ON THE $q$-GENOCCHI NUMBERS AND POLYNOMIALS WITH WEIGHT ZERO AND THEIR APPLICATIONS

Serkan Araci$^1$, Mehmet Acikgoz$^2$ and Feng Qi$^3$

$^1$Department of Mathematics, Faculty of Science and Arts
University of Gaziantep, 27310 Gaziantep, Turkey
e-mail: mtsrktn@hotmail.com

$^2$Department of Mathematics, Faculty of Science and Arts
University of Gaziantep, 27310 Gaziantep, Turkey
e-mail: acikgoz@gantep.edu.tr

$^3$Department of Mathematics, School of Science, Tianjin Polytechnic University
Tianjin City, 300387, China;
School of Mathematics and Informatics, Henan Polytechnic University
Jiaozuo City, Henan Province, 454010, China
e-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

Abstract. In the paper, the authors discuss properties of the $q$-Genocchi numbers and polynomials with weight zero. They discover some interesting relations via the $p$-adic $q$-integral on $\mathbb{Z}_p$ and familiar basis Bernstein polynomials and show that the $p$-adic log gamma functions are associated with the $q$-Genocchi numbers and polynomials with weight zero.

1. Preliminaries

Let $p$ be an odd prime number. Denote the ring of the $p$-adic integers by $\mathbb{Z}_p$, the field of rational numbers by $\mathbb{Q}$, the field of the $p$-adic rational numbers by $\mathbb{Q}_p$, and the completion of algebraic closure of $\mathbb{Q}_p$ by $\mathbb{C}_p$, respectively. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}^* = \{0\} \cup \mathbb{N}$ the set of all non-negative integers. Let $| \cdot |_p$ be the $p$-adic norm on $\mathbb{Q}$ with $|p|_p = p^{-1}$.

When one talks of a $q$-extension, $q$ can be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < 1$.  

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We use the notation $[x]_q = \frac{1-x^q}{1-q}$. Hence $\lim_{q \to 1} [x]_q = x$ for any $x \in \mathbb{C}$ in the complex case and any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. This is the hallmark of a $q$-analog: The limit as $q \to 1$ recovers the classical object.

A function $f$ is said to be uniformly differentiable at a point $a \in \mathbb{Z}_p$ if the divided difference $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ converges to $f'(a)$ as $(x, y) \to (a, a)$. The class of all the uniformly differentiable functions is denoted by $UD(\mathbb{Z}_p)$.

For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-analogue of Riemann sum for $f$ is defined by

$$\frac{1}{[p^n]_q} \sum_{0 \leq \xi < p^n} f(\xi)q^\xi = \sum_{0 \leq \xi < p^n} f(\xi)\mu_q(\xi + p^n\mathbb{Z}_p)$$

in [7, 9], where $n \in \mathbb{N}$. The integral of $f$ on $\mathbb{Z}_p$ is defined as the limit of (1.1) as $n$ tends to $\infty$, if it exists, and represented by

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi) \, d\mu_q(\xi).$$

The bosonic integral and the fermionic $p$-adic integral on $\mathbb{Z}_p$ are defined respectively by

$$I_1(f) = \lim_{q \to 1} I_q(f)$$

and

$$I_{-q}(f) = \lim_{q \to -q} I_q(f).$$

For a prime $p$ and a positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^n\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{(a,p)=1} a + dp\mathbb{Z}_p,$$

and

$$a + dp^n\mathbb{Z}_p = \{ x \in X \mid x \equiv a \mod dp^n \},$$

where $a \in \mathbb{Z}$ satisfies $0 \leq a < dp^n$ and $n \in \mathbb{N}$.

In this paper, we will discuss properties of the $q$-Genocchi numbers and polynomials with weight zero. Via the $p$-adic $q$-integral on $\mathbb{Z}_p$ and familiar basis Bernstein polynomials, we discover some interesting relations and show that the $p$-adic log gamma functions are associated with the $q$-Genocchi numbers and polynomials with weight zero.

