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On a general q-Fourier transformation with nonsymmetric kernels[1]

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Abstract

Wiener used the Poisson kernel for the Hermite polynomials to deal with the classical Fourier transform. Askey, Atakishiyev and Suslov used this approach to obtain a q-Fourier transform by using the continuous q-Hermite polynomials. Rahman and Suslov extended this result by taking the Askey–Wilson polynomials, considered to be the most general continuous classical orthogonal polynomials. The theory of q-Fourier transformation is further extended here by considering a nonsymmetric version of the Poisson kernel with Askey–Wilson polynomials. This approach enables us to obtain some new results, for example, the complex and real orthogonalities of these kernels.

Keywords: q-Fourier transform; Integral transforms; q-orthogonal polynomials; Hermite polynomials; Poisson kernels; Fourier transform; Basic hypergeometric series; Askey–Wilson polynomials; Al-Salam–Chihara polynomials

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1. Introduction

Two of the most frequently used formulas in the area of integral transforms are the classical Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} g(y)\, dy := F[g](x), \tag{1.1}$$

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and the corresponding formal inversion formula

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx := F^{-1}[f](y). \tag{1.2}$$

There are many excellent references on Fourier transforms (see e.g., Titchmarsh's classic book [20]), but, for our purposes the most important reference is Wiener [21], who looked at the Fourier transform from the point of view of classical orthogonal polynomials on R, in particular, the Hermite polynomials $H_n(x)$.

An elementary result in nonrelativistic quantum mechanics is that the normalized wave function of the Hamiltonian operator for the harmonic oscillator is the Hermite function

$$\Psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2}, \tag{1.3}$$

see, e.g., Landau and Liftschitz [13, p. 70]. The bilinear generating function (the Poisson kernel) for these functions is

$$\begin{aligned} K_t(x, y) &= \sum_{n=0}^{\infty} t^n \Psi_n(x) \Psi_n(y) \\ &= [\pi(1-t^2)]^{-1/2} \exp\left[\frac{4xyt - (x^2+y^2)(1+t^2)}{2(1-t^2)}\right], \end{aligned} \tag{1.4}$$

which is better known as Mehler's formula; see [19–21].

We make two important observations. First, the kernel $K_t(x, y)$ as given by the function on the right of the second equality in (1.4) has the property

$$\lim_{t \to i} K_t(x, y) = K_i(x, y) = \frac{e^{ixy}}{\sqrt{2\pi}}, \tag{1.5}$$

which, of course, is the kernel of the integral transform (1.1). In fact, $K_t(x, y)$ is analytic everywhere on the unit circle in the complex t-plane except at $t = \pm 1$. We shall set

$$\begin{aligned} \mathcal{K}_\tau(x, y) &= K_t(x, y)|_{t = e^{i\tau}} \\ &= \frac{e^{i(\pi/2 - \tau)/2}}{\sqrt{2\pi \sin\tau}} \exp\left[i\,\frac{2xy - (x^2+y^2)\cos\tau}{2\sin\tau}\right], \end{aligned} \tag{1.6}$$

$0 < \tau < \pi$, and define a generalized Fourier transform by the formula

$$f(x) := \int_{-\infty}^{\infty} \mathcal{K}_\tau(x, y) g(y)\,dy. \tag{1.7}$$

The corresponding formal inversion formula is

$$g(y) := \int_{-\infty}^{\infty} \mathcal{K}_\tau^*(x, y) f(x)\,dx, \tag{1.8}$$

where $*$ indicates the complex conjugate. For a sketch of the proof of (1.8) see [17].

The second observation is that the wave functions in (1.3) are the eigenfunctions of the Fourier operators in (1.1) and (1.7):

$$\mathrm{e}^{\mathrm{i}n\tau}\Psi_n(x) = \int_{-\infty}^{\infty} \mathscr{K}_\tau(x, y)\Psi_n(y)\,\mathrm{d}y, \tag{1.9}$$

where $0 < \tau < \pi$ for (1.7) and, $\tau = \pi/2$ for (1.1).

Our primary interest in this paper is a general q-analogue of (1.9) as well as of (1.7). As was explained in [17], a q-analogue of a formula contains a parameter q, usually complex and $|q| < 1$, such that the limit of the new formula as $q \to 1$ is the given formula. For example, a q-analogue of a complex number a is $(1 - q^a)/(1 - q)$ since

$$a = \lim_{q\to 1} \frac{1 - q^a}{1 - q},$$

the branch of q^a chosen here is the one that gives $\lim_{q\to 1} q^a = 1$.

There is a great deal of interest these days in q-analogues of important classical formulas. Sometimes the interest is nothing more than a curiosity, but in view of some recent developments in the generalizations of the classical harmonic oscillator problem (see, e.g., [4, 5, 8, 9, 14]) the interest in a q-version of the Fourier transform is much more than an academic curiosity. The question appears in a very natural way in the q-oscillator problem.

As was shown in [2] it is Wiener's treatment of the Fourier integrals that contains the key to a meaningful q-extension. Just as the Hermite polynomials are associated with the wave functions for the harmonic oscillator, the continuous q-Hermite polynomials,

$$H_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,\mathrm{e}^{\mathrm{i}(n-2k)\theta}, \quad x = \cos\theta, \tag{1.10}$$

are associated with the q-wave functions for the q-harmonic oscillator, namely,

$$\Psi_n(x|q) = [(q^{n+1};q)_\infty/2\pi]^{1/2}\sqrt{\rho_0(x)}\,H_n(x|q), \tag{1.11}$$

where

$$\rho_0(x) = 4\sqrt{1-x^2}\prod_{k=1}^{\infty}(1 - 2(2x^2-1)q^k + q^{2k}), \quad 0 < q < 1. \tag{1.12}$$

The q-shifted factorials in (1.10) and (1.11) are defined by

$$\begin{aligned} &(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}), \quad n = 1, 2, \ldots, \\ &(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad |q| < 1. \end{aligned} \tag{1.13}$$

The q-annihilation operator b and the q-creation operator b^+ that satisfy the commutation rule

$$bb^+ - q^{-1}b^+b = 1 \tag{1.14}$$

were introduced explicitly in [8]. They act on the q-wave function (1.11) in the following manner:

$$\begin{aligned} b\Psi_n(x|q) &= \left(\frac{1-q^{-n}}{1-q^{-1}}\right)^{1/2}\Psi_{n-1}(x|q), \\ b^+\Psi_n(x|q) &= \left(\frac{1-q^{-n-1}}{1-q^{-1}}\right)^{1/2}\Psi_{n+1}(x|q), \end{aligned} \tag{1.15}$$

see [2, 8]. One also has the orthogonality property

$$\int_{-1}^{1} \Psi_m(x\,|\,q)\Psi_n(x\,|\,q)\,\mathrm{d}x = \delta_{m,n}, \tag{1.16}$$

see, e.g., [3, 11]. The important difference between this formula and the corresponding formula for the Hermite functions (1.3), namely,

$$\int_{-\infty}^{\infty} \Psi_m(x)\Psi_n(x)\,\mathrm{d}x = \delta_{m,n}, \tag{1.17}$$

is that one is over the finite interval $(-1, 1)$ while the other is over the whole real line. So the use of (1.11) as an analogue of (1.3) has the advantage of orthogonality over a finite interval. This automatically ensures completeness of the system $\{\Psi_n(x\,|\,q)\}_{n=0}^{\infty}$, see [19, Theorem, 3.1.5]. Furthermore, we need not be tied to the q-analogues of the Hermite polynomials only, but can consider other generalizations of these polynomials, e.g., the q-ultraspherical [3], the continuous q-Jacobi [6, 15] or even the most general, the Askey–Wilson, polynomials [6], all of which are orthogonal on $(-1, 1)$. Following this direction, the ideas presented in [2] were extended in [17] to the case of the Askey–Wilson polynomials using the explicit formula for the Poisson kernel derived in [18]. As in [2] this led to a singular integral transformation, thus extending the results corresponding to the continuous q-Hermite polynomials.

The purpose of this paper is to do even more: to apply the ideas of Wiener's treatment of the Fourier integrals to the case of a nonsymmetric extension of the Poisson kernel for the Askey–Wilson polynomials derived here. This paper is organized as follows. In Section 2 we introduce the Askey–Wilson polynomials and then state our result for the corresponding Poisson kernel in Section 3. In Sections 4–6 we derive an explicit formula for this kernel which reveals the structure of the poles. The orthogonality property of these kernels is then proved in Sections 7–12. Finally, we introduce a new integral transformation and prove its inversion formula in Section 13. We close this paper by displaying some special cases of the kernel and a continuous orthogonality relation for the simplest of them, in Section 14.

2. The Askey–Wilson polynomials

We shall assume that a, b, c, d and q are real parameters such that $0 < q < 1$ and $\max(|a|, |b|, |c|, |d|) < 1$. The Askey–Wilson polynomials [6] are defined by

$$p_n(x) = p_n(x; a, b, c, d)$$
$$= {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{2.1}$$

$n = 0, 1, \ldots, x = \cos\theta, 0 \leqslant \theta \leqslant \pi$. This definition differs slightly from the one given in [6], but, for our purposes, this is the more convenient one. The symbol on the right-hand side is the $r = 3$ case of the basic hypergeometric series ${}_{r+1}\phi_r$ defined by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k, \tag{2.2} $$

with

$$ (a_1, a_2, \dots, a_m; q)_k = \prod_{j=1}^{m} (a_j; q)_k, \quad \text{see [11].} \tag{2.3} $$

The orthogonality property satisfied by $p_n(x; a, b, c, d)$ is

$$ \int_{-1}^{1} p_n(x) p_m(x) \rho(x)\, dx = \delta_{m,n}/h_n, \tag{2.4} $$

$$ h_0 = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty}, \tag{2.5} $$

$$ h_n = h_0 \frac{(1 - abcdq^{2n-1})(abcdq^{-1}, ab, ac, ad; q)_n}{(1 - abcdq^{-1})(q, cd, bd, bc; q)_n} a^{-2n}, \tag{2.6} $$

$$ \rho(x) = \rho(x; a, b, c, d) = \frac{h(x; 1, -1, q^{1/2}, -q^{1/2})}{h(x; a, b, c, d)} (1 - x^2)^{-1/2}, \tag{2.7} $$

and

$$ h(x; a_1, a_2, \dots, a_r) = \prod_{j=1}^{r} h(x; a_j), \tag{2.8} $$

$$ h(x; a) = \prod_{n=0}^{\infty} (1 - 2axq^n + a^2 q^{2n}). \tag{2.9} $$

3. The Poisson kernel and a nonsymmetric extension

The Poisson kernel for the Askey–Wilson polynomials is defined by

$$ P_t(x, y) = h_0 K_t(x, y), \tag{3.1} $$

with

$$ K_t(x, y) = \sum_{n=0}^{\infty} \frac{(1 - abcdq^{2n-1})(abcdq^{-1}, ab, ac, ad; q)_n}{(1 - abcdq^{-1})(q, cd, bd, bc; q)_n} a^{-2n} t^n \times p_n(x; a, b, c, d)\, p_n(y; a, b, c, d). \tag{3.2} $$

The series on the right-hand side of (3.2) converges for $|t| < 1$ and $|x| \leqslant 1, |y| \leqslant 1$. In [10], Gasper and Rahman found an explicit representation of $K_t(x, y)$ in the special case $ad = bc$, and proved its positivity when $a = q^{\alpha/2+1/4}$, $b = aq^{1/2}$, $c = -q^{\beta/2+1/4}$, $d = cq^{1/2}$, $\alpha, \beta > -1$ (this corresponds to Askey and Wilson's continuous q-Jacobi polynomials [6]). Rahman and Verma [18] found an expression for the most general kernel (without the condition $ad = bd$) which was reproduced in [17] since the original expression in [18] has many misprints.

In this paper we shall consider a nonsymmetric extension of the bilinear generating function (3.2) by replacing $p_n(y; a, b, c, d)$ by $p_n(y; \alpha, \beta, \gamma, \delta)$ such that

$$\alpha\gamma = ac, \qquad \beta\delta = bd, \qquad \max(|\alpha|, |\beta|, |\gamma|, |\delta|) < 1. \tag{3.3}$$

For ease of notation we shall use the labels λ and μ to refer to the parameter-quartets (a, b, c, d) and $(\alpha, \beta, \gamma, \delta)$, respectively. Thus h_0^λ means the same as h_0 in (2.5), but

$$h_0^\mu := \frac{(q, \alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta; q)_\infty}{2\pi(\alpha\beta\gamma\delta; q)_\infty}, \tag{3.4}$$

with similar meanings for h_n^λ and h_n^μ. Also, we shall use the notations

$$\begin{aligned} p_n^\lambda(x) &= p_n(x; a, b, c, d), \\ p_n^\mu(y) &= p_n(y; \alpha, \beta, \gamma, \delta), \end{aligned} \tag{3.5}$$

and

$$\varepsilon = (abcd)^{1/2}. \tag{3.6}$$

The extension of (3.1) and (3.2) that we have in mind is

$$P_t^{\lambda,\mu}(x, y) = h_0^\lambda K_t^{\lambda,\mu}(x, y), \tag{3.7}$$

where

$$K_t^{\lambda,\mu}(x, y) = (h_0^\lambda)^{-1} \sum_{n=0}^{\infty} h_n^\lambda t^n p_n^\lambda(x) p_n^\mu(y). \tag{3.8}$$

