Differential calculus on the Faber polynomials

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Abstract

The Faber polynomials are presented as a coordinate system to study the geometry of the manifold of coefficients of univalent functions.

Résumé

Les polynômes de Faber sont présentés comme un système de coordonnées pour étudier la géométrie de la variété des coefficients des fonctions univalentes.

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1. Introduction

We show how the methods introduced in [2] and [3] allow to do differential calculus on the manifold of coefficients of univalent functions. The Faber polynomials \((F_k)_{k \geq 1}\) are given by the identity [5,12]

\[
1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots = \exp \left( - \sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k \right).
\]  

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The polynomials \((G_m)_{m \geq 1}\) and \((K_p^n)_{n \geq 1}\), \(p \in \mathbb{Z}\) are given by

\[
1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots = 1 + \sum_{m=1}^{+\infty} G_m(b_1, b_2, \ldots, b_m) w^m, \tag{1.2}
\]

\[
(1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots)^p = 1 + \sum_{n \geq 1} K_p^n(b_1, b_2, \ldots, b_n) w^n, \tag{1.3}
\]

then \(G_m = K_m^{-1}\) and \(K_m^1 = b_m\). Important polynomials are also the \((P_k^n)_{n \geq 2}\), see \([1, (A.1.7)]\). If \(f(z) = zh(z)\),

\[
\left( \frac{zf'(z)}{f(z)} \right)^2 [h(z)]^k = \sum_{n \geq 2} P_n^{n+k} z^n \quad \text{for } k \in \mathbb{Z}. \tag{1.4}
\]

The polynomials \((F_n)_{n \geq 0}\), \((G_n)_{n \geq 0}\), \((K_p^n)_{n \geq 0}\), \((P_k^n)_{n \geq 2}\) are homogeneous of degree \(n\) in the variables \((b_1, b_2, \ldots)\) where \(b_k\) has weight \(k\). As in \([1–3,7]\) let the function of the infinite number of variables \((b_1, b_2, \ldots, b_k, \ldots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots\).

On the infinite dimensional manifold of coefficients \(\mathcal{M} = \{(b_1, b_2, \ldots, b_k, \ldots)\}\) of univalent functions, consider the operators \([3]\),

\[
W_j = -\frac{\partial}{\partial b_j} - b_1 \frac{\partial}{\partial b_{j+1}} - \cdots - b_k \frac{\partial}{\partial b_{j+k}} - \cdots \quad \text{for } j \geq 1. \tag{1.5}
\]

For \(j \geq 1\), it holds \(\frac{\partial}{\partial b_j} [h(z)] = z^j\) and \(W_j[h(z)] = -z^j h(z)\). We have (see \([3]\))

\[
W_j(F_m) = m \delta_{jm} \quad \text{and} \quad W_j(G_m) = G_{m-j} \times 1_{m \geq j}, \tag{1.6}
\]

\[
\frac{\partial}{\partial b_p} W_j - W_j \frac{\partial}{\partial b_p} = -\frac{\partial}{\partial b_{j+p}} \quad \text{and} \quad W_p W_q = W_q W_p, \tag{1.7}
\]

\[
W_p W_q + W_{p+q} = \sum_{k \geq 0} \sum_{m \geq 0} b_k b_m \frac{\partial^2}{\partial b_{q+m} \partial b_{k+p}}. \tag{1.8}
\]

For \(k \in \mathbb{Z}\), let

\[
V_k^j = -\sum_{n \geq 0} K_n^{k+1} \frac{\partial}{\partial b_{n+j}}. \tag{1.9}
\]

We consider for \(j \geq 1\) and \(a \in \mathbb{Z}\), the operators \((V^{aj}_j)_{j \geq 1}\) and for \(a = 1\), we put

\[
V_j = -\sum_{n \geq 0} K_n^{j+1} \frac{\partial}{\partial b_{n+j}}. \tag{1.10}
\]

Then \(W_j = V_j^0, V_j^{-1} = -\frac{\partial}{\partial b_j}, V_j = V_j^j, j \geq 1\) and

\[
V_j^k V_p^s - V_p^s V_j^k = (k - s)V_{j+p}^{k+s} \quad \text{for } p \geq 1, j \geq 1. \tag{1.11}
\]

The differential operators \((V^{aj}_j)_{j \geq 1}\), \(k \in \mathbb{Z}\) form an algebra and for \(a \in \mathbb{Z}\), the set of \((V^{aj}_j)_{j \geq 1}\) is a subalgebra since

\[
V_j^{aj} V_p^a - V_p^a V_j^{aj} = a(j - p)V_j^{a(j+p)}. \tag{1.12}
\]
Let \( f(z) = zh(z) \). For \( j \geq 1 \), the vector field \( V_j \) is the image through the map \( f \to f^{-1} \) of the Kirillov operator

\[
L_j = \frac{\partial}{\partial b_j} + \sum_{n \geq 1} (n + 1)b_n \frac{\partial}{\partial b_{n+j}}.
\] (1.13)

Let \( x_1, x_2, \ldots, x_n \), be the roots of \( \xi^n + b_1 \xi^{n-1} + b_2 \xi^{n-2} + \cdots + b_{n-1} \xi + b_n = 0 \) and consider Newton symmetric functions \( \pi_k = x_1^k + x_2^k + \cdots + x_n^k, k \geq 1 \), it was proved in [3] that

\[
\pi_k(b_1, b_2, \ldots, b_n) = F_k(b_1, b_2, \ldots, b_n) \quad \text{for} \quad k \leq n,
\] (1.14)

where \((F_k)_{k \geq 1}\) are the Faber polynomials. This is a consequence of

\[
\log(1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots) = -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k
\] (1.15)
or equivalently (1.1). With this identification, the exact coefficients of the polynomial \( F_k(b_1, b_2, \ldots, b_k), k \geq 1 \), have been calculated in [3]. The polynomials \((F_k)_{k \geq 1}\) are completely determined as homogeneous polynomial solutions of the system of partial differential equations (1.6) involving \((W_j)_{j \geq 1}\) (See [3]). The exact coefficients of the polynomials \((G_n)\) and of all the \((K_n^p)\) have been given in [3].

The object of this note is to prove that the polynomials \((K_n^p)\) are all obtained as partial derivatives of the Faber polynomials and show how some of the recursion formulae on the polynomials are related to elementary differential calculus on \( M \). This is a step towards the classification of Faber type polynomials (see [8]). In the last section, we give the example of the conformal map from the exterior of the unit disk onto the exterior of \([−2, +2]\). This shows how to introduce non trivial second order differential operators on the manifold \( M \).

**Main Theorem.** We have for \( n \geq 1, k \geq 1, \)

\[
\frac{\partial F_n}{\partial b_k} = -nG_{n-k} \times 1_{n \geq k},
\]

\[
\frac{\partial}{\partial b_k} G_n = \frac{\partial}{\partial b_k} K_{n-1}^{-1} = -K_{n-k}^{-2} \times 1_{k \leq n}, \quad \frac{\partial}{\partial b_k} K_n^p = pK_{n-k}^{p-1} \times 1_{k \leq n},
\]

\[
\frac{\partial^2 F_j}{\partial b_k \partial b_p} = jK_{j-(p+k)}^{-2} \times 1_{j \geq k+p}, \quad \frac{\partial^3 F_j}{\partial b_r \partial b_k \partial b_p} = -2jK_{j-(p+k+r)}^{-3} \times 1_{j \geq k+p+r}
\]

For \( j \geq k_1 + k_2 + \cdots + k_s, k_1 \geq 1, \ldots, k_s \geq 1 \) and \( s \geq 1, \)

\[
\frac{\partial^s F_j}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_s}} = (-1)^s (s - 1)! jK_{j-(k_1+k_2+\cdots+k_s)}^{-s}.
\] (T1)

Moreover for \( n \geq 1, k \geq 1, \)

\[
K_n^k(b_1, b_2, \ldots, b_n) = K_n^{-k}(G_1(b_1), G_2(b_1, b_2), \ldots, G_j(b_1, b_2, \ldots, b_j), \ldots, G_n(b_1, b_2, \ldots, b_n)).
\] (T2)

In the notation, all functions are functions of \((b_1, b_2, \ldots, b_n, \ldots)\).