2. Main results

Now we are in a position to state our main results.
Theorem 2.1. For \( n \in \mathbb{N} \), we have
\[
\frac{\widetilde{G}_{n+1,q}(x)}{n+1} = H_n(-q^{-1}, x).
\]  
(2.1)

Proof. In [2, 3], Araci, Acikgoz, and Seo considered the \( q \)-Genocchi polynomials with weight \( \alpha \) in the form
\[
\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} = \int_{\mathbb{Z}_p} \left[ x + \xi \right]_q^n \, d\mu_{-q}(\xi),
\]  
(2.2)

where \( \widetilde{G}_{n+1,q}^{(\alpha)} = \widetilde{G}_{n+1,q}(0) \) is called the \( q \)-Genocchi numbers with weight \( \alpha \). Taking \( \alpha = 0 \) in (2.2), we easily see that
\[
\frac{\widetilde{G}_{n,q}}{n+1} = \frac{\widetilde{G}_{n+1,q}^{(0)}}{n+1} = \int_{\mathbb{Z}_p} \xi^n \, d\mu_{-q}(\xi),
\]  
(2.3)

where \( \widetilde{G}_{n,q} \) are called the \( q \)-Genocchi numbers and polynomials with weight 0. From (2.3), it is simple to see
\[
\sum_{n=0}^{\infty} \frac{\widetilde{G}_{n,q} t^n}{n!} = t \int_{\mathbb{Z}_p} e^{\xi t} \, d\mu_{-q}(\xi).
\]  
(2.4)

By (1.4), we have
\[
q^n I_{-q}(f_n) + (-1)^n-1 I_{-q}(f) = [2]_q \sum_{0 \leq \ell < n} q^\ell (-1)^n-1-\ell \, f(\ell),
\]  
(2.5)

where \( f_n(x) = f(x+n) \) and \( n \in \mathbb{N} \) (see, [6, 8, 10]). Taking \( n = 1 \) in (2.5) leads to the well-known equality
\[
q I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).
\]  
(2.6)

When setting \( f(x) = e^{xt} \) in (2.6), we find
\[
\sum_{n=0}^{\infty} \frac{\widetilde{G}_{n,q} t^n}{n!} = \frac{[2]_q t}{q e^t + 1}.
\]  
(2.7)

By (2.7), we obtain the \( q \)-Genocchi polynomials with weight 0 as follows
\[
\sum_{n=0}^{\infty} \frac{\widetilde{G}_{n,q}(x) t^n}{n!} = \frac{[2]_q t}{q e^t + 1} \, e^{xt}.
\]  
(2.8)

By (2.8), we see that
\[
\sum_{n \geq 0} \frac{\widetilde{G}_{n,q}(x) t^n}{n!} = t \frac{1 - (-q^{-1})}{e^t - (-q^{-1})} \, e^{xt} = t \sum_{n \geq 0} H_n(-q^{-1}, x) \frac{t^n}{n!},
\]
where \( H_n(-q^{-1}, x) \) are the \( n \)-th Frobenius-Euler polynomials defined by
\[
\sum_{n=0}^{\infty} H_n(\lambda, x) \frac{t^n}{n!} = \frac{1 - \lambda}{e^t - \lambda}, \quad \lambda \in \mathbb{C} \setminus \{1\}.
\]
Equating coefficients of \( t^n \) on both sides of the above equality leads to the identity \((2.1)\). □

**Theorem 2.2.** For \( n \in \mathbb{N} \), the identity
\[
qH_n(-q^{-1}, x + 1) + H_n(-q^{-1}, x) = [2]_q x^n \tag{2.9}
\]
is valid.

**Proof.** By \((2.6)\), we discover that
\[
[2]_q \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = q \int_{\mathbb{Z}_p} e^{(x+\xi+1)t} d\mu_q(\xi) + \int_{\mathbb{Z}_p} e^{(x+\xi)t} d\mu_q(\xi)
\]
\[
= \sum_{n=0}^{\infty} [q \int_{\mathbb{Z}_p} (x+\xi+1)^n d\mu_q(\xi) + \int_{\mathbb{Z}_p} (x+\xi)^n d\mu_q(\xi)] \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} [qH_n(-q^{-1}, x + 1) + H_n(-q^{-1}, x)] \frac{t^n}{n!}.
\]
Equating coefficients of \( t^n \) on both sides of the above equation leads to the identity \((2.9)\). □

**Theorem 2.3.** The identities
\[
G_n(x + 1) + G_n(x) = 2nx^{n-1}, \quad n \geq 1 \tag{2.10}
\]
and
\[
q \tilde{G}_{n,q}(1) + \tilde{G}_{n,q} = \begin{cases} [2]_q, & n = 1 \\ 0, & n \neq 1 \end{cases} \tag{2.11}
\]
are true, where \( G_n(x) \) are called the Genocchi polynomials.