We shall prove in the following sections that

$$K_t^{\lambda,\mu}(x, y) = K_t^{(1)}(x, y) + K_t^{(2)}(x, y) + K_t^{(3)}(x, y), \tag{3.9}$$

where

$$\begin{aligned} &K_t^{(1)}(x, y) \\ &\quad = (1 - t^2) \frac{(-qt\varepsilon, \alpha\beta c/b, \alpha\beta d/b, a e^{i\theta}, \varepsilon^2 e^{-i\theta}/b; q)_\infty}{(-t/\varepsilon, ac, ad, \alpha\beta e^{i\theta}/b, \alpha\beta c d e^{-i\theta}/b; q)_\infty} \\ &\qquad \times \sum_{k=0}^{\infty} \frac{(\varepsilon, \varepsilon q^{1/2}, -\varepsilon q^{1/2}, -\varepsilon/ad; q)_k}{(q, bc, -qt\varepsilon, -q\varepsilon/t; q)_k} q^k \\ &\qquad \times \sum_{l=0}^{k} \frac{(q^{-k}, -\varepsilon, ad, \alpha e^{i\phi}, \alpha e^{-i\phi}, \alpha\beta e^{i\theta}/b, \alpha\beta e^{-i\theta}/b, \alpha\beta c d e^{-i\theta}/b; q)_l}{(q, -adq^{1-k}/\varepsilon, \alpha\beta, \alpha\delta, \alpha/\delta, \alpha\beta d/b, \alpha\beta c/b, \varepsilon^2 e^{-i\theta}/b; q)_l} q^l \\ &\qquad \times \sum_{m=0}^{l} \frac{(1 - \delta q^{2m-l}/\alpha)(\delta q^{-l}/\alpha, q^{1-l}/\alpha\beta, bq^{1-l}/\alpha\beta c, q^{-l}; q)_m}{(1 - \delta q^{-l}/\alpha)(q, \beta\delta, cd, q\delta/\alpha; q)_m} \\ &\qquad \times \frac{(\delta e^{i\phi}, \delta e^{-i\phi}, d e^{i\theta}, d e^{-i\theta}; q)_m}{(q^{1-l} e^{-i\phi}/\alpha, q^{1-l} e^{i\phi}/\alpha, bq^{1-l} e^{-i\theta}/\alpha\beta, bq^{1-l} e^{i\theta}/\alpha\beta; q)_m} \left(\frac{bcq}{\alpha\delta}\right)^m \\ &\qquad \times {}_8W_7\left(\frac{\alpha\beta c d}{b} q^{l-1} e^{-i\theta}; c e^{-i\theta}, d q^m e^{-i\theta}, \frac{\alpha\beta}{b} q^{l-m} e^{-i\theta}, c d q^l, \frac{\alpha\beta}{ab}; q, a e^{i\theta}\right), \end{aligned} \tag{3.10}$$

$$K_t^{(2)}(x, y)$$

$$= \frac{(\varepsilon^2, -\varepsilon/ad, t, -t\varepsilon/ad, \alpha\beta c/b, \alpha\beta d/b, a\mathrm{e}^{\mathrm{i}\theta}, \varepsilon^2\mathrm{e}^{-\mathrm{i}\theta}/b; q)_\infty}{(-\varepsilon, bc, t/ad, -\varepsilon/t, ac, ad, \alpha\beta\mathrm{e}^{\mathrm{i}\theta}/b, \alpha\beta cd\mathrm{e}^{-\mathrm{i}\theta}/b; q)_\infty}$$

$$\times \sum_{k=0}^{\infty} \frac{(-t, tq^{1/2}, -tq^{1/2}, t/ad; q)_k}{(q, qt^2, -t\varepsilon/ad, -qt/\varepsilon; q)_k} q^k$$

$$\times \sum_{l=0}^{\infty} \frac{(-\varepsilon q^{-k}/t, -\varepsilon, ad, \alpha\mathrm{e}^{\mathrm{i}\phi}, \alpha\mathrm{e}^{-\mathrm{i}\phi}, \alpha\beta\mathrm{e}^{\mathrm{i}\theta}/b, \alpha\beta e^{-\mathrm{i}\theta}/b, \alpha\beta cd\mathrm{e}^{-\mathrm{i}\theta}/b; q)_l}{(q, adq^{1-k}/t, \alpha\beta, \alpha\delta, \alpha/\delta, \alpha\beta c/b, \alpha\beta d/b, \varepsilon^2\mathrm{e}^{-\mathrm{i}\theta}/b; q)_l} q^l$$

$$\times \sum_{m=0}^{l} \frac{(1-\delta q^{2m-l}/\alpha)(\delta q^{-l}/\alpha, q^{1-l}/\alpha\beta, bq^{1-l}/\alpha\beta c, q^{-l}; q)_m}{(1-\delta q^{-l}/\alpha)(q, \beta\delta, cd, q\delta/\alpha; q)_m}$$

$$\times \frac{(\delta\mathrm{e}^{\mathrm{i}\phi}, \delta\mathrm{e}^{-\mathrm{i}\phi}, d\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}; q)_m}{(q^{1-l}\mathrm{e}^{-\mathrm{i}\phi}/\alpha, q^{1-l}\mathrm{e}^{\mathrm{i}\phi}/\alpha, bq^{1-l}\mathrm{e}^{-\mathrm{i}\theta}/\alpha\beta, bq^{1-l}\mathrm{e}^{\mathrm{i}\theta}/\alpha\beta; q)_m} \left(\frac{bcq}{\alpha\delta}\right)^m$$

$$\times {}_8W_7\left(\frac{\alpha\beta cd}{b} q^{l-1}\mathrm{e}^{-\mathrm{i}\theta}; c\mathrm{e}^{-\mathrm{i}\theta}, dq^m\mathrm{e}^{-\mathrm{i}\theta}, \frac{\alpha\beta}{b} q^{l-m}\mathrm{e}^{-\mathrm{i}\theta}, cdq^l, \frac{\alpha\beta}{ab}; q, a\mathrm{e}^{\mathrm{i}\theta}\right), \tag{3.11}$$

and

$$K_t^{(3)}(x, y)$$

$$= \frac{(\varepsilon^2, a\mathrm{e}^{-\mathrm{i}\theta}, c\mathrm{e}^{-\mathrm{i}\theta}, d\mathrm{e}^{-\mathrm{i}\theta}, \alpha\beta\mathrm{e}^{\mathrm{i}\theta}/b; q)_\infty}{(ac, bc, cd, \alpha\beta, ad/t, bc\delta t/\gamma; q)_\infty}$$

$$\times \frac{(ct\mathrm{e}^{\mathrm{i}\theta}, c\delta t\mathrm{e}^{\mathrm{i}\theta}/\gamma, bct\mathrm{e}^{\mathrm{i}\phi}/\gamma, bct\mathrm{e}^{-\mathrm{i}\phi}/\gamma; q)_\infty}{(\mathrm{e}^{-2\mathrm{i}\theta}, ct\mathrm{e}^{\mathrm{i}(\theta+\phi)}/\gamma, ct\mathrm{e}^{\mathrm{i}(\theta-\phi)}/\gamma; q)_\infty}$$

$$\times \sum_{k=0}^{\infty} \frac{(t, -\varepsilon/ad, -t\varepsilon/ad, bc\delta t/\gamma, ct\mathrm{e}^{\mathrm{i}(\theta+\phi)}/\gamma, ct\mathrm{e}^{\mathrm{i}(\theta-\phi)}/\gamma; q)_k}{(q, qt/ad, ct\mathrm{e}^{\mathrm{i}\theta}, c\delta t\mathrm{e}^{\mathrm{i}\theta}/\gamma, bct\mathrm{e}^{\mathrm{i}\phi}/\gamma, bct\mathrm{e}^{-\mathrm{i}\phi}/\gamma; q)_k} q^k$$

$$\times {}_4\phi_3\left[\begin{matrix} q^{-k}, -t, tq^{1/2}, -tq^{1/2} \\ qt^2, -t\varepsilon/ad, -adq^{1-k}/\varepsilon \end{matrix}; q, q\right]$$

$$\times \sum_{l=0}^{\infty} \frac{(ctq^k\mathrm{e}^{\mathrm{i}(\theta+\phi)}/\gamma, ctq^k\mathrm{e}^{\mathrm{i}(\theta-\phi)}/\gamma, a\mathrm{e}^{\mathrm{i}\theta}, c\mathrm{e}^{\mathrm{i}\theta}, d\mathrm{e}^{\mathrm{i}\theta}; q)_l}{(ctq^k\mathrm{e}^{\mathrm{i}\theta}, c\delta tq^k\mathrm{e}^{\mathrm{i}\theta}/\gamma, \alpha\beta\mathrm{e}^{\mathrm{i}\theta}/b, q\mathrm{e}^{2\mathrm{i}\theta}, q; q)_l} q^l$$

$$\times {}_8W_7\left(\frac{bc\delta}{\gamma} tq^{k-1}; \frac{bc}{\beta\gamma} tq^k, \frac{bc}{\alpha\gamma} tq^k, bq^{-l}e^{\mathrm{i}\theta}, \delta\mathrm{e}^{\mathrm{i}\phi}, \delta\mathrm{e}^{-\mathrm{i}\phi}; q, \frac{\alpha\beta}{b} q^l\mathrm{e}^{\mathrm{i}\theta}\right)$$

$$+ \text{idem } (\theta; -\theta), \tag{3.12}$$

where idem $(\alpha; \delta)$ means the same expression as the preceding one with α and δ interchanged (see [11]), with a similar meaning for idem $(\theta; -\theta)$. In (3.10)–(3.12), $x = \cos\theta$, $0 < \theta < \pi$, $y = \cos\phi$, $0 < \phi < \pi$, and the W-series are special cases of the very-well-poised ${}_{r+1}W_r$ series defined by

$$
\begin{aligned}
&{}_{r+1}W_r(a; b_1, b_2, \dots, b_{r-2}; q, z)\\
&= {}_{r+1}\phi_r\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b_1, b_2, \dots, b_{r-2} \\ a^{1/2}, -a^{1/2}, aq/b_1, aq/b_2, \dots, aq/b_{r-2} \end{matrix}; q, z\right],
\end{aligned}
\tag{3.13}
$$

see [11] for further details.

Looking at the horrendous expressions in (3.10)–(3.12), one might feel somewhat skeptical about their usefulness but, as we shell see later, the main purpose of these formulas is to isolate the poles of $K_t^{\lambda,\mu}(x, y)$, as a function of t, which are practically all we need for our subsequent analysis.

4. The q-integral representations

To prove (3.9)–(3.12) we will use, following the method of [18], the q-integral representation of the Askey–Wilson polynomials [11, Exercise 7.34]:

$$
\begin{aligned}
p_n(x; a, b, c, d) &= [A(\theta)]^{-1} \frac{(bc; q)_n}{(ad; q)_n}\\
&\quad \times \int_{qe^{i\theta}/d}^{qe^{-i\theta}/d} \frac{(due^{i\theta}, due^{-i\theta}, \varepsilon^2 u/q; q)_\infty}{(dau/q, dbu/q, dcu/q; q)_\infty}\\
&\quad \times \frac{(q/u; q)_n}{(\varepsilon^2 u/q; q)_n} (adu/q)^n \, d_q u,
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
A(\theta) &= A(\theta; a, b, c, d)\\
&= -i \frac{q(1-q)}{2d} (q, ab, ac, bc; q)_\infty h(x; d) \rho(x; a, b, c, d),
\end{aligned}
\tag{4.2}
$$

and the q-integral is defined by

$$
\begin{aligned}
&\int_a^b f(u)\, d_q u = \int_0^b f(u)\, d_q u - \int_0^a f(u)\, d_q u,\\
&\int_0^a f(u)\, d_q u = a(1-q) \sum_{m=0}^{\infty} f(aq^m) q^m,
\end{aligned}
\tag{4.3}
$$

for any function f such that the series on the right-hand side of (4.3) converge.

By using the symmetry property of the Askey–Wilson polynomials (2.1), see [6] and [11], and the q-integral representation (4.1), we obtain

$$
\begin{aligned}
K_t^{\lambda,\mu}(x, y) &= B(\theta, \phi) \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u \frac{(bue^{i\theta}, bue^{-i\theta}, \varepsilon^2 u/q; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty}\\
&\quad \times \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_q v \frac{(\gamma ve^{i\phi}, \gamma ve^{-i\phi}, \varepsilon^2 v/q; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q, \gamma\delta v/q; q)}\\
&\quad \times {}_6W_5(\varepsilon^2/q; ad, q/u, q/v; q, bcuvt/q^2),
\end{aligned}
\tag{4.4}
$$

where the λ and μ parameters are related by (3.3) and, of course, $\varepsilon^2 = abcd = \alpha\beta\gamma\delta$. Also,

$$B^{-1}(\theta, \phi) = -\frac{q^2(1-q)^2}{4b\gamma}(q, q, ac, ad, cd, \alpha\beta, \alpha\delta, \beta\delta; q)_\infty \times h(x; b)h(y; \gamma)\rho^\lambda(x)\rho^\mu(y), \tag{4.5}$$

where $\rho^\lambda(x) = \rho(x; a, b, c, d)$ and $\rho^\mu(y) = \rho(y; \alpha, \beta, \gamma, \delta)$.

Crucial to our calculations is the following formula for the ${}_6W_5$ series in (4.4) which was obtained in [18]:

$$\begin{aligned}
&{}_6W_5(\varepsilon^2/q; ad, q/u, q/v; q, bcuvt/q^2)\\
&= (1-t^2)\frac{(-qt\varepsilon; q)_\infty}{(-t/\varepsilon; q)_\infty}\sum_{k=0}^{\infty}\frac{(\varepsilon, \varepsilon q^{1/2}, -\varepsilon q^{1/2}, -\varepsilon/ad; q)_k}{(q, bc, -qt\varepsilon, -q\varepsilon/t; q)_k}q^k\\
&\times {}_4\phi_3\left[\begin{matrix} q^{-k}, -\varepsilon, ad, \varepsilon^2uv/q^2\\ -adq^{1-k}/\varepsilon, u\varepsilon^2/q, v\varepsilon^2/q\end{matrix}; q, q\right]\\
&+\frac{(\varepsilon^2, -\varepsilon/ad, t, -t\varepsilon/ad; q)_\infty}{(-\varepsilon, bc, t/ad, -\varepsilon/t; q)_\infty}\sum_{k=0}^{\infty}\frac{(-t, tq^{1/2}, -tq^{1/2}, t/ad; q)_k}{(q, qt^2, -t\varepsilon/ad, -qt/\varepsilon; q)_k}q^k\\
&\times {}_4\phi_3\left[\begin{matrix} -\varepsilon q^{-k}/t, -\varepsilon, ad, \varepsilon^2uv/q^2\\ adq^{1-k}/t, u\varepsilon^2/q, v\varepsilon^2/q\end{matrix}; q, q\right]\\
&+\frac{(\varepsilon^2, ad, bctu/q, bctv/q, uv\varepsilon^2/q^2; q)_\infty}{(bc, ad/t, u\varepsilon^2/q, v\varepsilon^2/q, bcuvt/q^2; q)_\infty}\\
&\times\sum_{k=0}^{\infty}\frac{(t, -\varepsilon/ad, -\varepsilon t/ad, bcuvt/q^2; q)_k}{(q, qt/ad, bctu/q, bctv/q; q)_k}q^k\\
&\times {}_4\phi_3\left[\begin{matrix} q^{-k}, -t, tq^{1/2}, -tq^{1/2}\\ qt^2, -\varepsilon t/ad, -adq^{1-k}/\varepsilon\end{matrix}; q, q\right].
\end{aligned}\tag{4.6}$$

