A FAMILY OF FIBONACCI-LIKE SEQUENCES

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We consider the recurrence relation

\[ G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} a_j n^j, \]

where \( G_0 = G_1 = 1 \), and we express \( G_n \) in terms of the Fibonacci numbers \( F_n \) and \( F_{n-1} \), and in the parameters \( a_0, \ldots, a_k \).

For integer values of \( k, a_0, \ldots, a_k \), the relation

\[ G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} a_j n^j, \]

(1)

where \( G_0 = G_1 = 1 \), forms a difference equation that can be solved by standard methods. In this note, we provide such a solution for equations of this type, in which we treat \( a_0, \ldots, a_k \) as parameters.

First, the solution \( G_n^{(h)} \) of the corresponding homogeneous equation equals

\[ G_n^{(h)} = C_1 \phi_1^n + C_2 \phi_2^n, \]

where \( \phi_1 = \frac{1}{2}(1 + \sqrt{5}) \) and \( \phi_2 = \frac{1}{2}(1 - \sqrt{5}) \); cf. e.g., [1] and [3].

Second, as a particular solution, we try

\[ G_n^{(p)} = \sum_{i=0}^{k} A_i n^i, \]

which yields

\[ \sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} A_i (n - 1)^i - \sum_{i=0}^{k} A_i (n - 2)^i - \sum_{i-0}^{k} a_i n^i = 0 \]

or

\[ \sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} \left( \sum_{j=0}^{i} A_j \left( \frac{i}{j} \right) (1 - 1)^{i-j} (1 + 2^{i-j}) n^j \right) - \sum_{i=0}^{k} a_i n^i = 0 \]

For each \( i \) (\( 0 \leq i \leq k \)), we have

\[ A_i - \sum_{m=i}^{k} \beta_{i,m} A_m - a_i = 0, \]

(2)

where, for \( m \geq i \),

\[ \beta_{i,m} = \left( \frac{m}{i} \right) (-1)^{m-i} (1 + 2^{m-i}). \]

From the recurrence relation (2), \( A_k, \ldots, A_0 \) can be computed (in that order): \( A_i \) is a linear combination of \( \alpha_i, \ldots, \alpha_k \). However, a more explicit expression for \( A_i \) can be obtained by setting...
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\[ A_i = -\sum_{j=0}^{k} a_{ij} \alpha_j. \]

(The minus sign happens to be convenient in the sequel.) Then (2) implies

\[ -\sum_{j=0}^{k} a_{ij} \alpha_j + \sum_{m=0}^{i} \beta_{im} \left( \sum_{k=0}^{i} a_{mk} \alpha_k \right) - a_i = 0. \]

Since \( \beta_{ii} = 2 \), we have, for \( 0 \leq i \leq k \),

\[ a_{ii} = 1 \]

\[ a_{ij} = -\sum_{m=0}^{i} \beta_{im} a_{mj}, \text{ if } j > i. \]

Hence,

\[ G^{(p)}_n = -\sum_{i=0}^{k} \sum_{j=0}^{k} a_{ij} \alpha_j n^i = -\sum_{j=0}^{k} a_{j} \left( \sum_{i=0}^{j} a_{ij} n^i \right). \]

Finally, we ought to determine \( C_1 \) and \( C_2 \): \( G_0 = G_1 = 1 \) implies

\[ C_1 + C_2 = 1 - G^{(p)}_0, \quad C_1 \phi_1 + C_2 \phi_2 = 1 - G^{(p)}_1. \]

These equalities yield

\[ C_1 = ((G^{(p)}_0 - 1) \phi_2 - 1) \phi_1 - G^{(p)}_0 (\sqrt{5})^{-1} \]

\[ = ((1 - G^{(p)}_0) \phi_1 - G^{(p)}_1 + G^{(p)}_0) (\sqrt{5})^{-1}, \]

\[ C_2 = ((G^{(p)}_0 - 1) \phi_1 + G^{(p)}_1 - 1) (\sqrt{5})^{-1} \]

\[ = (((1 - G^{(p)}_0) \phi_2 - G^{(p)}_1 + G^{(p)}_0) (\sqrt{5})^{-1}, \]

and

\[ G_n = (1 - G^{(p)}_0) F_n + (G^{(p)}_1 - G^{(p)}_0) F_{n-1} + G^{(p)}_n. \]

Summarizing, we have the following proposition.

**Proposition:** The solution of (1) can be expressed as

\[ G_n = (1 + \lambda_k) F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k} a_{j} p_{j}(n), \]

where \( \lambda_k \) is a linear combination of \( \alpha_0, \ldots, \alpha_k \), \( \lambda_k \) is a linear combination of \( \alpha_1, \ldots, \alpha_k \), and for each \( j \) (\( 0 \leq j \leq k \)), \( p_{j}(n) \) is a polynomial of degree \( j \):

\[ \lambda_k = \sum_{j=0}^{k} a_{j} \alpha_j, \quad \lambda_k = \sum_{j=0}^{k} \left( \sum_{t=1}^{j} a_{jt} \right) \alpha_j, \quad p_{j}(n) = \sum_{t=0}^{j} a_{jt} n^t. \]

**Remarks:**

(1) For \( j = 0, 1, \ldots, 8 \), the polynomials \( p_{j}(n) \) are given in Table 1.

(2) No assumptions on \( \alpha_0, \ldots, \alpha_k \) have been made; thus, they may be rational or real numbers as well.

(3) Changing \( G_1 = 1 \) into \( G_1 = \sigma \) only affects \( \lambda_k \); it has to be increased with \( \sigma - 1 \).
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Table 1

<table>
<thead>
<tr>
<th>j</th>
<th>$p_j(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$n + 3$</td>
</tr>
<tr>
<td>2</td>
<td>$n^2 + 6n + 13$</td>
</tr>
<tr>
<td>3</td>
<td>$n^3 + 9n^2 + 39n + 81$</td>
</tr>
<tr>
<td>4</td>
<td>$n^4 + 12n^3 + 78n^2 + 324n + 673$</td>
</tr>
<tr>
<td>5</td>
<td>$n^5 + 15n^4 + 130n^3 + 810n^2 + 3365n + 6993$</td>
</tr>
<tr>
<td>6</td>
<td>$n^6 + 18n^5 + 195n^4 + 1620n^3 + 10095n^2 + 41958n + 87193$</td>
</tr>
<tr>
<td>7</td>
<td>$n^7 + 21n^6 + 273n^5 + 2835n^4 + 23555n^3 + 146853n^2 + 610351n + 1268361$</td>
</tr>
<tr>
<td>8</td>
<td>$n^8 + 24n^7 + 364n^6 + 4536n^5 + 47110n^4 + 391608n^3 + 2441404n^2 + 10146888n + 21086113$</td>
</tr>
</tbody>
</table>

(4) The coefficients of $a_0$, $a_1$, $a_2$, ... in $\Lambda_k$ and of $a_1$, $a_2$, ... in $\lambda_k$ are independent of $k$. Thus, they give rise to two infinite sequences $\Lambda$ and $\lambda$ of natural numbers, as $k$ tends to infinity, of which the first few elements are

$$\Lambda: 1, 3, 13, 81, 673, 6993, 87193, 1268361, 21086113, ...$$

$$\lambda: 1, 7, 49, 415, 4321, 53887, 783889, 13031935, ...$$

Neither of these sequences is included in [2].

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REFERENCES