**Proof.** These follow from respectively letting \( q = 1 \) and \( x = 0 \) into the identity \((2.9)\). □

**Theorem 2.4.** The following identity holds
\[
\tilde{G}_{n,q^{-1}}(1 - x) = (-1)^{n+1} \tilde{G}_{n,q}(x). \tag{2.12}
\]
Proof. When we substitute $x$ by $1 - x$ and $q$ by $q^{-1}$ in (2.8), it follows that
\[
\sum_{n=0}^{\infty} \tilde{G}_{n,q-1}(1 - x) \frac{t^n}{n!} = t \frac{1 + q^{-1}}{q^{-1} e^t + 1} e^{(1-x)t} = \frac{1 + q}{e^t + q} e^{t x} = -\frac{[2]_q(-t)}{qe^{-t} + 1} e^{(-t)x} = \sum_{n=0}^{\infty} (-1)^{n+1} \tilde{G}_{n,q}(x) \frac{t^n}{n!}.
\]
From this, we procure the equality (2.12), the symmetric property of this type polynomials. □

**Theorem 2.5.** The identity
\[
\tilde{G}_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} \tilde{G}_{k,q} x^{n-k}
\]
(2.13) is true.

**Proof.** By using (2.2) for $\alpha = 0$ and the binomial theorem, we readily obtain that
\[
\tilde{G}_{n+1,q}(x) = \frac{1}{n+1} \int_{Z_p} (x + \xi)^n d\mu_q(\xi) = \sum_{k=0}^{n} \binom{n}{k} \left[ \int_{Z_p} \xi^k d\mu_q(\xi) \right] x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \tilde{G}_{k+1,q} x^{n-k}.
\]
Further using
\[
\frac{n+1}{k+1} \binom{n}{k} = \binom{n+1}{k+1},
\]
we obtain
\[
\tilde{G}_{n+1,q}(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} \tilde{G}_{k+1,q} x^{n-k} = \sum_{k=1}^{n+1} \binom{n+1}{k} \tilde{G}_{k,q} x^{n+1-k}.
\]
Thus, the equality (2.13) follows. □

**Proposition 2.1.** The identities
\[
\tilde{G}_{0,q} = 0 \quad \text{and} \quad q(\tilde{G}_{q} + 1)^n + \tilde{G}_{n,q} = \begin{cases} [2]_q, & n = 1 \\ 0, & n \neq 1 \end{cases}
\]
(2.14) are true, where the usual convention of replacing $(\tilde{G}_q)^n$ by $\tilde{G}_{n,q}$ is used.

**Proof.** These can be deduced from combining (2.11) with (2.13). □
Proposition 2.2. For \( n > 1 \),
\[
\hat{G}_{n+1,q}(2) = \frac{(n + 1)}{q} [2]_q + \frac{1}{q^2} \hat{G}_{n+1,q}.
\] (2.15)

Proof. From (2.13), it follows that
\[
q^2 \hat{G}_{n+1,q}(2) = q^2 (\hat{G}_q + 1 + 1)^{n+1} = q^2 \sum_{k=0}^{n+1} \binom{n + 1}{k} (\hat{G}_q + 1)^k
\]
\[
= (n + 1)q^2 (\hat{G}_q + 1) + q \sum_{k=2}^{n+1} \binom{n + 1}{k} q(\hat{G}_q + 1)^k
\]
\[
= (n + 1)q([2]_q - \hat{G}_1) - q \sum_{k=2}^{n+1} \binom{n + 1}{k} \hat{G}_k
\]
\[
= (n + 1)q[2]_q - \left[ q \sum_{k=0}^{n+1} \binom{n + 1}{k} \hat{G}_k + (n + 1)q\hat{G}_1, \right]
\]
\[
= (n + 1)q[2]_q - q \sum_{k=0}^{n+1} \binom{n + 1}{k} \hat{G}_k
\]
\[
= (n + 1)q[2]_q - q(\hat{G}_q + 1)^{n+1} = (n + 1)q[2]_q + \hat{G}_{n+1,q}
\]
for \( n > 1 \). Therefore, we deduce (2.15). \( \square \)

Theorem 2.6. The identity
\[
\int_{\mathbb{Z}_p} (1 - \xi)^n d\mu_{-q}(\xi) = [2]_q + q^3 \hat{G}_{n+1,q}^{-1} \frac{n+1}{n+1}
\] (2.16)
is valid.