The first \((F_n)\) for \( 1 \leq n \leq 11, \) as well as the first \((G_n)\) are shown in [3]. The \((K_n^p)\), \( p \in \mathbb{Z}, p \neq 0 \) and \( n \leq 5 \) are in [2]. In [3], an exact expression of the coefficients of all the polynomials \((K_n^p)\) has been given. We explicit the first \( K_n^p, \)
An expression of $K_p^\phi$ can be obtained as follows, let

$$\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n.$$
The line integral $D^k_n = \int_{\phi_n} \frac{\phi'_{n}(\xi)}{\xi^{n+1}} d\xi$ is equal to the coefficient of $z^n$ in $\phi_n(z)^k$ and is of course independent of $p$. For any $p \in \mathbb{Z}$, we have,

$$K^p_n = pb_n + \frac{p(p-1)}{2} D^2_n + \frac{p!}{(p-3)!} D^3_n + \frac{p!}{(p-4)!} D^4_n + \cdots + \frac{p!}{(p-n)!} D^n_n.$$  \hspace{1cm} (T3)

$D^k_n$ is the sum of terms having $k$ factors in $K^k_n$ and

$$C^p_n = \frac{p!}{n!(p-n)!} = \frac{p(p-1) \cdots (p-n+1)}{n!}$$

is the binomial coefficient. If $b_1 \neq 0$,

$$D^k_n(b_1, b_2, \ldots, b_n) = b_1^k K^k_{n-k} \left( b_1, b_2, \ldots, \frac{b_{n-k+1}}{b_1} \right).$$ \hspace{1cm} (T4)

Replacing in (T3) and iterating the procedure permits to obtain the exact expression on $K^p_n$, see [3] and Section 4.2 below.

We have relations between the partial derivatives of the Faber polynomials as

$$\frac{\partial G_n}{\partial b_k} = \frac{\partial G_{n+p}}{\partial b_{k+p}} = -K_{n-k}^{2} \times 1_{n \geq k} \quad \text{for all } n \geq 1, \ k \geq 1, \ \text{and } p \geq 0,$$  \hspace{1cm} (1.17)

$$\frac{\partial^2 F_n}{\partial b_1^2} = -n \frac{\partial G_n}{\partial b_2}, \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}}.$$  \hspace{1cm} (1.18)

Let

$$X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots = -\sum_{j \geq 1} G_j W_j,$$  \hspace{1cm} (1.19)

then

$$X_0 F_n = -n G_n \quad \text{and} \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r \geq 1, s \geq 1,$$  \hspace{1cm} (1.20)

$$\frac{\partial}{\partial b_j} X_0 = X_0 \frac{\partial}{\partial b_j} = -\frac{\partial}{\partial b_j}.$$  \hspace{1cm} (1.21)

This leads to the construction of differential operators on $\mathcal{M}$ which transform one polynomial into the other. See Section 7.

On the other hand, from (T1), we obtain, see (3.19) and Section 4,

**Main Corollary.** The coefficients of the Schwarzian derivative of $f(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_n z^{n+1} + \cdots$ are given in terms of Faber polynomials and their second derivatives as

$$z^2 S(f) = z^2 \left( \frac{f''}{f'} - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right) = \sum_{k \geq 2} \mathcal{P}_k z^k$$

where

$$\mathcal{P}_k = -(k-2) F_k(2b_1, 3b_2, \ldots, (j+1)b_j, \ldots)$$

$$- \frac{1}{2} K^2_k( F_1(2b_1), F_2(2b_1, 3b_2), \ldots, F_k(2b_1, 3b_2, \ldots, (k+1)b_k))$$  \hspace{1cm} (C1)

and $K^2_k( F_1(2b_1), F_2(2b_1, 3b_2), \ldots) = -\frac{1}{k+2} \frac{\partial^2 F_k}{\partial b_1^{k+2}}(c_1, c_2, \ldots, c_k)$ is the second derivative $F_{k+2}$ calculated at the point
The first polynomials \( (c_1, c_2, \ldots, c_k) = (G_1(F_1(2b_1)), G_2(F_1(2b_1), F_2(2b_1, 3b_2)), \ldots, G_k(F_1(2b_1), \ldots, F_k(2b_1, 3b_2, \ldots, (k+1)b_k)) \).

The tool is the composition of maps on the manifold \( \mathcal{M} \). We have

\[
\frac{h'(w)}{h(w)} = -\sum_{k=1}^{+\infty} F_k(b_1, b_2, \ldots, b_k) w^{k-1} = -(F_1 + F_2 w + F_3 w^2 + \cdots). \tag{1.22}
\]

The function \( f(w) = wh(w) = w + b_1 w^2 + b_2 w^3 + \cdots + b_n w^{n+1} + \cdots \) satisfies

\[
w \frac{f'(w)}{f(w)} = 1 + w \frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k) w^k \tag{1.23}
\]

and \( g(w) = \frac{w}{h(w)} = w + \sum_{n \geq 1} G_n(b_1, b_2, \ldots, b_n) w^{n+1} + \cdots \) satisfies

\[
w \frac{g'(w)}{g(w)} = 1 - w \frac{h'(w)}{h(w)} = 1 + \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k) w^k. \tag{1.24}
\]

From (1.23) and (1.24), we deduce

\[
F_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) + F_n(b_1, b_2, \ldots, b_n) = 0. \tag{1.25}
\]

We consider the following maps from \( \mathcal{M} \to \mathcal{M} \),

\[
F : (b_1, b_2, \ldots, b_n, \ldots) \to (F_1(b_1), F_2(b_1, b_2), \ldots, F_n(b_1, b_2, \ldots, b_n), \ldots)
\]

\[
F^{-1} : (b_1, b_2, \ldots, b_n, \ldots) \to (c_1, c_2, \ldots, c_n, \ldots) \text{ such that}
\]

\[
F_1(c_1) = b_1, \quad F_2(c_1, c_2) = (b_1, b_2), \quad \ldots,
\]

\[
F_n(c_1, c_2, \ldots, c_n) = (b_1, b_2, \ldots, b_n), \quad \ldots.
\]

\[
G : (b_1, b_2, \ldots, b_n, \ldots) \to (G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n), \ldots),
\]

\[
S : (b_1, b_2, \ldots, b_n, \ldots) \to (-b_1, -b_2, \ldots, -b_n, \ldots).
\]

The relation (1.25) means that \( G \) is obtained as the composition of maps

\[
G = F^{-1} \circ S \circ F. \tag{1.26}
\]