The contribution of the first term in (4.6) to $K_t^{\lambda,\mu}(x, y)$ is

$$\begin{aligned}
K_t^{(1)}(x, y) &= B(\theta, \phi)(1-t^2)\frac{(-qt\varepsilon; q)_\infty}{(-t/\varepsilon; q)_\infty}\\
&\times\sum_{k=0}^{\infty}\frac{(\varepsilon, \varepsilon q^{1/2}, -\varepsilon q^{1/2}, -\varepsilon/ad; q)_k}{(q, bc, -qt\varepsilon, -q\varepsilon/t; q)_k}q^k\\
&\times\sum_{l=0}^{k}\frac{(q^{-k}, -\varepsilon, ad; q)_l}{(q, -adq^{1-k}/\varepsilon; q)_l}q^l U_l,
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
U_l = &\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u\,\frac{(bue^{i\theta}, bue^{-i\theta}, u\varepsilon^2q^{l-1}; q)_\infty}{(bau/q, bcu/q, bdu/q; q)_\infty}\\
&\times\int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_q v\,\frac{(\gamma ve^{i\phi}, \gamma ve^{-i\phi}, v\varepsilon^2q^{l-1}, uv\varepsilon^2/q; q)_\infty}{(\gamma\alpha u/q, \gamma\beta v/q, \gamma\delta v/q, uv\varepsilon^2q^{l-2}; q)_\infty}.
\end{aligned}\tag{4.8}$$

Likewise, the contribution of the second term in (4.6) is

$$K_t^{(2)}(x,y) = B(\theta,\phi)\frac{(\varepsilon^2,\ -\varepsilon/ad,\ t,\ -t\varepsilon/ad;\ q)_\infty}{(-\varepsilon,\ bc,\ t/ad,\ -\varepsilon/t;\ q)_\infty} \times \sum_{k=0}^{\infty}\frac{(-t,\ tq^{1/2},\ -tq^{1/2},\ t/ad;\ q)_k}{(q,\ qt^2,\ -t\varepsilon/ad,\ -qt/\varepsilon;\ q)_k}q^k \times \sum_{l=0}^{\infty}\frac{(-\varepsilon q^{-k}/t,\ -\varepsilon,\ ad;\ q)_l}{(q,\ adq^{1-k}/t;\ q)_l}q^l U_l. \tag{4.9}$$

Finally, the last term on the right-hand side of (4.6) gives

$$K_t^{(3)}(x,y) = B(\theta,\phi)\frac{(\varepsilon^2,\ ad;\ q)_\infty}{(bc,\ ad/t;\ q)_\infty} \times \sum_{k=0}^{\infty}\frac{(t,\ -\varepsilon/ad,\ -t\varepsilon/ad;\ q)_k}{(q,\ qt/ad;\ q)_k}q^k \times {}_4\phi_3\left[\begin{matrix} q^{-k},\ -t,\ tq^{1/2},\ -tq^{1/2} \\ qt^2,\ -t\varepsilon/ad,\ -adq^{1-k}/\varepsilon \end{matrix};q,q\right]V_k, \tag{4.10}$$

where

$$V_k = \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u\,\frac{(bue^{i\theta},\ bue^{-i\theta},\ bctuq^{k-1};\ q)_\infty}{(bau/q,\ bcu/q,\ bdu/q;\ q)_\infty} \times \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_q v\,\frac{(\gamma ve^{i\phi},\ \gamma ve^{-i\phi},\ bctvq^{k-1},\ uv\varepsilon^2/q^2;\ q)_\infty}{(\gamma\alpha v/q,\ \gamma\beta v/q,\ \gamma\delta v/q,\ bcuvtq^{k-2};\ q)_\infty}. \tag{4.11}$$

5. Computation of $K_t^{(1)}(x,y)$ and $K_t^{(2)}(x,y)$

Using [11, (2.10.19)], we get

$$\int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma}\frac{(\gamma ve^{i\phi},\ \gamma ve^{-i\phi},\ v\varepsilon^2 q^{l-1},\ uv\varepsilon^2/q;\ q)_\infty}{(\gamma\alpha v/q,\ \gamma\beta v/q,\ \gamma\delta v/q,\ uv\varepsilon^2 q^{l-2};\ q)_\infty}d_q v = \frac{q(1-q)}{2i\gamma}(q,\ \alpha\beta,\ \alpha\delta,\ \beta\delta;\ q)_\infty h(y;\gamma)\rho^\mu(y) \times\frac{(\alpha e^{i\phi},\ \alpha e^{-i\phi},\ \alpha\beta u/q;\ q)_l}{(\alpha\beta,\ \alpha\delta,\ \alpha/\delta;\ q)_l} \times {}_8W_7(\delta q^{-l}/\alpha;\ \delta e^{i\phi},\ \delta e^{-i\phi},\ \beta\delta u/q,\ q^{1-l}/\alpha\beta,\ q^{-l};\ q,\ q^2/\alpha\delta u). \tag{5.1}$$

Substituting (5.1) in (4.8) we get

$$U_l = \frac{q(1-q)}{2i\gamma}(q,\ \alpha\beta,\ \alpha\delta,\ \beta\delta;\ q)_\infty h(y;\gamma)\rho^\mu(y)\frac{(\alpha e^{i\phi},\ \alpha e^{-i\phi};\ q)_l}{(\alpha\beta,\ \alpha\delta,\ \alpha/\delta;\ q)_l} \times \sum_{m=0}^{l}\frac{(1-\delta q^{2m-l}/\alpha)(\delta q^{-l}/\alpha,\ \delta e^{i\phi},\ \delta e^{-i\phi},\ q^{1-l}/\alpha\beta,\ q^{-l};\ q)_m}{(1-\delta q^{-l}/\alpha)(q,\ q^{1-l}e^{-i\phi}/\alpha,\ q^{1-l}e^{i\phi}/\alpha,\ \beta\delta,\ q\delta/\alpha;\ q)_m}I_m, \tag{5.2}$$

where

$$I_m = q^{lm-\binom{m}{2}}(-\beta/\delta)^m \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, u\varepsilon^2 q^{l-1}, \alpha\beta u/q; q)_\infty}{(bcu/q, bduq^{m-1}, \alpha\beta uq^{l-m-1}, abu/q; q)_\infty} d_q u$$

$$= q^{lm-\binom{m}{2}}(-\beta/\delta)^m \frac{q(1-q)}{b} e^{-i\theta}$$

$$\times \frac{(q, e^{2i\theta}, qe^{-2i\theta}, \alpha\beta dq^l/b, \alpha\beta cq^{l-m}/b, cdq^m, q^l\varepsilon^2 e^{-i\theta}/b, \alpha\beta e^{-i\theta}/b; q)_\infty}{(ce^{i\theta}, dq^m e^{i\theta}, \alpha\beta q^{l-m}e^{i\theta}/b, ce^{-i\theta}, dq^m e^{-i\theta}, \alpha\beta q^{l-m}e^{-i\theta}/b, ae^{-i\theta}, \alpha\beta cdq^l e^{-i\theta}/b; q)_\infty}$$

$$\times {}_8W_7(\alpha\beta cdq^{l-1}e^{-i\theta}/b; ce^{-i\theta}, dq^m e^{-i\theta}, \alpha\beta q^{l-m}e^{-i\theta}/b, cdq^l, \alpha\beta/ab; q, ae^{i\theta}), \tag{5.3}$$

by [11, (2.10.19)]. Simplifying the coefficients, we obtain from (5.2), (5.3) and (4.5)

$$U_l = \frac{B^{-1}(\theta, \phi)(\alpha\beta c/b, \alpha\beta d/b, ae^{i\theta}, \varepsilon^2 e^{-i\theta}/b; q)_\infty}{(ac, ad, \alpha\beta e^{i\theta}/b, \alpha\beta cde^{-i\theta}/b; q)_\infty}$$

$$\times \frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, \alpha\beta e^{i\theta}/b, \alpha\beta e^{-i\theta}/b, \alpha\beta cde^{-i\theta}/b; q)_l}{(\alpha\beta, \alpha\delta, \alpha/\delta, \alpha\beta d/b, \alpha\beta c/b, \varepsilon^2 e^{-i\theta}/b; q)_l}$$

$$\times \sum_{m=0}^{l} \frac{(1-\delta q^{2m-l}/\alpha)(\delta q^{-l}/\alpha, q^{1-l}/\alpha\beta, bq^{1-l}/\alpha\beta c, q^{-l}; q)_m}{(1-\delta q^{-l}/\alpha)(q, \beta\delta, cd, q\delta/\alpha; q)_m}$$

$$\times \frac{(\delta e^{i\phi}, \delta e^{-i\phi}, de^{i\theta}, de^{-i\theta}; q)_m}{(q^{1-l}e^{-i\phi}/\alpha, q^{1-l}e^{i\phi}/\alpha, bq^{1-l}e^{-i\theta}/\alpha\beta, bq^{1-l}e^{i\theta}/\alpha\beta; q)_m}(bcq/\alpha\delta)^m$$

$$\times {}_8W_7(\alpha\beta cdq^{l-1}e^{-i\theta}/b, ce^{-i\theta}, dq^m e^{-i\theta}, \alpha\beta q^{l-m}e^{-i\theta}/b, cdq^l, \alpha\beta/ab; q, ae^{i\theta}). \tag{5.4}$$

Use of (5.4) in (4.7) and (4.9) then gives (3.10) and (3.11).

6. Computation of $K_t^{(3)}(x, y)$

In order to compute $K_t^{(3)}(x, y)$, we have to express V_k of (4.11) in a form that is real in both θ and ϕ. First of all, by [11, (2.10.19)]

$$I(u) := \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} \frac{(\gamma ve^{i\phi}, \gamma ve^{i\phi}, bctvq^{k-1}, uv\varepsilon^2/q; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q, \gamma\delta v/q, bcuvtq^{k-2}; q)_\infty} d_q v$$

$$= \frac{q(1-q)}{2i\gamma}(q, \alpha\beta, \alpha\delta, \beta\delta; q)_\infty h(y; \gamma)\rho^\mu(y)$$

$$\times \frac{(bctq^k e^{-i\phi}/\gamma, u\varepsilon^2 e^{-i\phi}/\gamma q; q)_\infty}{(bcutq^{k-1}e^{-i\phi}/\gamma, \varepsilon^2 e^{-i\phi}/\gamma; q)_\infty}$$

$$\times {}_8W_7(\varepsilon^2 e^{-i\phi}/\gamma q; \alpha e^{-i\phi}, \beta e^{-i\phi}, \delta e^{-i\phi}, q/u, adq^{-k}/t; q, bcutq^{k-1}e^{i\phi}/\gamma). \tag{6.1}$$

Using Bailey's transformation formula [11, (2.10.1)] for a very-well poised ${}_8\phi_7$ series we find that

$$I(u) = \frac{q(1-q)}{2i\gamma}(q, \alpha\delta, \beta\delta; q)_\infty h(y;\gamma)\rho^\mu(y)$$
$$\times \frac{(bctq^k e^{i\phi}/\gamma, bctq^k e^{-i\phi}/\gamma, \alpha\beta u/q, bc\delta utq^{k-1}/\gamma; q)_\infty}{(bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k-1}e^{-i\phi}/\gamma, bc\delta tq^k/\gamma; q)_\infty}.$$
$$\times {}_8W_7(bc\delta tq^{k-1}/\gamma; \delta e^{i\phi}, \delta e^{-i\phi}, \alpha\delta tq^k/ad, bctq^k/\alpha\gamma, q/u; q, \alpha\beta u/q). \tag{6.2}$$

Now we break up the ${}_8W_7$ series in (6.2) into two balanced nonterminating ${}_4\phi_3$ series by use of [11, (2.10.10)]:

$$I(u) = \frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, \beta\delta, \alpha\beta u/q, bctq^k/\gamma\delta, bc\delta utq^{k-1}/\gamma; q)_\infty}{(\alpha/\delta, bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k-1}e^{-i\phi}/\gamma; q)_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix} bctq^k/\alpha\gamma, \delta e^{i\phi}, \delta e^{-i\phi}, \beta\delta u/q \\ q\delta/\alpha, \beta\delta, bc\delta utq^{k-1}/\gamma \end{matrix}; q, q\right] + \text{idem}(\alpha;\delta). \tag{6.3}$$

Use of (6.3) in (4.11) now gives

$$V_k = \frac{q(1-q)}{2i\gamma}(q;q)_\infty h(y;\gamma)\rho^\mu(y)\frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, \beta\delta, bctq^k/\gamma\delta; q)_\infty}{(\alpha/\delta; q)_\infty}$$
$$\times \sum_{l=0}^{\infty} \frac{(bctq^k/\alpha\gamma, \delta e^{i\phi}, \delta e^{-i\phi}; q)_l}{(q, \beta\delta, q\delta/\alpha; q)_l} q^l$$
$$\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, bctuq^{k-1}, bc\delta tuq^{k+l-1}/\gamma; q)_\infty}{(bcu/q, bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k-1}e^{-i\phi}/\gamma, \beta\delta uq^{l-1}; q)_\infty}$$
$$\times \frac{(\alpha\beta u/q, \delta\beta u/q; q)_\infty}{(abu/q, bdu/q; q)_\infty} d_q u$$
$$+ \text{idem}(\alpha;\delta). \tag{6.4}$$

Observe that the two q-integrals on the right-hand side of (6.4) differ only in that α and δ are interchanged. So we need to transform only one of them. By using the definition (4.3) we obtain

$$\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, bctuq^{k-1}, bc\delta tuq^{k+l-1}/\gamma; q)_\infty}{(bcu/q, bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k-1}e^{-i\phi}/\gamma, \beta\delta uq^{l-1}; q)_\infty}$$
$$\times \frac{(\alpha\beta u/q, \delta\beta u/q; q)_\infty}{(abu/q, bdu/q; q)_\infty} d_q u$$
$$= \frac{q(1-q)}{2ib}(q;q)_\infty h(x;b)\rho^\lambda(x)\frac{(\beta\delta e^{i\theta}/b; q)_l}{(c\delta te^{i\theta}q^k/\gamma; q)_l}$$
$$\times \frac{(ctq^k e^{i\theta}, c\delta tq^k e^{i\theta}/\gamma, ae^{-i\theta}, ce^{-i\theta}, de^{-i\theta}, \alpha\beta e^{i\theta}/b; q)_\infty}{(ctq^k e^{i(\theta+\phi)}/\gamma, ctq^k e^{i(\theta-\phi)}/\gamma, e^{-2i\theta}; q)_\infty}$$
$$\times {}_6\phi_5\left[\begin{matrix} ctq^k e^{i(\theta+\phi)}/\gamma, ctq^k e^{i(\theta-\phi)}/\gamma, \delta\beta q^l e^{i\theta}/b, ae^{i\theta}, ce^{i\theta}, de^{i\theta} \\ qe^{2i\theta}, ctq^k e^{i\theta}, c\delta tq^{k+l}e^{i\theta}/\gamma, \alpha\beta e^{i\theta}/b, \delta\beta e^{i\theta}/b \end{matrix}; q, q\right]$$
$$- \text{idem}(\theta; -\theta). \tag{6.5}$$