Proof. By virtue of (1.4), (2.12), and (2.15), we find
\[
(n + 1) \int_{\mathbb{Z}_p} (1 - \xi)^n d\mu_{-q}(\xi) = (n + 1)(-1)^n \int_{\mathbb{Z}_p} (\xi - 1)^n d\mu_{-q}(\xi)
\]
\[
= (-1)^n \hat{G}_{n+1,q}(-1) = \hat{G}_{n+1,q}^{-1}(2) = (n + 1)[2]_q + q^3 \hat{G}_{n+1,q}^{-1}.
\]
As a result, we conclude Theorem 2.6. \( \square \)
Theorem 2.7. The following identity holds:

\[ \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} (-1)^\ell \tilde{G}_{\ell+k+1, q} \]

\[ = \begin{cases} 
[2]_q + q^2 \tilde{G}_{n+1, q^{-1}}, & k = 0, \\
\sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} \left( [2]_q + q^2 \tilde{G}_{n-s+1, q^{-1}} \right), & k \neq 0.
\end{cases} \]

Proof. Let \( UD(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic analogue of Bernstein operator for \( f \) is defined by

\[ B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \]

where \( n, k \in \mathbb{N}^* \) and the \( p \)-adic Bernstein polynomials of degree \( n \) is defined by

\[ B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in \mathbb{Z}_p, \quad (2.17) \]

see [4, 11, 12, 13]. Via the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) and Bernstein polynomials in (2.17), we can obtain that

\[ I_1 = \int_{\mathbb{Z}_p} B_{k,n}(\xi) d\mu_q(\xi) \]

\[ = \binom{n}{k} \int_{\mathbb{Z}_p} \xi^k (1-\xi)^{n-k} d\mu_q(\xi) \]

\[ = \binom{n}{k} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} (-1)^\ell \int_{\mathbb{Z}_p} \xi^{\ell+k} d\mu_q(\xi) \]

\[ = \binom{n}{k} \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} (-1)^\ell \tilde{G}_{\ell+k+1, q}. \]

On the other hand, by symmetric properties of Bernstein polynomials, we have
\[ I_2 = \int_{\mathbb{Z}_p} B_{n-k,n}(1-\xi)\,d\mu_{-q}(\xi) \]
\[ = \left( \frac{n}{k} \right) \sum_{s=0}^{k} \left( \frac{k}{s} \right) (-1)^{k-s} \int_{\mathbb{Z}_p} (1-\xi)^{n-s}\,d\mu_{-q}(x) \]
\[ = \left( \frac{n}{k} \right) \sum_{s=0}^{k} \left( \frac{k}{s} \right) (-1)^{k-s} \left( \frac{[2]_q + q^2 \tilde{G}_{n-s+1,q^{-1}}}{n-s+1} \right) \]
\[ = \begin{cases} 
[2]_q + q^2 \tilde{G}_{n+1,q^{-1}}, & k = 0, \\
\left( \frac{n}{k} \right) \sum_{s=0}^{k} \left( \frac{k}{s} \right) (-1)^{k-s} \left( \frac{[2]_q + q^2 \tilde{G}_{n-s+1,q^{-1}}}{n-s+1} \right), & k \neq 0.
\end{cases} \]

Equating \( I_1 \) and \( I_2 \) yields Theorem 2.7. \( \square \)

**Theorem 2.8.** The identity
\[ \sum_{\ell=0}^{n_1+\cdots+n_m-mk} \left( \frac{n_1 + \cdots + n_m - mk}{\ell} \right) (-1)^{\ell} \frac{\tilde{G}_{\ell+mk+1,q}}{\ell+mk+1} \]
\[ = \begin{cases} 
[2]_q + q^2 \tilde{G}_{n_1+\cdots+n_m+1,q^{-1}}, & k = 0, \\
\sum_{\ell=0}^{mk} \left( \frac{mk}{\ell} \right) (-1)^{mk+\ell} \left( \frac{[2]_q + q^2 \tilde{G}_{n_1+\cdots+n_m+\ell+1,q^{-1}}}{n_1+\cdots+n_m+\ell+1} \right), & k \neq 0.
\end{cases} \quad (2.18) \]
is true.