The first polynomials \( (F_n^{-1})_{n \geq 1} \) defined by the map \( F^{-1} \) are given by

\[
F_1^{-1}(b_1) = -b_1 \quad \text{and} \quad F_2^{-1}(b_1, b_2) = \frac{1}{2}(b_1^2 - b_2),
\]

\[
F_3^{-1}(b_1, b_2, b_3) = \frac{1}{6}(-b_3^3 + 3b_1 b_2 - 2b_3),
\]

\[
F_4^{-1}(b_1, b_2, b_3, b_4) = \frac{1}{4!}(b_4^4 - 6b_1^2 b_2 + 3b_2^2 + 8b_1 b_3 - 6b_4),
\]

\[
F_5^{-1} = \frac{1}{5!}(-b_1^5 - 15b_1 b_2^2 + 10b_1^3 b_2 - 20b_1^2 b_3 + 20b_2 b_3 + 30b_1 b_4 - 24b_5),
\]

\[
F_6^{-1} = \frac{1}{6!}(b_1^6 + 144b_1 b_5 - 15b_2^3 + 45b_2 b_3^2 - 15b_2 b_4^2 + 120b_1 b_2 b_3 + 90b_2 b_4
\]

\[
+ 40b_1^3 b_2 - 120b_6 + 40b_2^3 - 90b_1^2 b_4),
\]
\[ F_7^{-1} = \frac{1}{7!} (-b_1^7 - 504b_1^2b_5 + 504b_2b_5 + 840b_1b_6 + 21b_2b_6 + 420b_2^2b_3 - 70b_3b_3 \\
- 280b_1b_3^2 - 210b_2^2b_3 - 105b_2b_1^3 + 105b_1b_2^3 + 420b_3b_4 - 720b_7 \\
+ 210b_1^2b_4 - 630b_1b_2b_4), \]
\[ F_8^{-1} = \frac{1}{8!} (b_1^8 - 4032b_1b_2b_5 + 1344b_3b_5 + 2688b_3b_5 - 3360b_1^2b_6 + 3360b_2b_6 \\
+ 5760b_1b_7 + 105b_2^4 - 420b_3b_1^2 + 1680b_1b_2^2b_3 - 1120b_1^2b_3 - 1120b_1b_3b_3 - 420b_1b_2^2 \\
- 1120b_2b_3^2 + 28b_1b_5b_3 - 28b_1^2b_5 \\
- 5040b_8 + 2520b_2^2b_4 - 1260b_2^2b_4 - 3360b_1b_3b_4). \]

We put \( F_0^{-1} = 1 \). We have \( \exp(- \sum_{j \geq 1} b_jz^j) = 1 + \sum_{k \geq 1} F_k^{-1}(b_1, b_2, \ldots, b_k)z^k \) and \( \frac{\partial}{\partial b_k} F_j^{-1} = -F_{j-1}^{-1}, \forall j \geq 2, \)
\[
\frac{\partial}{\partial b_k} F_p^{-1} = 0 \quad \text{if } k \geq p + 1, \tag{1.27}
\]
\[
\frac{\partial}{\partial b_k} F_1^{-1} = -\frac{1}{k} \quad \text{and} \quad \frac{\partial}{\partial b_k} F_p^{-1} = -\frac{1}{k} F_{p-k}^{-1} \quad \text{if } k \leq p. \tag{1.28}
\]

Differentiating (1.25), we obtain systems of partial differential equations satisfied by the \((F_n)_{n \geq 1}\) and the \((F_n^{-1})\). If \( p \geq 1 \) is an integer, we denote \( p \times S \) the map
\[
p \times S: (b_1, b_2, \ldots, b_n, \ldots) \rightarrow (-pb_1, -pb_2, \ldots, -pb_n, \ldots) \tag{1.29}
\]
and \( p \times I \) the map
\[
p \times I: (b_1, b_2, \ldots, b_n, \ldots) \rightarrow (pb_1, pb_2, \ldots, pb_n, \ldots). \tag{1.30}
\]

Consider the maps, for \( p \in Z, p \neq 0, \)
\[
K^p: (b_1, b_2, \ldots, b_n, \ldots) \rightarrow (K^p_1(b_1), K^p_2(b_1, b_2), \ldots, K^p_j(b_1, b_2, \ldots, b_j), \ldots). \]

We obtain for \( p \geq 1, \)
\[
K^{-p} = F^{-1} \circ (p \times S) \circ F, \tag{1.31}
\]
\[
K^p = F^{-1} \circ (p \times I) \circ F. \tag{1.32}
\]

This last relation shows that \( K^p \circ K^q = K^q \circ K^p = K^{pq} \) for \( p \neq 0, q \neq 0, p, q \in Z \). In particular, it is enough to know the \((K^p)\) when \( p \) are prime numbers, to obtain all the other \( K^p, p \in Z \), by composition of maps. From (1.31)–(1.32), we see that
\[
F_n(K^{-p}_1(b_1), K^{-p}_2(b_1, b_2), \ldots, K^{-p}_n(b_1, b_2, \ldots, b_n)) \\
+ F_n(K^p_1(b_1), K^p_2(b_1, b_2), \ldots, K^p_n(b_1, b_2, \ldots, b_n)) = 0
\]
which extends (1.25).

The coefficients of \( f^{-1}(z) \) the inverse map of \( f(z) = z + b_1z^2 + b_2z^3 + \cdots \) are given by
\[
f^{-1}(z) = z + \sum_{n \geq 1} \frac{1}{n+1} K_n^{-(n+1)} z^{n+1} \tag{1.33}
\]
(compare with [13]) and the coefficients of \( g^{-1}(z) \) the inverse map of \( g(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \ldots + \frac{b_{n+1}}{z^n} + \ldots \) are given by
\[
g^{-1}(z) = z - b_1 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{z^n}.
\] (1.13)

Thus the two maps \( \phi_f \) and \( \phi_g \) defined by
\[
\phi_f : (b_1, b_2, b_3, b_4, \ldots) \to \left( \frac{1}{2} K_1^{-2}, \frac{1}{3} K_2^{-3}, \frac{1}{4} K_3^{-4}, \frac{1}{5} K_4^{-5}, \ldots, \frac{1}{n+1} K_{n+1}^{-(n+1)} \right),
\]
\[
\phi_g : (b_1, b_2, b_3, \ldots) \to \left( -b_1, -K_2^1, -\frac{1}{2} K_3^2, -\frac{1}{3} K_4^3, -\frac{1}{4} K_5^4, \ldots, -\frac{1}{n} K_{n+1}^n \right)
\]
satisfy \( \phi_f \circ \phi_f = \text{Id}_M \) and \( \phi_g \circ \phi_g = \text{Id}_M \).

2. Identities between the polynomials

First, we recall the basic facts relative to the polynomials \((F_j)_{j \geq 0}, (G_j)_{j \geq 0}, (K_n^p), (P_n^p), n \geq 1, p \in \mathbb{Z}\) and the differential operators \((W_j)_{j \geq 1}\).