Using (6.5) and (4.5) in (6.4) we find that

$$V_k = B^{-1}(\theta, \phi)\left\{\frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, bct/\gamma\delta, cte^{i\theta}, c\delta te^{i\theta}/\gamma; q)_\infty}{(ac, ad, cd, \alpha\beta, \alpha\delta, \alpha/\delta; q)_\infty}\right.$$

$$\times\frac{(ae^{-i\theta}, ce^{-i\theta}, de^{-i\theta}, \alpha\beta e^{i\theta}/b; q)_\infty (cte^{i(\theta+\phi)}/\gamma, cte^{i(\theta-\phi)}/\gamma; q)_k}{(e^{-2i\theta}, cte^{i(\theta+\phi)}/\gamma, cte^{i(\theta-\phi)}/\gamma; q)_\infty (bct/\gamma\delta, cte^{i\theta}, c\delta te^{i\theta}/\gamma; q)_k}$$

$$\times\sum_{l=0}^{\infty}\frac{(bctq^k/\alpha\gamma, \delta e^{i\phi}, \delta e^{-i\phi}, \beta\delta e^{i\theta}/b; q)_l}{(q, q\delta/\alpha, \beta\delta, c\delta tq^k e^{i\theta}/\gamma; q)_l} q^l$$

$$\times\, {}_6\phi_5\left[\begin{matrix} ctq^k e^{i(\theta+\phi)}/\gamma, ctq^k e^{i(\theta-\phi)}/\gamma, \beta\delta q^l e^{i\theta}/b, ae^{i\theta}, ce^{i\theta}, de^{i\theta} \\ qe^{2i\theta}, ctq^k e^{i\theta}, c\delta tq^{k+l}e^{i\theta}/\gamma, \alpha\beta e^{i\theta}/b, \delta\beta e^{i\theta}/b \end{matrix}; q, q\right]$$

$$\left. + \operatorname{idem}(\theta; -\theta)\right\} + \operatorname{idem}(\alpha; \delta). \tag{6.6}$$

Changing the order of summation one can rewrite the sum over l in (6.6) in the following form:

$$\sum_{l=0}^{\infty}\frac{(ctq^k e^{i(\theta+\phi)}/\gamma, ctq^k e^{i(\theta-\phi)}/\gamma, ae^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_l}{(q, qe^{2i\theta}, ctq^k e^{i\theta}, \alpha\beta e^{i\theta}/b, c\delta tq^k e^{i\theta}/\gamma; q)_l} q^l$$

$$\times\, {}_4\phi_3\left[\begin{matrix} bctq^k/\alpha\gamma, \delta e^{i\phi}, \delta e^{-i\phi}, \delta\beta q^l e^{i\theta}/b \\ q\delta/\alpha, \beta\delta, c\delta tq^{k+l}e^{i\theta}/\gamma \end{matrix}; q, q\right],$$

where the ${}_4\phi_3$ function is now balanced. The two ${}_4\phi_3$ series that arise from (6.6) in this way can be combined into a single ${}_8\phi_7$ series by [11, III.36]. Substituting this result in (4.10) we get (3.12).

7. Multiplication law for the kernels

It follows from (2.4), (3.7) and (3.8) that

$$\int_{-1}^{1} P_t^{\lambda,\mu}(x, y) P_{t'}^{\mu,\lambda'}(y, x')\rho^\mu(y)\,dy = P_{tt'}^{\lambda,\lambda'}(x, x'), \tag{7.1}$$

when $\max(|\lambda|, |\mu|, |\lambda'|, |t|, |t'|) < 1$ where $\lambda = (a,b,c,d)$, $\lambda' = (a',b',c',d')$, $\mu = (\alpha, \beta, \gamma, \delta)$, and $|\lambda| < 1$ means $|a|, |b|, |c|, |d|$ are all numerically less than 1. The corresponding formula for the kernels $K_t^{\lambda,\mu}(x, y)$ is

$$\int_{-1}^{1} K_t^{\lambda,\mu}(x, y) K_{t'}^{\mu,\lambda'}(y, x')\rho^\mu(y)\,dy$$

$$= \frac{2\pi(\alpha\beta\gamma\delta; q)_\infty}{(q, \alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta; q)_\infty} K_{tt'}^{\lambda,\lambda'}(x, x'). \tag{7.2}$$

In the next sections we will consider an analytic continuation of (7.2) when the parameters t and t' go to the unit circle and tt' approaches 1, in order to obtain the orthogonality relation for the kernels $K_t^{\lambda,\mu}(x, y)$.

8. Computation of $K_1^{\lambda,\mu}(x, y)$

Setting $t = 1$ in (3.10) and (3.11) we can see that $K_1^{(1)}(x, y) = 0 = K_1^{(2)}(x, y)$. So, from (3.9), (3.12) and (4.10) we find that

$$K_1^{\lambda,\mu}(x, y) = K_1^{(3)}(x, y) = B(\theta, \phi)\frac{(\varepsilon^2; q)_\infty}{(bc; q)_\infty} V_0|_{t=1}. \tag{8.1}$$

However, from (4.11) we have

$$V_0|_{t=1} = \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_qv \frac{(\gamma ve^{i\phi}, \gamma ve^{-i\phi}, bcv/q; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q, \gamma\delta v/q; q)_\infty} \times \int_{qe^{i\theta}/\gamma}^{qe^{-i\theta}/\gamma} \frac{(bue^{i\theta}, bue^{-i\theta}, uv\varepsilon^2/q^2; q)_\infty}{(bau/q, bdu/q, bcuv/q^2; q)_\infty} d_qu \tag{8.2}$$

and, therefore, one can evaluate these q-integrals by [11, (2.10.18) and (2.10.19)], thus getting

$$V_0|_{t=1} = -\frac{(1-q)^2 q^2}{4b\gamma} h(x; b)h(x; c)h(y; \alpha)\, h(y; \gamma)\rho^\lambda(x)\rho^\mu(y) \times \frac{(q, q, ad, c^2/\gamma^2, c\delta e^{i\theta}/\gamma, c\delta e^{-i\theta}/\gamma, bce^{-i\phi}/\gamma, cde^{-i\phi}/\gamma, \beta e^{i\phi}; q)_\infty}{(ce^{i\theta+i\phi}/\gamma, ce^{i\theta-i\phi}/\gamma, ce^{i\phi-i\theta}/\gamma, ce^{-i\theta-i\phi}/\gamma, c^2\delta e^{-i\phi}/\gamma^2; q)_\infty} \times {}_8W_7\left(\frac{c^2\delta}{q\gamma^2}e^{-i\phi}; \frac{c}{\gamma}e^{i\theta-i\phi}, \frac{c}{\gamma}e^{-i\theta-i\phi}, \delta e^{-i\phi}, \frac{bc}{\beta\gamma}, \frac{cd}{\beta\gamma}; q, \beta e^{i\phi}\right). \tag{8.3}$$

Substituting (8.3) in (8.1) we obtain

$$\begin{aligned} K_1^{\lambda,\mu}(x, y) &= \frac{(abcd, c^2/\gamma^2; q)_\infty}{(ac, bc, cd, \alpha\beta, \alpha\delta, \beta\delta; q)_\infty} \times \frac{(ce^{i\theta}, ce^{-i\theta}, c\delta e^{i\theta}/\gamma, c\delta e^{-i\theta}/\gamma, \alpha e^{i\phi}, \alpha e^{-i\phi}, bce^{-i\phi}/\gamma, cde^{-i\phi}/\gamma, \beta e^{i\phi}; q)_\infty}{(ce^{i\theta+i\phi}/\gamma, ce^{i\theta-i\phi}/\gamma, ce^{i\phi-i\theta}/\gamma, ce^{-i\theta-i\phi}/\gamma, c^2\delta e^{-i\phi}/\gamma^2; q)_\infty} \\ &\quad \times {}_8W_7\left(\frac{c^2\delta}{q\gamma^2}e^{-i\phi}; \frac{c}{\gamma}e^{i\theta-i\phi}, \frac{c}{\gamma}e^{-i\theta-i\phi}, \delta e^{-i\phi}, \frac{bc}{\beta\gamma}, \frac{cd}{\beta\gamma}; q, \beta e^{i\phi}\right) \\ &= \frac{(abcd, cd/\beta\gamma; q)_\infty}{(ac, bc, cd, \alpha\beta, \alpha\delta, \beta bc/\gamma; q)_\infty} \times \frac{(ce^{i\theta}, ce^{-i\theta}, \beta ce^{i\theta}/\gamma, \beta ce^{-i\theta}/\gamma, \alpha e^{i\phi}, \alpha e^{-i\phi}, bce^{i\phi}/\gamma, bce^{-i\phi}/\gamma; q)_\infty}{(ce^{i\theta+i\phi}/\gamma, ce^{i\theta-i\phi}/\gamma, ce^{i\phi-i\theta}/\gamma, ce^{-i\theta-i\phi}/\gamma; q)_\infty} \\ &\quad \times {}_8W_7\left(\frac{\beta bc}{q\gamma}; be^{i\theta}, be^{-i\theta}, \beta e^{i\phi}, \beta e^{-i\phi}, \frac{bc}{\gamma\delta}; q, \frac{cd}{\beta\gamma}\right) \end{aligned} \tag{8.4}$$

by using [11, III.23].

All of our calculations so far have been done on the basis of only two conditions (3.3) connecting the λ- and μ-parameters. We shall now impose a third condition:

$$\beta\gamma = bc. \tag{8.5}$$

This still leaves us with enough freedom to compute $K_1^{\lambda,\mu}(x, y)$ without hitting any singularities while enabling us to simplify (8.4) enormously. Our final result is

$$K_1^{\lambda,\mu}(x,y) = \frac{(abcd;q)_\infty}{(\alpha\beta, ac, ad, bc, bd, cd; q)_\infty} \times \frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, \beta e^{i\phi}, \beta e^{-i\phi}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, (\beta/b)^2; q)_\infty}{(\beta e^{i\theta+i\phi}/b, \beta e^{i\theta-i\phi}/b, \beta e^{i\phi-i\theta}/b, \beta e^{-i\theta-i\phi}/b; q)_\infty}, \tag{8.6}$$

where, it has been assumed that $|\beta| < |b|$ and $\alpha\gamma = ac$, $\beta\delta = bd$, $\beta\gamma = bc$.

We would like to point out that it is possible to derive (8.6) directly from (3.12) by using [11, II.24].

9. Relation of special $K_1^{\lambda,\mu}(x, y)$ with the continuous q-Hermite polynomials

The Poisson kernel for the continuous q-Hermite polynomials is, see [3],

$$\sum_{n=0}^{\infty} H_n(\cos\theta\,|\,q) H_n(\cos\phi\,|\,q) \frac{r^n}{(q;q)_n} = \frac{(r^2;q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}; q)_\infty}, \quad |r| < 1. \tag{9.1}$$

So the right-hand side of (8.6) coincides with it (up to a factor independent of r) if we identify r with β/b. The q-Hermite polynomials are orthogonal on the compact interval $[-1, 1]$ and so are complete in $L^2[-1,1]$, [19, Theorem 3.1.5]. We also have

$$\lim_{r\to 1^-} \int_{-1}^{1} K_r^{(0)}(x,y) f(y)\,dy = f(x), \quad -1 < x < 1 \tag{9.2}$$

for every bounded function f that is continuous on $(-1, 1)$, where

$$K_r^{(0)}(x,y) = \frac{(q, r^2, e^{2i\phi}, e^{-2i\phi}; q)_\infty}{2\pi \sin\phi (re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}; q)_\infty}. \tag{9.3}$$

It also means that in the limit $r \to 1^-$ the kernel $K_r^{(0)}(x, y)$ behaves like the delta functional $\delta(x - y)$.

The proof of (9.2) follows by Wiener's arguments in lemma [21, Proposition X_{55}, p. 41]. Indeed, if we set $a = 0$, $b = \pi$ and $K_r(\theta, \phi) = \sin\phi\, K_r^{(0)}(\cos\theta, \cos\phi)$, then the hypotheses of this lemma are satisfied by this kernel. Here

$$\int_0^{\pi} K_r(\theta,\phi)\,d\phi = \sum_{n=0}^{\infty} \frac{r^n}{(q;q)_n} H_n(\cos\theta\,|\,q) \times \frac{(q;q)_\infty}{2\pi} \int_0^{\pi} H_n(\cos\phi\,|\,q)(e^{2i\phi}, e^{-2i\phi}; q)_\infty\,d\phi = 1 \tag{9.4}$$

due to uniform convergence of the series for $|r| < 1$ and the orthogonality of the q-Hermite polynomials. Thus the property (4.12) of [21] is established. Again if $2\pi - \psi > \theta + \phi \geqslant |\theta - \phi| > \psi > 0$; $\psi < \theta < \pi - \psi$ and $0 < \phi < \pi$, then

$$K_r(\theta, \phi) < \frac{(q, r^2, e^{2i\phi}, e^{-2i\phi}; q)_\infty}{2\pi(re^{i\psi}, re^{-i\psi}; q)_\infty^2} < (\text{constant}) \frac{(r^2; q)_\infty}{(re^{i\psi}, re^{-i\psi}; q)_\infty^2}, \tag{9.5}$$

since

$$(re^{i(\theta \mp \phi)}, re^{-i(\theta \mp \phi)}; q)_\infty = \prod_{k=0}^{\infty} (1 - 2r\cos(\theta \mp \phi) q^k + r^2 q^{2k}) > \prod_{k=0}^{\infty} (1 - 2r\cos\psi q^k + r^2 q^{2k}) = (re^{i\psi}, re^{-i\psi}; q)_\infty.$$

The validity of [21, (4.13)] is thus assured. The positivity of $K_r(\theta, \phi)$ is also obvious. Therefore, by [21, Proposition X_{55}]

$$\lim_{r \to 1^-} \int_0^\pi K_r(\theta, \phi) g(\phi)\, d\phi = g(\theta), \quad 0 < \theta < \pi \tag{9.6}$$

for every bounded function g that is continuous on $(0, \pi)$, which is equivalent to (9.2).

10. Complex orthogonality of the kernels

Set $t = re^{i\tau}$, $t' = re^{-i\tau}$ with $0 < \tau < \pi$ in Eq. (7.2) and consider its limiting form when $r \to 1^-$ and then $\lambda' \to \lambda$. With the aid of (8.6), (9.2) and (9.3) we can write (in the distribution sense), for the right-hand side of (7.2):

$$K_1^{\lambda, \lambda'}(x, x') \to \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \times \frac{\delta(x - x')}{\rho^\lambda(x)} \quad \text{as } \lambda' \to \lambda. \tag{10.1}$$

Here $\lambda = (a, b, c, d)$, $\lambda' = (a', b', c', d')$ with $ac = a'c'$, $bd = b'd'$, $bc = b'c'$ and $b' < b$.