**Proof.** The \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) of the product of several Bernstein polynomials can be calculated as
\[ I_3 = \int_{\mathbb{Z}_p} \prod_{s=1}^{m} B_{k,n_s}(\xi)\,d\mu_{-q}(\xi) \]
\[ = \prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \int_{\mathbb{Z}_p} \xi^{mk} (1-\xi)^{n_1+\cdots+n_m-mk}\,d\mu_{-q}(\xi) \]
\[ = \prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \sum_{\ell=0}^{n_1+\cdots+n_m-mk} \left( \frac{n_1 + \cdots + n_m - mk}{\ell} \right) (-1)^{\ell} \left[ \int_{\mathbb{Z}_p} \xi^{\ell+mk}\,d\mu_{-q}(\xi) \right] \]
\[ = \prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \sum_{\ell=0}^{n_1+\cdots+n_m-mk} \left( \frac{n_1 + \cdots + n_m - mk}{\ell} \right) (-1)^{\ell} \frac{\tilde{G}_{\ell+mk+1,q}}{\ell+mk+1} \]
On the other hand, by symmetric properties of Bernstein polynomials and the equality (2.16), we have

\[
I_4 = \int_{\mathbb{Z}_p} \prod_{s=1}^{m} B_{n_s-k, n_s} (1 - \xi) \, d\mu_q(\xi)
\]

\[
= \prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \sum_{\ell=0}^{mk} \left( -1 \right)^{mk-\ell} \int_{\mathbb{Z}_p} (1 - \xi)^{n_1+\cdots+n_m-\ell} \, d\mu_q(\xi)
\]

\[
= \prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \sum_{\ell=0}^{mk} \left( -1 \right)^{mk-\ell} \left( [2]_q + q^2 \frac{G_{n_1+\cdots+n_m-\ell+1,q^{-1}}}{n_1+\cdots+n_m-\ell+1} \right)
\]

\[
= \begin{cases} 
[2]_q + q^2 \frac{G_{n_1+\cdots+n_m+1,q^{-1}}}{n_1+\cdots+n_m+1}, & k = 0, \\
\prod_{s=1}^{m} \left( \frac{n_s}{k} \right) \sum_{\ell=0}^{mk} \left( -1 \right)^{mk-\ell} \left( [2]_q + q^2 \frac{G_{n_1+\cdots+n_m-\ell+1,q^{-1}}}{n_1+\cdots+n_m-\ell+1} \right), & k \neq 0.
\end{cases}
\]

Equating \( I_3 \) and \( I_4 \) results in an interesting identity (2.18) for the \( q \)-analogue of Genocchi polynomials with weight 0. \( \square \)

3. AN IDENTITY ON \( p \)-ADIC LOCALLY ANALYTIC FUNCTIONS

In this section, we consider Kim’s \( p \)-adic \( q \)-log gamma functions related to the \( q \)-analogue of Genocchi polynomials.

Definition 3.1. ([5, 7]) For \( x \in \mathbb{C}_p \setminus \mathbb{Z}_p \),

\[
(1 + x) \log(1 + x) = x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1}.
\]

Kim’s \( p \)-adic locally analytic function on \( x \in \mathbb{C}_p \setminus \mathbb{Z}_p \) can be defined as follows.

Definition 3.2. ([5, 7]) For \( x \in \mathbb{C}_p \setminus \mathbb{Z}_p \),

\[
G_{p,q}(x) = \int_{\mathbb{Z}_p} [x + \xi]_q \log[x + \xi]_q - 1) \, d\mu_q(\xi).
\]

By considering Kim’s \( p \)-adic \( q \)-log gamma function, we introduce the following \( p \)-adic locally analytic function

\[
G_{p,1}(x) \triangleq G_p(x) = \int_{\mathbb{Z}_p} (x + \xi) \log(x + \xi) - 1) \, d\mu_q(\xi).
\]
Theorem 3.1. For \( x \in \mathbb{C}_p \setminus \mathbb{Z}_p \),
\[
G_p(x) = \left( x + \frac{\tilde{G}_{2,q}}{2} \right) \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)(n+2)} \frac{\tilde{G}_{n+2,q}}{x^n} - x. \tag{3.2}
\]

Proof. Replacing \( x \) by \( \frac{x}{\xi} \) in (3.1) leads to
\[
(x + \xi)[\log(x + \xi) - 1] = (x + \xi) \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\xi^{n+1}}{n(n+1)} \frac{\xi}{x^n} - x. \tag{3.3}
\]
From (3.1) and (3.3), we can establish an interesting formula (3.2). \(\square\)

Remark 3.1. This is a revised version of the preprint [1].

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