; \beta+1) - M_k(-2; \beta+2)]
\]

\[
= \frac{c^i(c-1)}{i!} [M_i(i-1; \beta-i+1) - M_i(i-2; \beta-i+2)]
\]

\[
= \frac{c^{i-1}(c-1)}{i!} M_{i-1}(i-2; \beta-i+2).
\] (5.6)

Hence using (2.5) we obtain

\[
k_i = (-1)^i \frac{1-c}{i!} P_{i-1}^{(-1, \beta-i+1)}(1-2c)
\]

\[
= \frac{c-1}{i!} P_{i-1}^{(-1, \beta-i+1)}(2c-1).
\]

In particular, \( k_1 = c-1 \). Expressing the Jacobi polynomials in terms of \( _2F_1 \)'s and using Euler's transformation,

\[
_2F_1 \left( \begin{array}{c} a, & b \\ c & \end{array} \bigg| z \right) = (1-z)^{-a-b} _2F_1 \left( \begin{array}{c} c-a, & c-b \\ c & \end{array} \bigg| z \right),
\]

we obtain the relation

\[
2nP_{n}^{(\beta, -1)}(z) = (n + \beta)(z + 1) P_{n-1}^{(\beta, 1)}(z).
\]

Hence

\[
k_i(\beta, c) = \frac{\beta c(c-1)}{i!(i-1)} P_{i-2}^{(\beta-i+1, 1)}(2c-1), \quad \text{for } i \geq 2.
\] (5.7)

In order to show that the difference equation (4.1) is of infinite order we prove that no pair of consecutive coefficients \( (k_i, k_{i+1}) \) can vanish simultaneously. This even holds for arbitrary complex values of \( \beta \) and \( c \), provided that \( \beta c(c-1) \neq 0 \).
We start with two known relations for Jacobi polynomials (see [5, p. 173, formula (33), with \( n \) replaced by \( n - 1 \), and formula (35)]):

\[
n \frac{d}{dz} P_n^{(\alpha, \beta)}(z) = - (\beta + n) P_n^{(\alpha, \beta)}(z) + \frac{1}{2} (\alpha + \beta + 2n)(1 + z) P_{n+1}^{(\alpha, \beta+1)}(z),
\]

\[
(\alpha + \beta + 2n) P_n^{(\alpha-1, \beta)}(z) = (\alpha + \beta + n) P_n^{(\alpha, \beta)}(z) - (\beta + n) P_n^{(\alpha, \beta-1)}(z).
\]

We eliminate \( P_n^{(\alpha, \beta)}(z) \) from these two relations to find

\[
(\beta + n) P_{n-1}^{(\alpha, \beta)}(z) + n P_n^{(\alpha-1, \beta)}(z) = \frac{1}{2} (\alpha + \beta + n)(1 + z) P_{n+1}^{(\alpha, \beta+1)}(z)
\]

\[
= (1 + z) \frac{d}{dz} P_n^{(\alpha-1, \beta)}(z),
\]

and we take the special case \( \beta := 1, n := i - 1, \alpha := \beta - i + 1 \):

\[
i P_{i-2}^{(i, i+1)}(z) + (i - 1) P_{i-1}^{(i, i+1)}(z) = (1 + z) \frac{d}{dz} P_{i-2}^{(i-1, i+1)}(z).
\tag{5.8}
\]

Now suppose that \( k_{i+1}(\beta, c) = k_{i+1}(\beta, c) = 0 \) for some \( i \geq 2 \). Putting for convenience \( 2c - 1 = z \), we then get from (5.7)

\[
P_{i-2}^{(i, i+1)}(z) = P_{i-1}^{(i-1, i+1)}(z) = 0 \quad \text{for some} \quad i \geq 2.
\]

Hence from (5.8) we conclude that both \( P_{i-2}^{(i, i+1)}(z) \) and its derivative vanish. Since the Jacobi polynomial satisfies a second-order differential equation, this is impossible if \( z \) is a regular point of the differential equation, for then all its derivatives would vanish. The only singular points are \( z = -1 \) and \( z = 1 \) corresponding to \( c = 0 \) and \( c = 1 \).

It is well known that the Charlier polynomials \( C_n^{(a)}(x) \) can be regarded as limits of Meixner polynomials:

\[
C_n^{(a)}(x) = \lim_{\beta \to \infty} \left( - \frac{a}{\beta} \right)^n M_n(x; \beta, \frac{a}{\beta}).
\]

By using this relation in (4.5), (5.3), and (5.6), we retrieve the results of [3] for generalized Charlier polynomials.

6. A SECOND-ORDER DIFFERENCE EQUATION

In this section we show that the polynomials \( M_n(x; \beta, c) \) satisfy a second-order difference equation with polynomial coefficients, \( n \)-dependent,
and of at most second degree. We construct this difference equation in a way similar to the method derived in [7, Prop. 6.1], for obtaining differential equations for systems of orthogonal polynomials.

First, using (2.10) twice, we write (3.2) in the form

$$M_n^N(x; \beta) = \left[ Nc^{n-1}M_{n-1}(-1; \beta + 1) + 1 \right] M_n(x-1; \beta)$$

$$+ \left[ 1 - N \frac{c^n}{c-1} M_{n}(-1; \beta) \right] \Delta M_n(x-1; \beta), \quad n = 0, 1, 2, \ldots$$

(6.1)

and from (2.7) it is not difficult to derive that

$$c(x + \beta) \Delta^2 M_n(x-1; \beta) + [(c - 1)(x-n) + c\beta] \Delta M_n(x-1; \beta)$$

$$+ (1 - c) n M_n(x-1; \beta) = 0, \quad n = 0, 1, 2, \ldots$$

(6.2)

If we put

$$u := M_n^N(x; \beta), \quad p := 1 + Nc^{n-1}M_{n-1}(-1; \beta + 1),$$

$$q := 1 - N \frac{c^n}{c-1} M_{n}(-1; \beta), \quad y := M_n(x-1; \beta),$$

(6.1) and (6.2) can be rewritten as

$$u = py + q \Delta y,$$

(6.4)

$$(cx + c\beta) \Delta^2 y + [(c - 1)x + c(\beta - n) + n] \Delta y + (1 - c) ny = 0.$$  

(6.5)

We assume $q \neq 0$. We eliminate $\Delta y$ and $\Delta^2 y$ from (6.4), (6.5), and the equation obtained by taking the difference of (6.4). This leads to the relation

$$q(cx + c\beta) \Delta u + (ax + b) u + (dx + f) y = 0,$$

(6.6)

with

$$a := q(c - 1) - pc, \quad b := \beta c(q - p) + qn(1-c), \quad d := p^2c - pq(c - 1),$$

$$f := p^2c\beta - pq(c\beta - cn + n) + q^2(1-c)n.$$  

(6.7)

Next we eliminate $y$ and $\Delta y$ from (6.4), (6.6), and the first difference of (6.6). We finally obtain

$$q^2 c(dx + f)(x + \beta + 1) \Delta^2 u$$

$$+ [q(ax + a + b + qc)(dx + f) + qc(x + \beta)(dp x + dp + fp - dq)] \Delta u$$

$$+ [(dx + aq + d + f)(dx + f) + (ax + b)(dp x + dp + fp - dq)] u = 0,$$

where $u$, $p$, and $q$ are given in (6.3) and $a$, $b$, $d$, and $f$ in (6.7).
REFERENCES


7. T. H. Koornwinder, Orthogonal polynomials with weight function \((1-x)^a (1+x)^b + M\delta(x+1) + N\delta(x-1)\), Canad. Bull. Math. 27, No. 2 (1984), 205–214.