2.1. Zeroes and particular values of the polynomials

We have
\[
G_1(1) = -1, \quad G_n(1, 1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 2,
\]
\[
G_1(2) = -2, \quad G_2(2, 3) = 1,
\]
\[
G_1(3) = -3, \quad G_n(2, 3, 4, \ldots, k, \ldots, n+1) = 0 \quad \text{for } n \geq 2,
\]
\[
G_2(3, 6) = 3, \quad G_3(3, 6, 10) = -1,
\]
\[
G_4(3, 6, 10, 15) = 0,
\]
\[
G_n \left( 3, 6, 10, 15, 21, \ldots, \frac{(n+1)(n+2)}{2} \right) = 0 \quad \text{for } n \geq 5,
\]
\[
G_n \left( 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!} \right) = (-1)^n \frac{1}{n!} \quad \text{for } n \geq 1,
\] (2.1)
\[
K_1^{-2}(1) = -2, \quad K_2^{-2}(1, 1) = 1,
\]
\[
K_2^{-3}(1, 1) = 3, \quad K_3^{-3}(1, 1, 1) = -1,
\]
\[
K_n^{-2}(1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 3,
\]
\[
K_n^{-3}(1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 4,
\]
\[
K_n^{-2}(-4, -2, -4, -2, -4, -2, \ldots) = \begin{cases} 8(n-2) & \text{if } n \text{ is odd,} \\ 2(5n-4) & \text{if } n \text{ is even.} \end{cases}
\] (2.2)

For \( n \geq 1 \),
\[
F_n(1, 1, 1, \ldots, 1) = -1,
\]
\[
F_n(-1, 1, 1, \ldots, (-1)^n) = (-1)^{n+1},
\]
\[
F_n(4, 9, \ldots, (n+1)^2) = -3 + (-1)^n.
\] (i) (ii) (iii)
For any $p \in \mathbb{Z}$, $p \neq 0$, $n \geq 1$

\[
F_n^{-1}(p, p, \ldots, p) = (-1)^p C_n^p \quad \text{with} \quad C_n^p = \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!},
\]

(vi)

\[
F_n^{-1}(b_1 + 1, b_2 + 1, \ldots, b_n + 1) = F_n^{-1}(b_1, b_2, \ldots, b_n) - F_{n-1}^{-1}(b_1, b_2, \ldots, b_{n-1}).
\]

(vii)

\[
F_n^{-1}(-p, -p, \ldots, -p) = C_n^{n+p-1},
\]

(viii)

\[
K_n^p(2, 3, 4, 5, \ldots, n + 1) = C_n^{2p+n-1} = (-1)^n C_n^{-2p},
\]

(ix)

\[
K_n^p(-1, 1, -1, \ldots, (-1)^n) = C_n^{-p}.
\]

(x)

**Proof.** We take the particular case of the function

\[
h(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1 - z} \quad \text{for} \quad |z| < 1.
\]

Since $\frac{1}{h(z)} = 1 - z$, we find $G_n(1, 1, 1, \ldots, 1)$ for all $n \geq 1$. In the same way, for $|z| < 1$, consider

\[
h'(z) = 1 + 2z + 3z^2 + \cdots + (n + 1)z^n + \cdots.
\]

Since

\[
\frac{1}{h'(z)} = (1 - z)^2 = 1 - 2z + z^2,
\]

we find $G_n(2, 3, 4, \ldots, n + 1)$. We continue with

\[
\frac{1}{(1 - z)^3} = \frac{h''(z)}{2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + \cdots + \frac{(n + 1)(n + 2)}{2}z^n + \cdots.
\]

For $K_n^{-2}$, since $\frac{1}{h(z)^2} = (1 - z)^2$, we obtain $K_n^{-2}(1, 1, 1, \ldots, 1)$. For $K_n^{-3}(1, 1, 1, \ldots)$, we use

\[
\frac{1}{h(z)^3} = 1 - 3z + 3z^2 - z^3.
\]

In this way, we find particular values of $(b_1, b_2, \ldots, b_n, \ldots)$ such that the functions $G_n$ and $K_n^{-p}$, $p \geq 1$ are zero. We obtain the zeros of $(K_n^p)_{n \geq 1}$, $p \geq 1$, using the identity (T2) in the Main theorem. To find zeros of $(F_n)_{n \geq 1}$, we take $h(z) = \exp(z)$. Moreover since for a homogeneous polynomial $P_n$ of degree $n$,

\[
P_n(rb_1, r^2b_2, \ldots, r^nb_n) = r^n P_n(b_1, b_2, \ldots, b_n) \quad \forall r \in \mathbb{C}
\]

we obtain for the polynomials $(G_n)_{n \geq 1}$, $(K_n^{-p})_{n \geq 1}$ and $(F_n)_{n \geq 1}$, curves of zeros in the manifold $\mathcal{M}$. Of course for these polynomials, there are many other manifolds of zeros. See [4]. To find the special values for $(F_n)_{n \geq 1}$, for (i), we consider $h(z) = \frac{1}{1 - z}$ which gives $h' = \frac{1}{1 - z}$. For (ii), we take $h(z) = \frac{1}{1 + z}$. For (iii), (iv), (v) and (2.1), we take the Koebe function

\[
f(z) = \frac{z}{(1 - z)^2}, \quad \text{then} \quad \frac{zf'}{f} = 1 + \frac{2z}{1 - z} = \frac{1 + z}{1 - z},
\]

\[
h(z) = f'(z) = \frac{1 + z}{1 - z},
\]

\[
\frac{h'}{h} = \frac{f''}{f'} = \frac{1}{1 + z} + \frac{3}{1 - z} = 4 + 2z + 4z^2 + 2z^3 + 4z^4 + \cdots, \quad \text{iii}'
\]
We calculate (iv) with $\frac{f''}{f'}$ and (iii) with (iii'). To calculate (2.2), we use

$$
\sum_{k \geq 0} K^2_k(-4, -2, -4, -2, -4, -2, \ldots) z^k = \left(1 - z \frac{f''}{f'}\right)^2 = \left(\frac{1}{1+z} - \frac{3z}{1-z}\right)^2.
$$

To prove (vi),

$$
\exp\left(-p \sum_{j \geq 1} \frac{z^j}{j}\right) = \exp[p \log(1-z)] = (1-z)^p = 1 + \sum_{n \geq 1} F^{-1}_n(p, p, \ldots, p) z^n.
$$

To prove (vii),

$$
\exp\left(-\sum_{j \geq 1} (b_j + 1) \frac{z^j}{j}\right) = 1 + \sum_{n \geq 1} F^{-1}_n(b_1 + 1, b_2 + 1, \ldots, b_n + 1) z^n
$$

$$
\quad = \exp\left(-\sum_{j \geq 1} b_j \frac{z^j}{j}\right) \times \exp\left(-\sum_{j \geq 1} \frac{z^j}{j}\right) = (1-z) \times \left(1 + \sum_{n \geq 1} F^{-1}_n(b_1, b_2, \ldots, b_n) z^n\right)
$$

and we identify equal powers of $z$. The identity (vii) generalizes the classical identity $C^p_n = C^p_{n-1} + C^p_{n}$ for the binomial coefficients. See for example [6, vol. 1, II-12].

To prove (viii), we have to calculate $F^{-1} \circ S$, it comes from

$$
\exp\left(\sum_{j \geq 1} \frac{b_j z^j}{j}\right) = 1 + \sum_{k \geq 1} F^{-1}_k(-b_1, -b_2, \ldots, -b_k) z^k.
$$

Taking $b_j = n$ for all $j \geq 1$, $\exp(n \sum_{j \geq 1} \frac{1}{z^j}) = \frac{1}{(1-z)^n} = \sum_{j \geq 0} C^{n+j-1}_n z^j$.