To consider the analytic continuation of the left-hand side of (7.2) when $t = (t')^* = re^{i\tau} = e^{i\tau - \kappa}$ and $\kappa \to 0^+$, let us rewrite this formula in terms of the contour integral (Fig. 1):

$$\int_{-1}^{1} K_{re^{i\tau}}^{\lambda, \mu}(x, y) K_{re^{-i\tau}}^{\mu, \lambda'}(y, x') \rho^\mu(y)\, dy = \frac{\log q^{-1}}{i} \int_{C^-} \frac{L_{re^{i\tau}}^{\lambda, \mu}(z, s) L_{re^{-i\tau}}^{\mu, \lambda'}(s, z')(q^{2s}, q^{-2s}; q)_\infty\, ds}{(\alpha q^s, \alpha q^{-s}, \beta q^s, \beta q^{-s}, \gamma q^s, \gamma q^{-s}, \delta q^s, \delta q^{-s}; q)_\infty}, \tag{10.2}$$

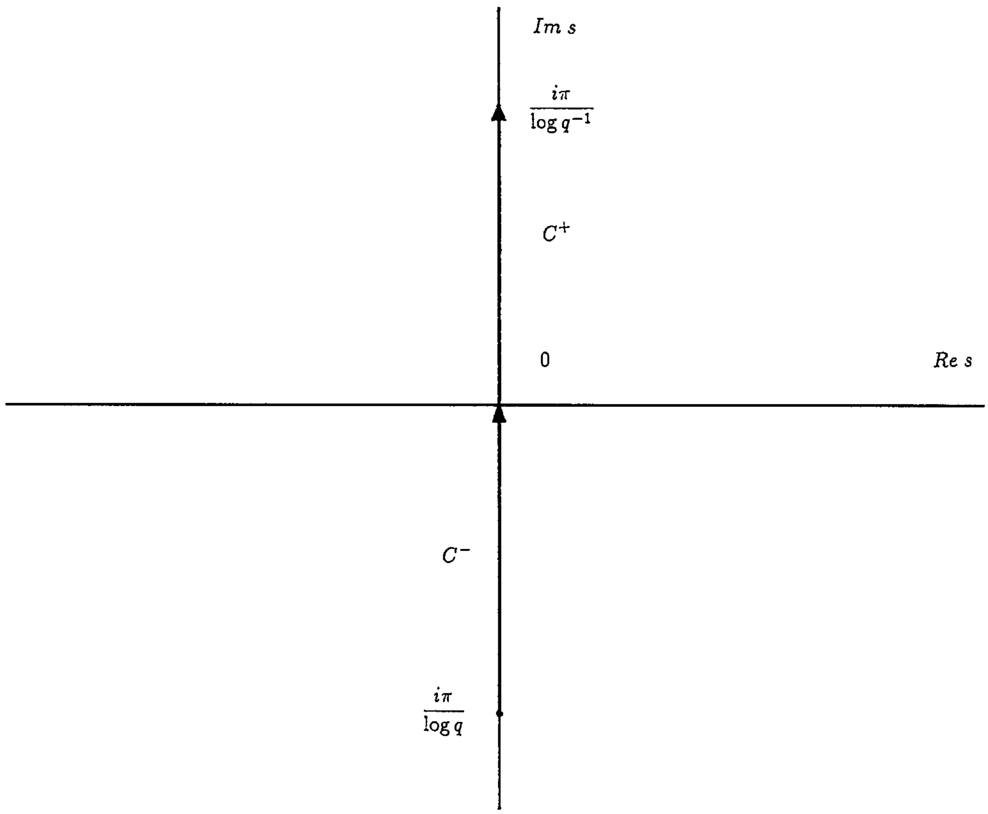

Fig. 1.

where $x = \frac{1}{2}(q^z + q^{-z})$, $q^z = e^{i\theta}$; $y = \frac{1}{2}(q^s + q^{-s})$, $q^s = e^{i\phi}$, and $L_t^{\lambda,\mu}(z, s) = K_t^{\lambda,\mu}(x, y)$. An examination of the series on the right-hand sides of (3.10)–(3.12) reveals that the poles only originate from the function $K_t^{(3)}(x, y)$ given by (3.12). Even here the only singular term corresponds to $k = 0$ on the right-hand side of (3.12). The poles of the integrand which are located in the lower half plane, are given by

$$s_1 = \frac{\omega + \kappa - i(\tau + \theta)}{\log q^{-1}}, \quad s_1 + 1, \dots,$$
$$s_2 = \frac{\omega + \kappa - i(\tau - \theta)}{\log q^{-1}}, \quad s_2 + 1, \dots; \qquad (10.3)$$

and

$$s_1' = \frac{\omega' - \kappa - i(\tau + \theta')}{\log q^{-1}}, \quad s_1' - 1, \dots,$$
$$s_2' = \frac{\omega' - \kappa - i(\tau - \theta')}{\log q^{-1}}, \quad s_2' - 1, \dots . \qquad (10.4)$$

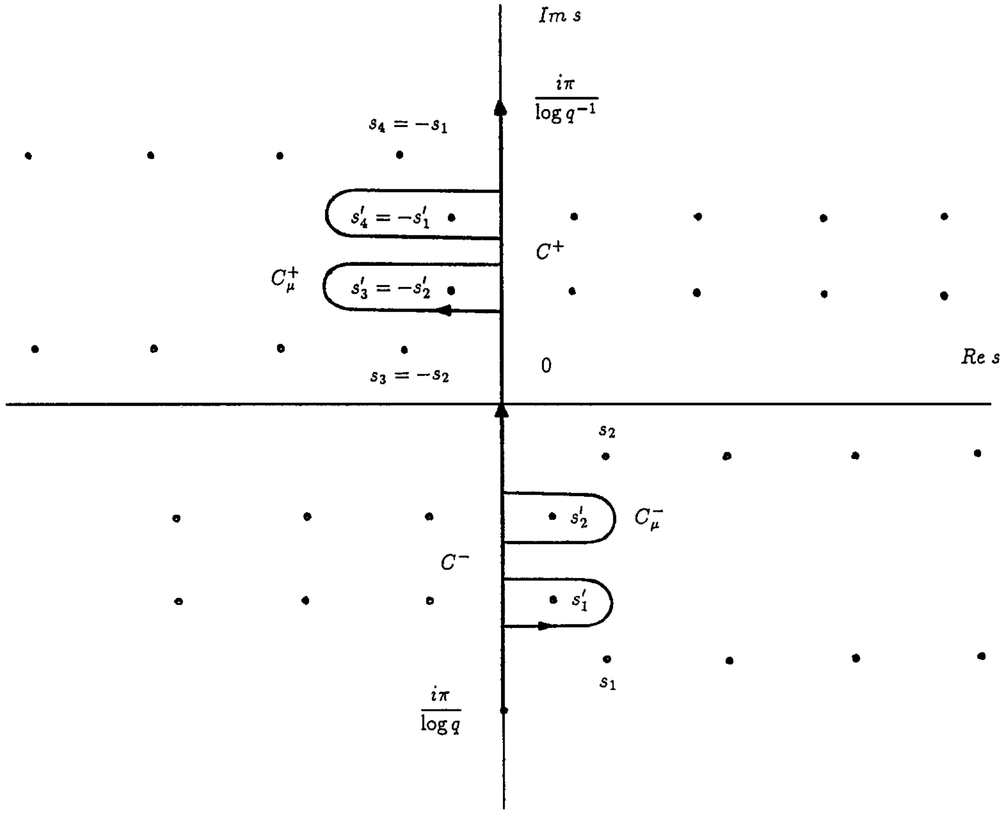

Fig. 2.

Here $c/\gamma = \mathrm{e}^{-\omega}$, $c'/\gamma = \mathrm{e}^{-\omega'}$, $r = \mathrm{e}^{-\kappa}$ and $c < c' < \gamma$. In the limit $\kappa \to 0^{+}$ the first poles of (10.4), s_1' and s_2', will move from the left half plane to the right plane, thus invalidating the formula (10.2). Therefore, we have to replace our original contour C^{-} by a contour C_{μ}^{-} with certain indentations (see Fig. 2), thus separating the increasing and decreasing sequences of poles (10.3) and (10.4). After the analytic continuation of both sides of (7.2) we arrive at the complex orthogonality property of the kernels:

$$\frac{\log q^{-1}}{\mathrm{i}} \int_{C_{\mu}^{-}} \frac{L_{r\mathrm{e}^{\mathrm{i}t}}^{\lambda,\mu}(z,s) L_{\mathrm{e}^{-\mathrm{i}t}}^{\mu,\lambda}(s,z')(q^{2s}, q^{-2s}; q)_{\infty}\, \mathrm{d}s}{(\alpha q^{s}, \alpha q^{-s}, \beta q^{s}, \beta q^{-s}, \gamma q^{s}, \gamma q^{-s}, \delta q^{s}, \delta q^{-s}; q)_{\infty}}$$

$$= \frac{2\pi(\alpha\beta\gamma\delta; q)_{\infty}}{(q, \alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta; q)_{\infty}}$$

$$\times \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}} \frac{\delta(x - x')}{\rho(x; a, b, c, d)}. \tag{10.5}$$

For the P-kernels, the orthogonality relation (10.5) has the more compact form

$$\int_{\Gamma_\mu} P_{e^{i\tau}}^{\lambda,\mu}(x,y)\, P_{e^{-i\tau}}^{\mu,\lambda}(y,x')\rho^\mu(y)\,dy = \frac{\delta(x-x')}{\rho^\lambda(x)}. \tag{10.6}$$

Here $C_\mu^- \to \Gamma_\mu$ when $y = \frac{1}{2}(q^s + q^{-s})$.

11. Real orthogonality of the kernels: the case of one additional mass-point

The structure of poles in (10.3) and (10.4) of the integrand in (10.5) was shown in Fig. 2, where we now let $\omega' \to \omega$ (or $c' \to c$). First we consider the simplest case when $0 < \omega/\log q^{-1} < 1$ or $1 < \gamma/c < q^{-1}$, so only the first pole from each sequence has appeared in the lower half s-plane after analytic continuation. Let us denote the integrand on the left-hand side of (10.5) as $F(z, z', s)$. Then

$$\int_{C_\mu^-} F(z,z',s)\,ds = \int_{C^-} F(z,z',s)\,ds + 2\pi i\left(\operatorname{Res} F(z,z',s)|_{s=s_1'} + \operatorname{Res} F(z,z',s)|_{s=s_2'}\right), \tag{11.1}$$

where

$$\operatorname{Res} F(z,z',s)|_{s=s_\alpha'} = \lim_{s\to s_\alpha'}(s-s_\alpha')F(z,z',s), \quad \alpha = 1,2. \tag{11.2}$$

In view of (3.9)–(3.12) we get

$$\operatorname{Res} F(z,z',s)|_{s=s_1'} = \left.\frac{L_{e^{i\tau}}^{\lambda,\mu}(z,s)(q^{2s}, q^{-2s}; q)_\infty}{(\alpha q^s, \alpha q^{-s}, \beta q^s, \beta q^{-s}, \gamma q^s, \gamma q^{-s}, \delta q^s, \delta q^{-s}; q)_\infty}\right|_{s=s_1'} \times \lim_{s\to s_1'}(s-s_1')L_{e^{-i\tau}}^{\lambda,\mu}(s,z'), \tag{11.3}$$

and

$$\lim_{s\to s_1'}(s-s_1')L_{e^{-i\tau}}^{\lambda,\mu}(s,z') = \frac{(\varepsilon^2, aq^{z'}, cq^{z'}, dq^{z'}, \alpha q^{-s_1'}, \gamma q^{-s_1'}, \delta q^{-s_1'}; q)_\infty}{\log q^{-1}(q, \alpha\delta e^{i\tau}, \alpha\gamma, \beta\gamma, \delta\gamma, ad, q^{-2s_1'}, q^{2z'}; q)_\infty}$$
$$\times\left\{\frac{(\beta\gamma e^{-i\tau}/cd, aq^{-z'}, abq^{s_1'}/\beta; q)_\infty}{(ab, a/d; q)_\infty}\, {}_3\phi_2\left[\begin{matrix}\beta\gamma e^{-i\tau}/ac, dq^{-z'}, bdq^{s_1'}/\beta \\ qd/a, bd\end{matrix}; q, q\right]\right.$$
$$\left. + \frac{(\beta\gamma e^{-i\tau}/ac, dq^{-z'}, bdq^{s_1'}/\beta; q)_\infty}{(bd, d/a; q)_\infty}\, {}_3\phi_2\left[\begin{matrix}\beta\gamma e^{-i\tau}/cd, aq^{-z'}, abq^{s_1'}/\beta \\ qa/d, ab\end{matrix}; q, q\right]\right\}, \tag{11.4}$$

where $q^{s_1'} = cq^{z'}e^{i\tau}/\gamma$. We can now sum the expression above by [11, II.24], thus getting the final result

$$\lim_{s\to s_1'}(s-s_1')L^{\lambda,\mu}_{e^{-i\tau}}(s,z')$$
$$=\frac{(\varepsilon^2,\alpha\gamma e^{-i(\tau+\theta')}/c,\beta\gamma e^{-i(\tau+\theta')}/c,\gamma^2 e^{-i(\tau+\theta')}/c,\delta\gamma e^{-i(\tau+\theta')}/c,ae^{i\theta'},be^{i\theta'},ce^{i\theta'},de^{i\theta'};q)_\infty}{\log q^{-1}(q,\alpha\gamma,\beta\gamma,\delta\gamma,ab,ad,bd,e^{2i\theta'},\gamma^2e^{-2i(\tau+\theta')}/c^2;q)_\infty}, \tag{11.5}$$

since $q^{z'} = e^{i\theta'}$ and $q^{s_1'} = ce^{i\tau}q^{z'}/\gamma$. The value of $\operatorname{Res} F(z,z',s)|_{s=s_2'}$ is obtained from (11.3) and (11.5) by simply replacing θ' by $-\theta'$. Thus we have shown that

$$\int_{-1}^{1} P^{\lambda,\mu}_{e^{i\tau}}(x,y)P^{\lambda,\mu}_{e^{-i\tau}}(y,x')\rho^\mu(y)\,dy$$
$$+\frac{(\alpha\beta,\alpha\delta,\beta\delta;q)_\infty}{(ab,ad,bd;q)_\infty}P^{\lambda,\mu}_{e^{i\tau}}(\cos\theta,\cos(\tau+\theta'+i\omega))$$
$$\times\frac{(ae^{i\theta'},be^{i\theta'},ce^{i\theta'},de^{i\theta'},e^{2i(\tau+\theta')-2\omega};q)_\infty}{(\alpha e^{i(\tau+\theta')-\omega},\beta e^{i(\tau+\theta')-\omega},\gamma e^{i(\tau+\theta')-\omega},\delta e^{i(\tau+\theta')-\omega},e^{2i\theta'};q)_\infty}$$
$$+\frac{(\alpha\beta,\alpha\delta,\beta\delta;q)_\infty}{(ab,ad,bd;q)_\infty}P^{\lambda,\mu}_{e^{i\tau}}(\cos\theta,\cos(\tau-\theta'+i\omega))$$
$$\times\frac{(ae^{-i\theta'},be^{-i\theta'},ce^{-i\theta'},de^{-i\theta'},e^{2i(\tau-\theta')-2\omega};q)_\infty}{(\alpha e^{i(\tau-\theta')-\omega},\beta e^{i(\tau-\theta')-\omega},\gamma e^{i(\tau-\theta')-\omega},\delta e^{i(\tau-\theta')-\omega},e^{-2i\theta'};q)_\infty}$$
$$=\frac{\delta(x-x')}{\rho^\lambda(x)}. \tag{11.6}$$

Here $x=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\cos\theta$, $\gamma/c=e^\omega$, with $1<\gamma/c<q^{-1}$.