To prove (ix), we take the Koebe function $f(z)$, then $h(z) = \frac{f(z)}{z} = \frac{1}{(1-z)^2}$ and $[h(z)]^p = \frac{1}{(1-z)^{2p}} = 1 + \sum_{n \geq 1} C^{2p+n-1}_n z^n$. We can also deduce (ix) using the composition of maps:

$$
K^p(2b_1, 3b_2, \ldots, (n+1)b_n, \ldots) = F^{-1} \circ pI \circ F(2b_1, 3b_2, \ldots). \quad \text{For } b_1 = b_2 = \ldots = 1, \text{ we replace } F(2, 3, \ldots) \text{ using (iv), then we use (viii). To prove (x), we take } h(z) = \frac{1}{1+z^2}. \quad \square
$$

**Remark 2.1.** We verify the main corollary (C1) when $f(z)$ is the Koebe function. From (C1), for the Koebe function,

$$
\mathcal{P}_k = -(k-2) F_k(4, 9, 16, \ldots, (k+1)^2) - \frac{1}{2} K^2_k(-4, -2, -4, -2, -4, \ldots).
$$

If $k$ is odd, from (iii), we have $F_k(4, 9, 16, \ldots, (k+1)^2) = -4$, thus from (2.2), we find that $\mathcal{P}_k = 0$. If $k$ is even, from (iii), $F_k(4, 9, 16, \ldots, (k+1)^2) = -2$ and using (2.2), we find $\mathcal{P}_k = -(k-2) \times (-2) - (5k-4) = -3k$. Thus $\mathcal{P}_{2p} = -6p$. Compare with (iii)’’.

We obtain values of $F_n$ and $G_n$ when the $(b_j)_{j \geq 1}$ are binomial coefficients,
Proposition 2.1. We have

\[ F_n\left(-\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \ldots, (-1)^n \binom{n}{n}\right) = n, \]

\[ G_n\left(-\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \ldots, (-1)^n \binom{n}{n}\right) = \binom{2n-1}{n}, \]

where \( \binom{n}{k} \) is the binomial coefficient. More generally, let \( q \in C \), and let

\[ \left[ n \right] = \frac{(1-q^n)(1-q^{-n-1})\cdots(1-q^{-n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} \]

be the Gaussian polynomial, then

\[ G_n\left(-\left[ n \right], q\left[ n \right], -q^3\left[ n \right], \ldots, (-1)^n q^{n(n-1)/2} \left[ n \right]\right) = \binom{2n-1}{n}. \]

Proof. We obtain \( F_n \) with

\[-n \sum_{j \geq 1} \frac{w_j}{j} = \log \left(1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \cdots + (-1)^n \binom{n}{n} w^n\right) = n \log(1-w).\]

We can also deduce the identity for \( F_n \) from (2.2)–(vi) and the composition of maps \( F \circ F^{-1} = \text{Id} \). We obtain \( G_n \) with the relation

\[ \left[ 1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \cdots + (-1)^n \binom{n}{n} w^n \right]^{-1} \]

\[ = \frac{1}{(1-w)^n} = \sum_{j \geq 0} \binom{n+j-1}{j} w^j \]

when \( \binom{n}{k} \) is the binomial coefficient, and in the case of the Gaussian polynomial,

\[ \left[ 1 - \left[ n \right] w + q\left[ n \right] w^2 - q^3\left[ n \right] w^3 + \cdots + (-1)^n q^{n(n-1)/2} \left[ n \right] w^n \right]^{-1} \]

\[ = \frac{1}{\prod_{k=0}^{n-1} (1-q^k w)} = \sum_{k \geq 0} \left[ n+k-1 \right] w^k. \]

From (1.26), we know that \( G = F^{-1} \circ S \circ F \), then \( G \circ F^{-1} = F^{-1} \circ S \), we can deduce the first identity for \( G_n \) from (2.2)–(viii).

2.2. Basic identities

The function \( h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots \) satisfies

\[ h(z) - zh'(z) = 1 - \sum_{n \geq 2} (n-1)b_n z^n, \]

\[ h(z) - zh'(z) + \frac{z^2 h''(z)}{2} = 1 + \sum_{n \geq 3} \frac{(n-1)(n-2)}{2} b_n z^n, \]
\[ h(z) - zh'(z) + \frac{z^2 h''(z)}{2} - \frac{z^3 h'''(z)}{3!} = 1 - \sum_{n \geq 4} \frac{(n - 1)(n - 2)(n - 3)}{3!} b_n z^n, \]

\[ \ldots \]

2.3. Relations between the polynomials

We differentiate (1.1) with respect to \( w \),

\[ b_1 + 2b_2 w + \cdots + kb_k w^{k-1} + \cdots = (1 + b_1 w + b_2 w^2 + \cdots + b_p w^p + \cdots) \times \left( -\sum_{j \geq 1} F_j w^{j-1} \right). \]

We equal coefficients of same powers of \( w \), it gives the recurrence for the polynomials \((F_k)_{k \geq 0}\), \( F_0 = 1 \),

\[ -kb_k = \sum_{1 \leq j \leq k} F_j b_{k-j}. \quad (2.3) \]

With the same approach, one find other relations between the polynomials as

**Proposition 2.2.**

\[ F_{j+1} = -\sum_{0 \leq r \leq j} (r + 1)b_{r+1} G_{j-r} \quad (2.4) \]

\[ nG_n = \sum_{1 \leq j \leq n} F_j G_{n-j} \quad (2.5) \]

\[ \frac{n}{p-1} K_n^{1-p} = \frac{1}{r-1} \sum_{1 \leq j \leq n} j K_j^{1-r} K_n^{r-j} \quad \text{for } 2 \leq r < p, \quad (2.6) \]

\[ \frac{n}{p-1} K_n^{1-p} = \sum_{1 \leq j \leq n} F_j K_n^{1-p-j} \quad \forall p \neq 1, \ p \in \mathbb{Z}, \quad (2.7) \]

\[ K_n^p = \sum_{0 \leq j \leq n} K_j^r K_n^{p-r-j}. \quad (2.8) \]

**Proof of the identities (2.4)–(2.8).** To find (2.4), we consider \( h(w) = 1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots + b_k w^k + \cdots \),

\[ h'(w) = b_1 + 2b_2 w + 3b_3 w^2 + \cdots + nb_n w^{n-1} + \cdots. \]

Since \( \frac{1}{h(w)} = \sum_{n \geq 0} G_n w^n \), multiplying by \( h'(w) \) gives

\[ \frac{h'(w)}{h(w)} = \sum_{r \geq 0} \left[ \sum_{0 \leq r \leq j} (r + 1)b_{r+1} G_{j-r} \right] w^r. \]

To obtain (2.4), we compare with (1.22). For (2.5), we identify the two following expansions

\[ \frac{h'(w)}{h(w)^2} = \frac{h'(w)}{h(w)} \times \frac{1}{h(w)} = \left( \sum_{j \geq 0} -F_{j+1} w^j \right) \times \left( \sum_{p \geq 0} G_p w^p \right). \]
\[ \frac{h'(w)}{h(w)^2} = - \frac{d}{dw} \frac{1}{h(w)} = - \sum_{n \geq 0} (n + 1)G_{n+1}w^n. \]

To find (2.6) and (2.7), we use that for \(0 \leq r \leq p\),

\[ \frac{h'(w)}{h(w)^p} = \frac{h'(w)}{h(w)^r} \times \frac{1}{h(w)^{p-r}}. \] \hspace{1cm} (i)

If \(p \neq 1\) and \(r \neq 1\), (2.6) comes from

\[ \frac{1}{(p - 1)} \frac{d}{dw} \frac{1}{h(w)^{p-1}} = \frac{1}{(r - 1)} \left( \frac{d}{dw} \frac{1}{h(w)^{r-1}} \right) \times \sum_{j \geq 0} K_j^{r-p} w^j. \]