12. Real orthogonality of the kernels: the general case

Let us now consider the case of more than one pole, say N poles, appearing in the lower half s-plane which happens when $N-1<\omega/\log q^{-1}<N$, i.e. $q^{1-N}<\gamma/c<q^{-N}$, since $e^\omega=\gamma/c$. Here

$$\int_{C_\mu^-}F(z,z',s)\,ds=\int_{C^-}F(z,z',s)\,ds$$
$$+2\pi i\sum_{j=0}^{N-1}(\operatorname{Res}F(z,z',s)|_{s=s_1'}+\operatorname{Res}F(z,z',s)|_{s=s_2'}), \tag{12.1}$$

where Eqs. (11.2) and (11.3) are still valid but with $q^{s_1'} = (c/\gamma)\mathrm{e}^{\mathrm{i}\tau}q^{z'-j}$ and $q^{s_2'} = (c/\gamma)\mathrm{e}^{\mathrm{i}\tau}q^{-z'-j}$, $j = 0, 1, \ldots, N-1$. So,

$$\lim_{s\to s_1'}(s - s_1')L^{\mu,\lambda}_{\mathrm{e}^{-\mathrm{i}\tau}}(s, z')$$
$$= \frac{(\varepsilon^2, cq^{z'-j}, \alpha\gamma\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c, \gamma^2\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c, \gamma\delta\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c; q)_\infty}{\log q^{-1}(q^{-j}; q)_j(q, \alpha\delta\mathrm{e}^{\mathrm{i}\tau}, \alpha\gamma, \beta\gamma, \delta\gamma, ad, \gamma^2\mathrm{e}^{-2\mathrm{i}\tau}q^{2j-2z'}/c^2, q^{2z'-j}; q)_\infty}$$
$$\times\left\{\frac{(\beta\gamma\mathrm{e}^{-\mathrm{i}\tau}/cd, aq^{z'}, aq^{-z'}, dq^{z'-j}, abc\mathrm{e}^{\mathrm{i}\tau}q^{z'-j}/\beta\gamma; q)_\infty}{(ab, a/d; q)_\infty}\right.$$
$$\times\,{}_4\phi_3\left[\begin{matrix}\beta\gamma\mathrm{e}^{-\mathrm{i}\tau}/ac, dq^{z'}, dq^{-z'}, bcd\mathrm{e}^{\mathrm{i}\tau}q^{z'-j}/\beta\gamma \\ qd/a, bd, dq^{z'-j}\end{matrix}; q, q\right]$$
$$+\frac{(\beta\gamma\mathrm{e}^{-\mathrm{i}\tau}/ac, dq^{z'}, dq^{-z'}, aq^{z'-j}, bcd\mathrm{e}^{\mathrm{i}\tau}q^{z'-j}/\beta\gamma; q)_\infty}{(bd, d/a; q)_\infty}$$
$$\left.\times\,{}_4\phi_3\left[\begin{matrix}\beta\gamma\mathrm{e}^{-\mathrm{i}\tau}/cd, aq^{z'}, aq^{-z'}, abc\mathrm{e}^{\mathrm{i}\tau}q^{z'-j}/\beta\gamma \\ qa/d, ab, aq^{z'-j}\end{matrix}; q, q\right]\right\}. \tag{12.2}$$

Both ${}_4\phi_3$ series above are balanced and their coefficients are so matched that the sum in the curly brackets can be transformed to a terminating ${}_8\phi_7$ series by [11, (2.10.10)] which, in turn, can be transformed back to a terminating and balanced ${}_4\phi_3$. The final result is

$$\lim_{s\to s_1'}(s - s_1')L^{\mu,\lambda}_{\mathrm{e}^{-\mathrm{i}\tau}}(s, z')$$
$$= \frac{(\varepsilon^2, \alpha\delta\mathrm{e}^{\mathrm{i}\tau}q^{-j}, \alpha\gamma\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c, \beta\gamma\mathrm{e}^{-\mathrm{i}\tau}q^{-z'}/c, \gamma^2\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c, \delta\gamma\mathrm{e}^{-\mathrm{i}\tau}q^{j-z'}/c; q)_\infty}{\log q^{-1}(q^{-j}; q)_j(q, \alpha\delta\mathrm{e}^{\mathrm{i}\tau}, \alpha\gamma, \beta\gamma, \delta\gamma, ab, ad, bd; q)_\infty}$$
$$\times\frac{(aq^{z'}, bq^{z'}, cq^{z'-j}, dq^{z'}; q)_\infty}{(\gamma^2\mathrm{e}^{-2\mathrm{i}\tau}q^{2j-2z'}/c^2, q^{2z'}; q)_\infty}$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-j}, aq^{-z'}, bq^{-z'}, dq^{-z'} \\ q^{1-2z'}, \beta\gamma\mathrm{e}^{-\mathrm{i}\tau}q^{-z'}/c, \alpha\delta\mathrm{e}^{\mathrm{i}\tau}q^{-j}\end{matrix}; q, q\right]. \tag{12.3}$$

So, from (3.7), (10.5), (12.1) and (12.3) we obtain the following general orthogonality property of the kernels:

$$\int_{-1}^{1} P^{\lambda,\mu}_{\mathrm{e}^{\mathrm{i}\tau}}(x, y)P^{\mu,\lambda}_{\mathrm{e}^{-\mathrm{i}\tau}}(y, x')\rho^\mu(y)\,\mathrm{d}y$$
$$+\frac{(\alpha\beta, \alpha\delta, \beta\delta, a\mathrm{e}^{\mathrm{i}\theta'}, b\mathrm{e}^{\mathrm{i}\theta'}, c\mathrm{e}^{\mathrm{i}\theta'}, d\mathrm{e}^{\mathrm{i}\theta'}, \mathrm{e}^{2\mathrm{i}(\theta+\tau)-2\omega}; q)_\infty}{(ab, ad, bd, \alpha\mathrm{e}^{\mathrm{i}(\tau+\theta')-\omega}, \beta\mathrm{e}^{\mathrm{i}(\tau+\theta')-\omega}, \gamma\mathrm{e}^{\mathrm{i}(\tau+\theta')-\omega}, \delta\mathrm{e}^{\mathrm{i}(\theta'+\tau)-\omega}, \mathrm{e}^{2\mathrm{i}\theta'}; q)_\infty}$$
$$\times\sum_{j=0}^{N-1}\frac{(\beta\gamma\mathrm{e}^{-\mathrm{i}(\tau+\theta')}/c, q\mathrm{e}^{-\mathrm{i}\tau}/\alpha\delta, q\mathrm{e}^{-\mathrm{i}\theta'}/c; q)_j(q\mathrm{e}^{2\omega-2\mathrm{i}(\tau+\theta')}; q)_{2j}}{(q, q\mathrm{e}^{\omega-\mathrm{i}(\tau+\theta')}/\alpha, q\mathrm{e}^{\omega-\mathrm{i}(\tau+\theta')}/\beta, q\mathrm{e}^{\omega-\mathrm{i}(\tau+\theta')}/\gamma, q\mathrm{e}^{\omega-\mathrm{i}(\tau+\theta')}/\delta; q)_j}$$
$$\times\left(-\frac{c}{\beta\gamma}\mathrm{e}^{\mathrm{i}(\tau+\theta')}\right)^j q^{-\binom{j}{2}}\,{}_4\phi_3\left[\begin{matrix}q^{-j}, \alpha\mathrm{e}^{-\mathrm{i}\theta'}, b\mathrm{e}^{-\mathrm{i}\theta'}, d\mathrm{e}^{-\mathrm{i}\theta'} \\ q\mathrm{e}^{-2\mathrm{i}\theta'}, \beta\gamma\mathrm{e}^{-\mathrm{i}(\tau+\theta')}/c, \alpha\delta\mathrm{e}^{\mathrm{i}\tau}q^{-j}\end{matrix}; q, q\right]$$

$$\times P^{\lambda,\mu}_{e^{i\tau}}(\cos\theta, \cos(\tau + \theta' + i\omega + ij\log q))$$
$$+ \operatorname{idem}(\theta'; -\theta')$$
$$= \frac{\delta(x - x')}{\rho^{\lambda}(x)}, \tag{12.4}$$

where, as before, $x = \cos\theta$, $\gamma/c = e^{\omega}$, $q^{1-N} < \gamma/c < q^{-N}$ and $\alpha\gamma = ac$, $\beta\delta = bd$. When $N = 1$, the formula above reduces to (11.6).

13. General nonsymmetric q-Fourier transformation and its inversion formula

From (3.7), (3.8) and the orthogonality property (2.4) of the Askey–Wilson polynomials one can write

$$t^m r^{\lambda}_m(x) = \int_{-1}^{1} P^{\lambda,\mu}_t(x, y) r^{\mu}_m(y)\rho^{\mu}(y)\,dy, \tag{13.1}$$

where $|t| < 1$ and $r^{\lambda}_m(x) = h^{\lambda}_m p^{\lambda}_m(x)$, $r^{\mu}_m(y) = h^{\mu}_m p^{\mu}_m(y)$. To consider the analytic continuation of (13.1) to the unit circle $|t| = 1$, when $t = re^{i\tau} = e^{i\tau-\kappa}$, $\kappa \to 0^+$ and $\gamma > c$, let us rewrite this formula in the contour integral form (Fig. 2):

$$t^m r^{\lambda}_m(x) = \frac{\log q^{-1}}{i} h^{\lambda}_0 \int_{C^-} \frac{L^{\lambda,\mu}_t(z, s) r^{\mu}_m(\frac{1}{2}(q^s + q^{-s}))(q^{2s}, q^{-2s}; q)_\infty\,ds}{(\alpha q^s, \alpha q^{-s}, \beta q^s, \beta q^{-s}, \gamma q^s, \gamma q^{-s}, \delta q^s, \delta q^{-s}; q)_\infty}, \tag{13.2}$$

where $x = \cos\theta$, $q^z = e^{i\theta}$ and $y = \cos\phi$, $q^s = e^{i\phi}$. The singular term of $L^{\lambda,\mu}_t(z, s) = K^{\lambda,\mu}_t(x, y)$ is the $k = 0$ term of $K^{(3)}_t(x, y)$ in (3.12), and the poles are defined by (10.3). Therefore, by analytic continuation, from (13.2) we obtain

$$e^{i\tau m} r^{\lambda}_m(x) = \int_{-1}^{1} P^{\lambda,\mu}_{e^{i\tau}}(x, y) r^{\mu}_m(y)\rho^{\mu}(y)\,dy, \tag{13.3}$$

provided that $\gamma > c$.

To consider the analytic continuation of (13.1) with λ and μ interchanged, t replaced by $t' = re^{-i\tau} = e^{-i\tau-\kappa}$ and $\kappa \to 0^+$, let us rewrite this formula in terms of a contour integral (Fig. 3):

$$(t')^m r^{\mu}_m(y) = \frac{\log q^{-1}}{i} h^{\mu}_0 \int_{C^-} \frac{L^{\mu,\lambda}_{t'}(s, z) r^{\lambda}_m(\frac{1}{2}(q^z + q^{-z}))(q^{2z}, q^{-2z}; q)_\infty\,dz}{(aq^z, aq^{-z}, bq^z, bq^{-z}, cq^z, cq^{-z}, dq^z, dq^{-z}; q)_\infty}. \tag{13.4}$$

The essential poles of the integrand are now at

$$z_1 = \frac{\omega - \kappa - i(\tau + \phi)}{\log q^{-1}}, \quad z_1 - 1, \ldots,$$
$$z_2 = \frac{\omega - \kappa - i(\tau + \phi)}{\log q^{-1}}, \quad z_2 - 1, \ldots . \tag{13.5}$$

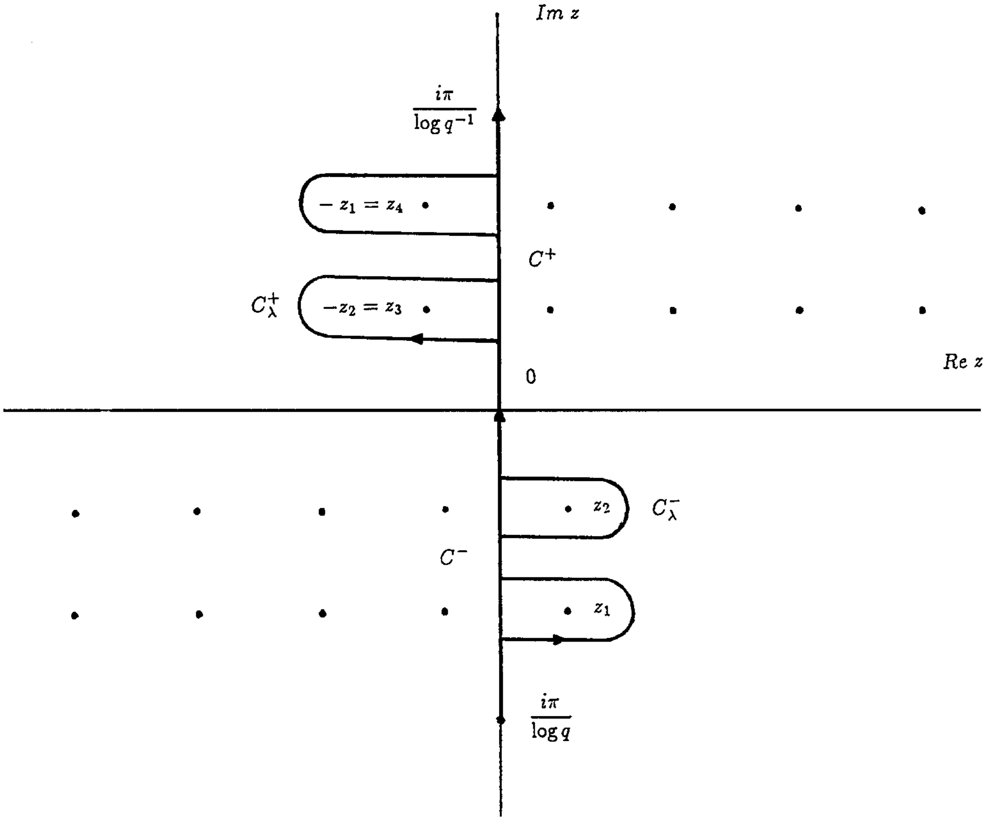

Fig. 3.