In (i), we take \(r = 1\) and we obtain (2.7) with

\[ \frac{1}{p - 1} \frac{d}{dw} \frac{1}{h(w)^{p-1}} = \left( \sum_{j \geq 0} F_{j+1} w^j \right) \times \frac{1}{h(w)^{p-1}}. \quad \square \]

**Remark 2.1.** For \(k \geq 1\), (1.22) yields

\[ \frac{w^{1-k}h'(w)}{h(w)} + F_1 w^{1-k} + F_2 w^{2-k} + \cdots + F_m w^{m-k} + \cdots + F_{k-1} w^{-1} + F_k \]

\[ = -(F_{k+1} w + \cdots + F_{k+r} w^r + \cdots) \]

and

\[ (F_{k+1} w + \cdots + F_{k+r} w^r + \cdots) \times h(w) \]

\[ = \sum_{j \geq 1} (F_{k+1} b_{j-1} + F_{k+2} b_{j-2} + \cdots + F_{j+k} b_0) w^j. \]

The relations (2.4), (2.5), (2.6) and (2.7) involve the first derivative of \(h\), we can find other relations by multiplying powers of \(h\). For example (2.8) comes from \(h(w)^p = h(w)^r \times h(w)^{p-r}\).

Below, we give further identities between the polynomials. We consider the polynomials \((P^n_k)_{n \geq 1}, k \in \mathbb{Z}\) defined by (1.4). We define \(B^n_k\) by

\[ \left( \frac{z h'(z)}{h(z)} \right)^2 \left[ h(z) \right]^k = \sum_{n \geq 2} B^n_{n+k} z^n. \] \hspace{1cm} (2.9)

Since \(zf' = 1 + \frac{b'}{h}\), we have

\[ p_k^{n+k} = B^n_{n+k} \times 1_{n \geq 2} + \frac{2n + k}{k} K^n_k. \] \hspace{1cm} (2.10)

**Proposition 2.3.** Let \(f(\xi) = \xi(1 + b_1 \xi + b_2 \xi^2 + \cdots)\). For \(k \neq 0\), we have

\[ \phi_k(\xi) = \frac{\xi f'(\xi)}{f(\xi)} \times h(\xi)^k = \sum_{n \geq 0} \frac{k + n}{k} k^n K^n_k \xi^n. \] \hspace{1cm} (2.11)

If \(k \geq 1\),

\[ \phi_k(\xi) = \frac{(-1)^k}{k!} \left[ \frac{\partial^k}{\partial b_1^k} \left( \sum_{n \geq 0} F_{k+n} \xi^n \right) \right] (G_1(b_1), G_2(b_1, b_2), \ldots) \] \hspace{1cm} (2.12)
and the function $\sum_{n \geq 0} F_{k+n}(b_1, b_2, \ldots, b_{k+n})z^n$, $k \geq 1$, is given by the line integral

$$\sum_{n \geq 0} F_{k+n}(b_1, b_2, \ldots)z^n = -\frac{1}{2i\pi} \int \frac{h'(\zeta)}{\zeta^{k-1} (\zeta - z) h(\zeta)} \, d\zeta. \quad (2.13)$$

If $k \geq 1$, the function $\phi_k(\zeta) = \frac{\xi'f(\zeta)}{f(\zeta)} \times \frac{\zeta^k}{f(\zeta)^k} = \sum_{n \geq 0} \frac{k-n}{k} K_{n-k} \zeta^n$ is given by

$$\phi_k(\zeta) = \frac{(-1)^k}{k!} \left( \frac{\partial^k}{\partial b_1^k} \left( \sum_{n \geq 0} \frac{k-n}{k+n} F_{k+n} \zeta^n \right) \right) (b_1, b_2, \ldots). \quad (2.14)$$

**Proof.** For (2.11), we use the recursion formula (2.7) or give a direct proof since $\phi_k(\zeta) = h(\zeta)^k + \zeta^k \frac{d}{d\zeta} h(\zeta)^k$. For (2.12), we have from (T1), for $k > 0$ and $j \geq 0$,

$$\left( j + \frac{k}{k} \right) K_{j-k} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j} \quad (2.15)$$

The proof of (2.13) is classical for Laurent series: let $l(\zeta) = \sum_{n \in \mathbb{Z}} \alpha_n \zeta^n$, then

$$\sum_{n \geq k} \alpha_n \zeta^n = \sum_{n \geq k} \frac{\zeta^n}{2i\pi} \int \frac{l(\zeta)}{\zeta^{n+1}} \, d\zeta = \frac{\zeta^k}{2i\pi} \int \frac{l(\zeta)}{\zeta^k (\zeta - z)} \, d\zeta.$$  

We take $l(\zeta) = -\frac{\zeta h'(\zeta)}{h(\zeta)}$. For $\phi_k$, we proceed in the same way. \[\square\]

**Remark 2.2.** By composition of maps, see (1.31)–(1.32), we can define $(K^k_n)_{n \geq 0}$ for any $k \in \mathbb{R}$ with $K^k_0 = 1$ and $(K^k_1, K^k_2, \ldots, K^k_n, \ldots) = F^{-1} \circ k \times \text{Id} \circ F$. From Proposition 2.3, we see that for fixed $n \geq 1$, we have $\lim_{k \to 0} n-k \cdot K_{n-k}^k = F_n$. Since $\lim_{k \to 0} K_{n-k}^k = 0$, we obtain for

$$\lim_{k \to 0} \frac{1}{k} K_{n-k}^k = \frac{1}{n} \times F_n$$  

for $n \geq 1$. \[ (2.16) \]

**Proposition 2.4.** Assume that $f^{-1}$ is the inverse of $f$, $f(f^{-1}(z)) = z$. For $f(\zeta) = \zeta [1 + \sum_{n \geq 1} b_n \zeta^n]$ and $\phi_k(\zeta) = \frac{\xi'f(\zeta)}{f(\zeta)} \times h(\zeta)^k$, then

$$\phi_k f^{-1}(z) = 1 + \sum_{n \geq 1} P^k_n (b_1, b_2, \ldots, b_n) \zeta^n \quad \text{for } k \neq 0, \quad (2.17)$$

$$P^k_n (b_1, \ldots, b_n) = \sum_{0 \leq s \leq n} \left( F_1 (b_1), \ldots, -F_s (b_1, \ldots, b_n) \right) \times K_{n-s}^{k-n} (b_1, \ldots, b_{n-s}). \quad (2.18)$$

**Proof.** See [1, (A.1.2)]. In particular,

$$P^0_n (b_1, \ldots, b_n) = K^2_n (-F_1 (b_1), \ldots, -F_n (b_1, \ldots, b_n)), \quad P^0_n (G_1 (b_1), \ldots, G_n (b_1, \ldots, b_n)) = K^2_n (F_1 (b_1), \ldots, F_n (b_1, \ldots, b_n))$$

and $P^0_n (b_1, \ldots, b_n) - P^0_n (G_1 (b_1), \ldots, G_n (b_1, \ldots, b_n)) = -4 F_n (b_1, b_2, \ldots, b_n).$ \[\square\]
(2.19) – Expressions of \((P^k_n)\)