In the limit $\kappa \to 0^+$ the first poles, z_1 and z_2, will move from the left half plane to the right half plane, thus invalidating the formula (13.4). Therefore, in analogy with Section 10, we need to replace our contour C^- by a contour C_λ^- with certain indentations (see Fig. 3). This gives the analytic continuation of (13.4), and for the kernel P we can write

$$e^{-i\tau m} r_m^{\mu}(y) = \int_{\Gamma_\lambda} P^{\lambda;\mu}_{e^{-i\tau}}(y, x) r_m^{\lambda}(x) \rho^{\lambda}(x)\, dx. \tag{13.6}$$

Here $C_\lambda^- \to \Gamma_\lambda$ when $x = \frac{1}{2}(q^z + q^{-z})$.

Now we can define a general q-Fourier transformation by

$$f(x) = \int_{-1}^{1} P^{\lambda;\mu}_{e^{i\tau}}(x, y) g(y) \rho^{\mu}(y)\, dy := F_q[g](x), \tag{13.7}$$

and may guess that its inversion formula has the form

$$g(y) = \int_{\Gamma_\lambda} P^{\mu;\lambda}_{e^{-i\tau}}(y, x) f(x) \rho^{\lambda}(x)\, dx := F_q^{-1}[f](y). \tag{13.8}$$

From this point of view, Eqs. (13.3) and (13.6) are just mappings, up to a phase factor, of two systems of Askey–Wilson polynomials, $\{p_m^{\lambda}(x)\}$ and $\{p_m^{\mu}(y)\}$, corresponding to two different sets of parameters $\lambda = (a, b, c, d)$ and $\mu = (\alpha, \beta, \gamma, \delta)$ with the conditions $\alpha\gamma = ac$ and $\beta\delta = bd$.

To prove the inversion formula (13.8), let us interchange the order of integration in the double integral

$$\int_{\Gamma_\mu} \mathrm{d}y \rho^\mu(y) P_{e^{i\tau}}^{\lambda,\mu}(x, y) \int_{\Gamma_{\lambda'}} P_{e^{-i\tau}}^{\mu,\lambda'}(y, x') f(x') \rho^\lambda(x') \,\mathrm{d}x'$$
$$= \int_{\Gamma_{\lambda'}} \mathrm{d}x' \rho^{\lambda'}(x') f(x') \int_{\Gamma_\mu} P_{e^{i\tau}}^{\lambda,\mu}(x, y) P_{e^{-i\tau}}^{\mu,\lambda'}(y, x') \rho^\mu(y) \,\mathrm{d}y$$

and apply the multiplication law (7.1) for the kernels to get

$$\int_{\Gamma_\mu} \mathrm{d}y \rho^\mu(y) P_{e^{i\tau}}^{\lambda,\mu}(x, y) \int_{\Gamma_{\lambda'}} P_{e^{-i\tau}}^{\mu,\lambda'}(y, x') f(x') \rho^{\lambda'}(x') \,\mathrm{d}x'$$
$$= \int_{-1}^{1} P_1^{\lambda,\lambda'}(x, x') f(x') \rho^{\lambda'}(x') \,\mathrm{d}x'. \tag{13.9}$$

In the limit $\lambda' \to \lambda$, under the same conditions as in (10.1), we obtain the inversion formula (13.8) for bounded continuous functions.

By using the same considerations as in Section 12 one can finally obtain the inversion formula in terms of a real integral with additional mass points:

$$\begin{aligned} g(y) = &\int_{-1}^{1} P_{e^{-i\tau}}^{\mu,\lambda}(y, x) f(x) \rho^\lambda(x) \,\mathrm{d}x \\ &+ \frac{(\alpha\beta, \alpha\delta, \beta\delta, \alpha e^{i\phi}, \beta e^{i\phi}, \gamma e^{i\phi}, \delta e^{i\phi}, e^{2i(\tau+\phi)-2\omega}; q)_\infty}{(ab, ad, bd, ae^{i(\tau+\phi)-\omega}, be^{i(\tau+\phi)-\omega}, ce^{i(\tau+\phi)-\omega}, de^{i(\tau+\phi)-\omega}, e^{2i\phi'}; q)_\infty} \\ &\times \sum_{j=0}^{N-1} \frac{(\gamma e^{i(\tau-\phi)}, qe^{-i\tau}/\alpha\delta, qe^{-i\phi}/\beta; q)_j (qe^{2\omega-2i(\tau+\phi)}; q)_{2j}}{(q, qe^{\omega-i(\tau+\phi)}/a, qe^{\omega-i(\tau+\phi)}/b, qe^{\omega-i(\tau+\phi)}/c, qe^{\omega-i(\tau+\phi)}/d; q)_j} \\ &\times (-\gamma e^{i(\tau+\phi)})^j q^{-\binom{j}{2}} {}_4\phi_3\left[\begin{matrix} q^{-j}, ae^{i(\tau+\phi)-\omega}q^{-j}, be^{i(\tau+\phi)-\omega}q^{-j}, de^{i(\tau+\phi)-\omega}q^{-j} \\ q^{1-2j}e^{2i(\tau+\phi)-2\omega}, \beta\gamma e^{i\phi-\omega}q^{-j}/c, \alpha\delta e^{i\tau}q^{-j} \end{matrix}; q, q\right] \\ &\times f(\cos(\tau + \phi + i\omega + ij\log q)) \\ &+ \mathrm{idem}(\phi; -\phi), \end{aligned} \tag{13.10}$$

where $y = \cos\phi$, $\gamma/c = e^\omega$, $q^{1-N} < \gamma/c < q^{-N}$, and $\alpha\gamma = ac$, $\beta\delta = bd$.

14. Special cases of the kernel

Although the general form of the kernel $K_t^{\lambda,\mu}(x, y)$ given in (3.9)–(3.12) is quite formidable it does expose all its poles which are all that are needed for the purposes of this paper. However, in view of its multiplicative property (7.2) and its continuous orthogonality when $|t| \to 1$ it may be useful to list some special cases that correspond to orthogonal polynomials which are lower than the Askey–Wilson polynomials in the general scheme of classical orthogonal polynomials.

Case I: *Continuous dual q-Hahn polynomials.* Setting $d = 0$ and $\delta = 0$, $\alpha\gamma = ac$ in (3.8) one can obtain

$$\begin{aligned}
&K_t^{\lambda,\mu}(x, y)\\
&:= \sum_{n=0}^{\infty} \frac{(ab, ac; q)_n}{(q, bc; q)_n} (ta^{-2}) p_n(x; a, b, c) p_n(y; \alpha, \beta, \gamma)\\
&= \frac{(c^2t^2/\gamma^2, \alpha cte^{i\theta}/\gamma, \alpha cte^{-i\theta}/\gamma; q)_\infty}{(\alpha\beta, ac, bc, \alpha c^2t^2e^{i\phi}/\gamma^2; q)_\infty}\\
&\quad\times \frac{(\beta e^{i\phi}, bcte^{-i\phi}/\gamma, \gamma e^{-i\phi}, c^2te^{i\phi}/\gamma, \alpha te^{i\phi}; q)_\infty}{(cte^{i\theta+i\phi}/\gamma, cte^{i\theta-i\phi}/\gamma, cte^{i\phi-i\theta}/\gamma, cte^{-i\theta-i\phi}/\gamma; q)_\infty}\\
&\quad\times \sum_{k=0}^{\infty} (-bc)^k q^{\binom{k}{2}} \frac{(\alpha c^2t^2e^{i\phi}/\gamma^2; q)_{2k}}{(c^2t^2/\gamma^2; q)_{2k}}\\
&\quad\times \frac{(t, cte^{i\theta+i\phi}/\gamma, cte^{i\theta-i\phi}/\gamma, cte^{i\phi-i\theta}/\gamma, cte^{-i\theta-i\phi}/\gamma; q)_k}{(q, c^2te^{i\phi}/\gamma, \alpha te^{i\phi}, bcte^{-i\phi}/\gamma, \alpha cte^{i\theta}/\gamma, \alpha cte^{-i\theta}/\gamma; q)_k}\\
&\quad\times \sum_{l=0}^{\infty} (\beta e^{i\phi})^l \frac{(\alpha e^{-i\phi}, bctq^k/\beta\gamma, \alpha c^2t^2q^{2k}e^{i\phi}/\gamma^2; q)_l}{(q, bct^ke^{-i\phi}/\gamma, c^2t^2q^{2k}/\gamma^2; q)_l}\\
&\quad\times \frac{(ctq^ke^{i\theta-i\phi}/\gamma, ctq^ke^{-i\theta-i\phi}/\gamma; q)_l}{(\alpha ctq^ke^{i\theta}/\gamma, \alpha ctq^ke^{-i\theta}/\gamma; q)_l}\\
&\quad\times {}_8W_7\left(\frac{\alpha c^2t^2}{\gamma^2} q^{2k+l-1}e^{i\phi}; \frac{ct}{\gamma} q^ke^{i\theta+i\phi}, \frac{ct}{\gamma} q^ke^{i\phi-i\theta}, \alpha e^{i\phi}, \frac{\alpha t}{\gamma} q^{k+l}, \frac{c^2t}{\gamma^2} q^{k+l}; q, \gamma e^{-i\phi}\right).
\end{aligned} \tag{14.1}$$

Here $\lambda = (a, b, c)$ and $\mu = (\alpha, \beta, \gamma)$, $\alpha\gamma = ac$. It would be nice if this formula would follow from (3.10)–(3.12) by simply setting $d = \delta = 0$, but it doesn't, unless one goes through a series of transformations followed by careful use of the limits. We shall give a separate proof of (14.1) in the Appendix.

The polynomials in (14.1) are, of course, the continuous dual q-Hahn polynomials

$$p_n(x; a, b, c) = {}_3\phi_2\left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, \end{matrix}; q, q\right], \tag{14.2}$$

see, e.g., [6, 7]. In the limit $d \to 0$, $\delta \to 0$ and $d/\delta = \beta/b$ it is easy to see from (8.4) that

$$\begin{aligned}
K_1^{\lambda,\mu}(x, y) &= \frac{(c^2/\gamma^2, ce^{i\theta}, ce^{-i\theta}, \alpha e^{i\phi}, \alpha e^{-i\phi}, bce^{-i\phi}/\gamma, \beta e^{i\phi}; q)_\infty}{(\alpha\beta, ac, bc, ce^{i\theta+i\phi}/\gamma, ce^{i\theta-i\phi}/\gamma, ce^{i\phi-i\theta}/\gamma, ce^{-i\theta-i\phi}/\gamma; q)_\infty}\\
&\quad\times {}_3\phi_2\left[\begin{matrix} ce^{i\theta-i\phi}/\gamma, ce^{-i\theta-i\phi}/\gamma, bc/\beta\gamma \\ c^2/\gamma^2, bce^{-i\phi}/\gamma \end{matrix}; q, \beta e^{i\phi}\right],
\end{aligned}$$

$c \neq \gamma$ and $\alpha\gamma = ac$. Under the additional condition $\beta\gamma = bc$ this reduces to

$$K_1^{\lambda,\mu}(x, y) = \frac{(\alpha e^{i\phi}, \alpha e^{-i\phi}, \beta e^{i\phi}, \beta e^{-i\phi}, ce^{i\theta}, ce^{-i\theta}, (c/\gamma)^2; q)_\infty}{(\alpha\beta, ac, bc, ce^{i\theta+i\phi}/\gamma, ce^{i\theta-i\phi}/\gamma, ce^{i\phi-i\theta}/\gamma, ce^{-i\theta-i\phi}/\gamma; q)_\infty}, \tag{14.3}$$

and we can apply the above method to get an inversion formula for the corresponding transformation.

Case II. *Al-Salam–Chihara polynomials.* The Al-Salam and Chihara polynomials [6, 12] may be defined by simply setting $c = 0$ in (14.2), i.e.

$$\begin{aligned} p_n(x; a, b) &:= {}_3\phi_2\left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}; q, q\right] \\ &= \frac{(be^{-i\theta}; q)_n}{(ab; q)_n}(ae^{i\theta})^n\, {}_2\phi_1\left[\begin{matrix} q^{-n}, ae^{i\theta} \\ q^{1-n}e^{i\theta}/b \end{matrix}; q, qe^{-i\theta}/b\right]. \end{aligned} \tag{14.4}$$

So, with $\lambda = (a, b)$ and $\mu = (\alpha, \beta)$, $\alpha\beta = ab$, we find that

$$\begin{aligned} K_t^{\lambda,\mu}(x, y) &:= \sum_{n=0}^\infty \frac{(ab; q)_n}{(q; q)_n}(ta^{-2})^n p_n(x; a, b) p_n(y; \alpha, \beta) \\ &= \frac{(\alpha^2t^2/a^2, be^{-i\theta}, \alpha^2te^{i\theta}/a, bte^{i\theta}, \alpha te^{i\phi}, \alpha te^{-i\phi}; q)_\infty}{(ab, \alpha^2t^2e^{i\theta}/a, \alpha te^{i\theta+i\phi}/a, \alpha te^{i\theta-i\phi}/a, \alpha te^{i\phi-i\theta}/a, \alpha te^{-i\theta-i\phi}/a; q)_\infty} \\ &\quad \times {}_8W_7(\alpha^2t^2e^{i\theta}/aq; t, \alpha t/\beta, ae^{i\theta}, \alpha te^{i\theta+i\phi}/a, \alpha te^{i\theta-i\phi}/a; q, be^{-i\theta}), \end{aligned} \tag{14.5}$$

which in the case $t = 1$ becomes

$$K_1^{\lambda,\mu}(x, y) = \frac{(be^{i\theta}, be^{-i\theta}, \alpha e^{i\phi}, \alpha e^{-i\phi}, \alpha^2/a^2; q)}{(ab, \alpha e^{i\theta+i\phi}/a, \alpha e^{i\theta-i\phi}/a, \alpha e^{i\phi-i\theta}/a, \alpha e^{-i\theta-i\phi}/a; q)_\infty}. \tag{14.6}$$

With a different normalization, namely,

$$p_n(x;a, b) := \frac{(ab; q)_n}{(q; q)_n} a^{-n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}; q, q\right], \tag{14.7}$$

the corresponding kernel is

$$\begin{aligned} K_t^{\lambda,\mu}(x, y) &:= \sum_{n=0}^\infty \frac{(q; q)_n}{(ab; q)_n} t^n p_n(x; a, b) p_n(y; \alpha, \beta) \\ &= \frac{(t^2, be^{-i\theta}, \alpha te^{i\theta}, \beta te^{i\theta}, ate^{i\phi}, ate^{-i\phi}; q)_\infty}{(ab, at^2e^{i\theta}, te^{i\theta+i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, te^{-i\theta-i\phi}; q)_\infty} \\ &\quad \times {}_8W_7(at^2q^{-1}e^{i\theta}; \alpha t/b, \beta t/b, ae^{i\theta}, te^{i\theta+i\phi}, te^{i\theta-i\phi}; q, be^{-i\theta}) \\ &= \frac{(\beta t/a, \alpha te^{i\theta}, \alpha te^{-i\theta}, ate^{i\phi}, ate^{-i\phi}; q)_\infty}{(\alpha at, te^{i\theta+i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, te^{-i\theta-i\phi}; q)_\infty} \\ &\quad \times {}_8W_7(\alpha atq^{-1}; \alpha t/b, ae^{i\theta}, ae^{-i\theta}, \alpha e^{i\phi}, \alpha e^{-i\phi}; q, \beta t/a) \end{aligned} \tag{14.8}$$

with $ab = \alpha\beta$. Eqs. (14.5) and (14.8) can be derived as a limiting case of (14.1).