If \(n \neq k\),

\[
P^k_n(b_1, \ldots, b_n) = -\frac{1}{k - n} \sum_{j=0}^{n} (k - j) F_j(b_1, \ldots, b_j) K_{n-j}^{k-n}(b_1, \ldots, b_{n-j})
\]

\[
= \frac{k}{k - n} K_n^{k-n} - \frac{1}{k - n} \sum_{j=1}^{n} (k - j) F_j(b_1, b_2, \ldots, b_j)
\]

\[
\times K_{n-j}^{k-n}(b_1, b_2, \ldots, b_{n-j}). \tag{E_1}
\]

If \(n = k\) (with \(F_0 = -1\))

\[
P^n_n(b_1, b_2, \ldots, b_n) = \sum_{j=0}^{n} F_j(b_1, b_2, \ldots, b_j) \times F_{n-j}(b_1, b_2, \ldots, b_{n-j}). \tag{E_2}
\]

**Remark 2.3.** If \(k = 1\), \(f'(f^{-1}(z)) = 1 + \sum_{n \geq 1} P^1_n(b_1, b_2, \ldots) z^n\) with

\[
P^1_n = \frac{1}{n - 1} \sum_{0 \leq j \leq n} (1 - j) F_j K_{n-j}^{1-n}. \tag{E_3}
\]

**Proof of (E_1 and E_2).** From \([1, (A.1.1)]\),

\[
\frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p = 1 + \sum_{n \geq 1} P^{n+k+p}_n z^n.
\]

On the other hand,

\[
\frac{zf'(z)}{f(z)} [h(z)]^k = z^{1-k} f'(z) f(z)^{k-1} = z^{1-k} \frac{1}{k} \frac{d}{dz} f(z)^k = \frac{1}{k} \sum_{n \geq 0} (n + k) K_n z^n.
\]

Multiplying \(\frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p\), we obtain for any \(k, p \in \mathbb{Z}\), \(k \neq 0, p \neq 0\),

\[
P^{n+k+p}_n = \sum_{0 \leq j \leq n} (j + k)(n - j + p) \times \frac{1}{pk} K_j^k K_{n-j}^p. \tag{E_4}
\]

We make \(p \to 0\) as in Remark 2.2, it gives \(P^k_n\). To obtain \(P^n_n\), we make \(k \to 0\). \(\square\)

**Remark 2.4.** From \(\frac{(j+k)}{k} K_j^k = \frac{(-1)^j}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j}\) and from (E_4), we deduce that for \(k > 0, p > 0\),

\[
P^{n+k+p}_n(b_1, b_2, \ldots)
\]

\[
= \sum_{0 \leq j \leq n} \frac{(-1)^{k+p}}{k! p!} \left( \frac{\partial^k}{\partial b_1^k} F_{k+j} \right)(c_1, c_2, \ldots) \times \left( \frac{\partial^p}{\partial b_1^p} F_{p+n-j} \right)(c_1, c_2, \ldots). \tag{E_5}
\]

From (E_1), we obtain for \(k > n\),
For the Koebe function, we have \( P_{n-k}^{-k} = P_{n+k}^k \) for \( n \geq 1, k \geq 1 \),

\[
P_{n-k}^{-k}(b_1, b_2, \ldots, b_{n-k}) = P_{n+k}^k(b_1, b_2, \ldots, b_{n+k})
\]

for \((b_1, b_2, \ldots) = (2, 3, 4, 5, 6, \ldots)\), \(b_n = n + 1\). \hspace{1cm} (2.20)

Conversely, let \( f(z) = z + b_1 z^2 + \cdots + b_n z^{n+1} + \cdots \), if \( P_{n-k}^{-k} = P_{n+k}^k \) for \( n \geq 1, k \geq 1 \), then \( f(z) = \frac{1}{1-\xi^2} \), \( \epsilon = 1 \) or \( \epsilon = -1 \).

Moreover, for the Koebe function, we have \( K_{n-j}^{-n} \times 1_{n \geq j} = K_{n-j}^{-n} \times 1_{n+j \geq 0}, n, j \in \mathbb{Z} \). This last relation is the same as the classical \( C_{n-j}^{2n} = C_{n+j}^{2n} \) on the binomial coefficients.

We verify (2.20) with \( n = 3, k = 2 \). With [1, (A.1.7)], we calculate \( P_1^{-2} = -b_1 \),

\[
P_3^2 = 7b_5 - 20b_1 b_4 + 3b_1 b_2^2 + 35b_1^2 b_3 - 50b_1^3 b_2 + 14b_1^5 - 16b_2 b_3.
\]

When \( b_n = (n + 1) \), we find \( P_1^{-2} = P_3^2 = -2 \). With [1, (A.1.7)],

\[
P_4^3(b_1, b_2, b_3, b_4) = 6b_4 - 12b_1 b_3 - 5b_2^2 + 16b_1^2 b_2 - 5b_1^4
\]

and \( P_4^3(2, 3, 4, 5) = P_0^{-2} = 1 \).

**Proof of Theorem 2.5.** Let \( f(z) = \frac{z}{(1-z)^2} \), we have \( \frac{zf'(z)}{f(z)} = \frac{1+2z}{1-z} \) and

\[
\left( \frac{zf'(z)}{f(z)} \right)^2 [h(z)]^k = \frac{(1+z)^2}{(1-z)^{2k+2}} = 1 + \sum_{n \geq 1} P_{n+k}^n \xi^n.
\] \hspace{1cm} (i)

It gives the line integral

\[
P_{n+k}^n = \frac{1}{2i\pi} \int \left( \frac{zf'(z)}{f(z)} \right)^2 [h(z)]^k \frac{d\xi}{\xi^{n+1}} - \sum_{\xi \neq 0} \text{Residue}
\]

\[
= \frac{1}{2i\pi} \int \frac{(1+z)^2}{(1-z)^{2k+2}} \frac{d\xi}{\xi^{n+1}} - \text{Residue at } \xi = 1.
\] \hspace{1cm} (ii)

With (ii), we find \( P_{n-j}^{-j} = \frac{1}{2i\pi} \int \lambda(\xi) \xi^j \frac{d\xi}{\xi} \) and \( P_{n+j}^j = \frac{1}{2i\pi} \int \lambda(\xi) \xi^{-j} \frac{d\xi}{\xi} \) with

\[
\lambda(\xi) = \left( \frac{zf'(z)}{f(z)} \right)^2 [f(\xi)]^{-n} = \frac{(1+z)^2}{(1-\xi)^{2-2n} \xi^n}.
\] \hspace{1cm} (iii)

since for \( n \geq 1 \), the function \( \lambda(\xi) \) has only a pole at \( \xi = 0 \). We can calculate the two line integrals on the circle \( |\xi| = 1 \). Since the function \( \lambda(\xi) \) is such that \( \lambda(\xi) = \lambda(1/\xi) \), we put \( \xi = 1/\xi \) and we see that the two line integrals \( P_{n-j}^{-j} \) and \( P_{n+j}^j \) are equal. Consider any \( f(z) \) and let the function...
then $f(z) = f(\frac{1}{z})$, then $(\frac{zk'(z)}{k(z)})^2 = (\frac{uf'(u)}{f(u)})^2$ at $u = \frac{1}{z}$. The Koebe function satisfies $f(z) = f(\frac{1}{z})$. This proves (2.20).

Conversely, assume that $P_{n-k}^k = P_{n+k}^k$ for $n \geq 1, k \geq 1$.

Taking $n = 1$, we find $P_1^2 = P_0^{-1} = 1$ and $P_{1+j}^j = 0$ for $j \geq 2$. It gives $P_{n}^{n-1} = 0$ for $n \geq 3$ and

$$\frac{1}{f(z)} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z} + P_1^0 + z$$

then $f(z) = f(\frac{1}{z})$.