The special case of the polynomials (14.7) are the so-called continuous q-Laguerre polynomials,

$$L_n^\alpha(x|q) = p_n(x; q^{(2\alpha+1)/4}, q^{(2\alpha+3)/4}) \tag{14.9}$$

(see [6]), which go to the classical Laguerre polynomials in the limit $q \to 1$,

$$\lim_{q\to 1} L_n^\alpha(1-(1-q^{1/2})x|q) = L_n^\alpha(x). \tag{14.10}$$

From this point of view, one can consider (14.8) as a q-version of the well-known Poisson kernel for the Laguerre polynomials,

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} t^n L_n^\alpha(x) L_n^\alpha(y)$$
$$= (1-t)^{-1} \exp\left(-t\frac{x+y}{1-t}\right)(xyt)^{-\alpha/2} I_\alpha\left(2\frac{(xyt)^{1/2}}{1-t}\right), \quad |t|<1, \tag{14.11}$$

where

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(-;\nu+1;z^2/4)$$
$$= \frac{(z/2)^\nu}{\Gamma(\nu+1)} e^{-z}\, {}_1F_1(\nu+1/2; 2\nu+1; 2z). \tag{14.12}$$

So, one can consider the corresponding transformation with the kernel (14.8) as a q-version of the Fourier–Bessel transform.

Case III: *Continuous big q-Hermite polynomials.* Setting $b=0$ in (14.4) one obtains the so-called continuous big q-Hermite polynomials [12]:

$$p_n(x;a) := {}_3\phi_2\left[\begin{matrix} q^{-n}, a e^{i\theta}, a e^{-i\theta} \\ 0, 0 \end{matrix}; q, q\right]$$
$$= (a e^{i\theta})^n \sum_{k=0}^{n} \frac{(q^{-n}, a e^{i\theta}; q)_k}{(q;q)_k} (-q^n e^{-2i\theta})^k q^{-\binom{k}{2}}. \tag{14.13}$$

The nonsymmetric Poisson kernel for these polynomials can be derived as a limiting case of (14.5) by [11, (3.2.11)],

$$K_t^{a,\alpha}(x,y) := \sum_{n=0}^{\infty} \frac{(ta^{-2})^n}{(q;q)_n} p_n(x;a) p_n(y;\alpha)$$
$$= \frac{(\alpha^2 t^2/a^2, \alpha t e^{i\phi}, \alpha e^{-i\phi}; q)_\infty}{(\alpha t e^{i\theta+i\phi}/a, \alpha t e^{i\theta-i\phi}/a, \alpha t e^{i\phi-i\theta}/a, \alpha t e^{-i\theta-i\phi}/a; q)_\infty}$$
$$\times {}_3\phi_2\left[\begin{matrix} t, \alpha t e^{i\theta+i\phi}/a, \alpha t e^{i\phi-i\theta}/a \\ \alpha^2 t^2/a^2, \alpha t e^{i\phi} \end{matrix}; q, \alpha e^{-i\phi}\right]. \tag{14.14}$$

On the other hand, setting $b = \beta = 0$ and $b/\beta = \alpha/a$ in (14.7)–(14.8) one obtains the kernel

$$K_t^{a,\alpha}(x, y) := \sum_{n=0}^{\infty} t^n (q;q)_n p_n(x;a) p_n(y;\alpha)$$

$$= \frac{(t^2, ate^{i\phi}, \alpha e^{-i\phi}; q)_\infty}{(te^{i\theta+i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, te^{-i\theta-i\phi}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{matrix} at/\alpha, te^{i\theta+i\phi}, te^{i\phi-i\theta} \\ t^2, ate^{i\phi} \end{matrix} ; q, \alpha e^{-i\phi} \right]. \tag{14.15}$$

Observe that

$$\lim_{a \to 0} e^{in\theta} \sum_{n=0}^{\infty} \frac{(q^{-n}, ae^{i\theta}; q)_k}{(q;q)_k} (-e^{-2i\theta})^k q^{-\binom{k}{2}} = H_n(x|q), \tag{14.16}$$

where $H_n(x|q)$ is the continuous q-Hermite polynomial defined in (1.10). See also [1] for the proof of this limit directly from the first expression in (14.13). Eq. (14.15) then reduces further to

$$K_t^{0,\alpha}(x, y) = \frac{(t^2, \alpha e^{-i\phi}; q)_\infty}{(te^{i\theta+i\phi}, te^{i\theta-i\phi}, te^{i\phi-i\theta}, te^{-i\theta-i\phi}; q)_\infty}$$

$$\times {}_2\phi_1 \left[\begin{matrix} te^{i\theta+i\phi}, te^{i\phi-i\theta} \\ t^2 \end{matrix} ; q, \alpha e^{-i\theta} \right]. \tag{14.17}$$

This ${}_2\phi_1$ series is related to an addition formula [16] for the q-Bessel functions of Jackson [11, Exercise 1.24]. Denoting

$$J_t(x, y) = {}_2\phi_1 \left[\begin{matrix} te^{i\theta+i\phi}, te^{i\theta-i\phi} \\ t^2 \end{matrix} ; q, \alpha e^{-i\theta} \right], \tag{14.18}$$

we find that a special limiting case of (10.6) is

$$\int_\Gamma J_{e^{i\tau}}(x, y) J_{e^{i\tau}}^*(x', y) \rho(y; e^{i\tau+i\theta}, e^{i\tau-i\theta}, e^{i\theta'-i\tau}, e^{-i\theta'-i\tau})\, dy$$

$$= \left[\frac{2\pi}{(q;q)_\infty} \right]^2 \frac{\delta(x - x')}{|(e^{2i\tau}, e^{2i\theta}; q)_\infty|^2} (1 - x^2)^{1/2}, \tag{14.19}$$

where Γ is a limiting symmetric form of the contour Γ_μ in (10.6)

We would like to add, in conclusion, that there are many more interesting special and limiting cases of the Askey–Wilson polynomials and that the same technique as we have developed in this paper can be applied to most of them.

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Appendix. Proof of (14.1)

The q-integral representation (4.4) in the case of the dual q-Hahn polynomials has the simpler form

$$K_t^{\lambda,\mu}(x, y) = B_1(\theta, \phi) \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u \frac{(bue^{i\theta}, bue^{-i\theta}; q)_\infty}{(bau/q, bcu/q; q)_\infty}$$
$$\times \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_q v \frac{(\gamma ve^{i\phi}, \gamma ve^{-i\phi}; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q; q)_\infty}$$
$$\times {}_2\phi_1\left[\begin{matrix} q/u, q/v \\ bc \end{matrix}; q, \frac{bc}{q^2} uvt\right], \tag{15.1}$$

where

$$B_1^{-1}(\theta, \phi) = -\frac{q^2(1-q)^2}{4b\gamma} (q, q, ac, \alpha\beta; q)_\infty$$
$$\times h(x; b) h(y; \gamma) \rho^\lambda(x) \rho^\mu(y), \tag{15.2}$$

and $\lambda = (a, b, c)$ and $\mu = (\alpha, \beta, \gamma)$, $\alpha\gamma = ac$. The troublesome ${}_6W_5$ function in (4.4) now becomes a ${}_2\phi_1$ series which is first transformed to a ${}_2\phi_2$ series by

$${}_2\phi_1\left[\begin{matrix} q/u, q/v \\ bc \end{matrix}; q, \frac{bc}{q^2} uvt\right] = \frac{(bctu/q, bctv/q; q)_\infty}{(bc, bcuvt/q^2; q)_\infty} {}_2\phi_2\left[\begin{matrix} t, bcuvt/q^2 \\ bctu/q, bctv/q \end{matrix}; q, bc\right] \tag{15.3}$$

(this formula can be obtained as a consequence of [11, III.2 and III.4]). So,

$$K_t^{\lambda,\mu}(x, y) = \frac{B_1}{(bc; q)_\infty} \sum_{k=0}^{\infty} (-bc)^k q^{\binom{k}{2}} \frac{(t; q)_k}{(q; q)_k} V_k, \tag{15.4}$$

where

$$V_k = \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u \frac{(bue^{i\theta}, bue^{-i\theta}, bctuq^{k-1}; q)_\infty}{(bau/q, bcu/q; q)_\infty}$$
$$\times \int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} d_q v \frac{(\gamma ve^{i\phi}, \gamma ve^{-i\phi}, bctvq^{k-1}; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q, bcuvtq^{k-2}; q)_\infty}. \tag{15.5}$$

The limiting form the q-integral (6.1) now is

$$\int_{qe^{i\phi}/\gamma}^{qe^{-i\phi}/\gamma} \frac{(\gamma v e^{i\phi}, \gamma v e^{-i\phi}, bctvq^{k-1}; q)_\infty}{(\gamma\alpha v/q, \gamma\beta v/q, bcuvtq^{k-2}; q)_\infty} d_q v$$
$$= \frac{q(1-q)}{2i\gamma}(q, \alpha\beta; q)_\infty h(y;\gamma)\rho^\mu(y)\frac{(bctq^k e^{-i\phi}/\gamma; q)_\infty}{(bcutq^{k-1}e^{-i\phi}/\gamma; q)_\infty}$$
$$\times {}_3\phi_2\left[\begin{matrix}\alpha e^{-i\phi}, \beta e^{-i\phi}, q/u \\ \alpha\beta, bctq^k e^{-i\phi}/\gamma\end{matrix}; q, \frac{bcut}{\gamma}q^{k-1}e^{i\phi}\right]$$
$$= \frac{q(1-q)}{2i\gamma}(q, \beta e^{i\phi}; q)_\infty h(y;\gamma)\rho^\mu(y)\frac{(bctq^k e^{-i\phi}/\gamma, \alpha bctuq^{k-1}/\gamma; q)_\infty}{(bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k-1}e^{-i\phi}/\gamma; q)_\infty}$$
$$\times {}_3\phi_2\left[\begin{matrix}\alpha e^{-i\phi}, bctq^k/\beta\gamma, bctuq^{k-1}e^{-i\phi}/\gamma \\ bctq^k e^{-i\phi}/\gamma, \alpha bctuq^{k-1}/\gamma\end{matrix}; q, \beta e^{i\phi}\right]. \qquad (15.6)$$

Here we have used a limiting case of [11, (2.10.19)],

$$\int_a^b \frac{(qt/a, qt/b, ct; q)_\infty}{(dt, et, ft; q)_\infty} d_q t$$
$$= b(1-q)\frac{(q, a/b, qb/a, bc, abde; q)_\infty}{(ad, ae, bd, be, bf; q)_\infty}$$
$$\times {}_3\phi_2\left[\begin{matrix}bd, be, c/f \\ bc, abde\end{matrix}; q, af\right], \quad |af| < 1, \qquad (15.7)$$

and then transformed the ${}_3\phi_2$ series by [11, III.9]. Therefore,

$$V_k = \frac{q(1-q)}{2i\gamma}h(y;\gamma)\rho^\mu(y)(q, \beta e^{i\phi}, bctq^k e^{-i\phi}/\gamma; q)_\infty$$
$$\times \sum_{l=0}^{\infty}(\beta e^{i\phi})^l \frac{(\alpha e^{-i\phi}, bctq^k/\beta\gamma; q)_l}{(q, bctq^k e^{-i\phi}/\gamma; q)_l}$$
$$\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bue^{i\theta}, bue^{-i\theta}, bctuq^{k-1}, \alpha bctuq^{k+l-1}/\gamma; q)_\infty}{(bau/q, bcu/q, bcutq^{k-1}e^{i\phi}/\gamma, bcutq^{k+l-1}e^{-i\phi}/\gamma; q)_\infty} d_q u$$
$$= B_1^{-1}\frac{(\beta e^{i\phi}, bctq^k e^{-i\phi}/\gamma, ctq^k e^{-i\theta}; q)_\infty}{(\alpha\beta, ctq^k e^{i\phi - i\theta}/\gamma; q)_\infty}$$
$$\times \sum_{l=0}^{\infty}(\beta e^{i\phi})^l \frac{(\alpha e^{-i\phi}, bctq^k/\beta\gamma; q)_l}{(q, bctq^k e^{-i\phi}/\gamma; q)_l}$$
$$\times \frac{(\alpha ctq^{k+l}e^{-i\theta}/\gamma, \alpha tq^{k+l}e^{-i\phi}, \alpha ctq^{k+l}e^{-i\phi}/a; q)_\infty}{(ctq^{k+l}e^{i\theta - i\phi}/\gamma, ctq^{k+l}e^{-i\theta - i\phi}/\gamma, \alpha ctq^{k+l}e^{-i\theta - i\phi}; q)_\infty}$$
$$\times {}_8W_7\left(\alpha ctq^{k+l-1}e^{-i\theta - i\phi}; \frac{ct}{\gamma}q^{k+l}e^{-i\theta - i\phi}, ae^{-i\theta}, ce^{-i\theta}, \gamma e^{-i\phi}, \alpha q^l e^{-i\phi}; q, \frac{ct}{\gamma}q^k e^{i\theta + i\phi}\right) \qquad (15.8)$$

by [11, (2.10.19)]. Substituting this result in (15.4) with the aid of [11, III.24] we finally get (14.1).

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