Taking $n = 2$, we obtain $P_4^2 = 1$ and $P_{2+j}^j = 0$ for $j \geq 3$. Thus $P_{n}^{n-2} = 0$ for $n \geq 4$ and from (1.4),

$$\frac{1}{f(z)^2} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z^2} + P_1^{-1} \frac{1}{z} + P_2^0 + P_3^1 z + z^2.$$  \hspace{1cm} (2.21)

We have $P_{n-1}^{-1} = P_1^3 \ (n = 2, \ j = 1)$. Taking the ratio of (2.20) by (2.21), we see that

$$f(z) = z \times \frac{1 + P_0^0 z + z^2}{1 + P_1^{-1} z + P_2^0 z^2 + P_3^{-1} z^3 + z^4}.$$  

Let $f(z) = zh(z)$ and $l(z) = \frac{1}{h(z)} = \frac{1 + P_1^{-1} z + P_2^0 z^2 + P_3^{-1} z^3 + z^4}{1 + P_1^0 z + z^2}$. With (2.21), $l(z) = 1 + G_1 z + G_2 z^2 + \cdots + G_n z^n + \cdots$ must satisfy

$$l(z) (1 - \frac{z l'(z)}{l(z)})^2 = 1 + P_0^0 z + z^2$$

then

$$\left( l(z) - z l'(z) \right)^2 = 1 + P_1^{-1} + P_2^0 z^2 + P_3^{-1} z^3 + z^4.$$  \hspace{1cm} (ii)

Using the identity (2.2), we have $P_{1}^{-1} = 0$. With (ii), we see that $1 + P_0^0 z + z^2$ must have a double root. It implies that $(P_0^0)^2 = 4$. Identifying the coefficients in (ii) gives $P_2^0 = -2$ and $f(z) = \frac{z}{(1 - \epsilon z)^2}, \ \epsilon = 1$ or $\epsilon = -1$. □

For the Koebe function, we prove $K_{n-j}^{-n} = K_{n+j}^{-n}$ in the same way and then apply (2.2)(ix).

**Remark 2.5.** Eq. (2.21) has other solutions than the Koebe function, but (2.21) and $K_{1-j}^{-1} = K_{1+j}^{-1}, \ \forall j \geq 1$ or equivalently $K_2^{-1} = 1, K_n^{-1} = 0, \ \forall n \geq 3$, implies that $f(z) = \frac{z}{(1 - \epsilon z)^2}, \ \epsilon = +1, -1$. According to (4.8) below, the condition $K_n^{-1} = 0$ for $n \geq 3$ means that $\frac{\partial}{\partial b_1} F_n = 0$ for $n \geq 4$.

**Remark 2.6.** When we write the expressions (E)1, (E)2, \ldots of $(P_0^k)$ in the case of the Koebe function, with (2.2)(iii), (iv), (x) we obtain relations between the binomial coefficients.

### 3. The composition of maps

We consider the polynomials

$$(b_1, b_2, \ldots, b_n) \to F_n (b_1, b_2, \ldots, b_n)$$

$$(b_1, b_2, \ldots, b_n) \to G_n (b_1, b_2, \ldots, b_n) \quad n \geq 1,$$
as functions of \((b_1, b_2, \ldots, b_n)\) and we take composition of maps. We denote \(F_n(G_1, \ldots, G_n)\) the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow F_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)).
\]

In the same way, \(G_n(G_1, \ldots, G_n)\) is the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow G_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n))
\]

and \(G_n(F_1, \ldots, F_n)\) is the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow G_n(F_1(b_1), F_2(b_1, b_2), \ldots, F_n(b_1, b_2, \ldots, b_n)).
\]

For example, we have

\[
F_1(G_1)(b_1) = F_1(-b_1) = b_1,
\]

\[
G_1(F_1)(b_1) = b_1,
\]

\[
G_2(F_1, F_2)(b_1, b_2) = F_1^2 - F_2 = 2b_2.
\]

\[
G_3(F_1, F_2, F_3)(b_1, b_2, b_3) = b_1 b_2 + 3b_3,
\]

\[
G_4(F_1, F_2, F_3, F_4) = 2b_2^2 + 2b_1 b_3 + 4b_4,
\]

\[
G_5(F_1, F_2, F_3, F_4, F_5) = b_1 b_2^3 + 7b_3 b_2 + 3b_1 b_4 + 5b_5,
\]

\[
G_6(F_1, F_2, F_3, F_4, F_5, F_6) = 4b_1 b_2 b_3 + 2b_2^3 + 6b_3^2 + 10b_2 b_4 + 4b_1 b_5 + 6b_6,
\]

\[
G_7(F_1, F_2, F_3, F_4, F_5, F_6, F_7) = 17b_3 b_4 + b_1 b_2^3 + 13b_2 b_5 + 11b_2^3 b_3 + 4b_1 b_2^2 + 5b_1 b_6 + 6b_1 b_2 b_4 + 7b_7,
\]

\[
G_8(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8) = 6b_3 b_1 b_2^3 + 8b_5 b_1 b_2 + 12b_4 b_1 b_3 + 2b_2^4 + 12b_4 b_2^2 + 16b_4 b_2 + 16b_6 b_2
\]

\[+ 22b_5 b_3 + 20b_2^2 b_2 + 6b_1 b_7 + 8b_8.
\]

**Proposition 3.1.** For \(n \geq 1\),

\[
G_n(G_1, G_2, \ldots, G_n) = b_n,
\]

\[
F_n(b_1, b_2, \ldots, b_n) + F_n(G_1, G_2, \ldots, G_n) = 0.
\]

**Proof of (3.1) and (3.2).** For (3.1), let \(\tilde{h}(w) = \frac{1}{\tilde{h}(w)} = \sum_{n \geq 0} G_n w^n\), then \(\frac{1}{\tilde{h}(w)} = h(w)\). In particular, writing \(G_2(G_1, G_2) = b_2\) and \(G_3(G_1, G_2, G_3) = b_3\), we obtain \(G_7^2 - G_2 = b_2\) and \(-G_1^2 + 2G_1 G_2 - G_3 = b_3, \ldots\). The second relation is an immediate consequence of (1.23) and (1.24). \(\Box\)

**Remark 3.1.** 1. The second formula (3.2) can also be proved by recurrence: it is true for \(n = 1\). We shall use the two recursion formulae (2.5) and (2.3),

\[
F_n + b_1 F_{n-1} + b_2 F_{n-2} + \cdots + b_{n-1} F_1 + n b_n = 0,
\]

\[
F_n + G_1 F_{n-1} + G_2 F_{n-2} + \cdots + G_{n-1} F_1 - n G_n = 0.
\]

From the first relation, we have \(F_2(G_1, G_2) + G_1 F_1(G_1) + 2G_2 = 0\) and from the second relation \(F_2(b_1, b_2) + G_1 F_1(b_1) - 2G_2 = 0\). Using that (3.2) is true for \(n = 1\) and adding the two relations above, we obtain (3.2) for \(n = 2\) and the formula by recurrence on \(n\).
2. We can also find this formula with the recursion formula for the \((G_n)_n\geq 0\),
\[
G_1 + b_1 = 0, \quad G_2 + b_1 G_1 + b_2 = 0, \quad G_3 + b_1 G_2 + b_2 G_1 + b_3 = 0
\]
and in general, we have \(G_n + b_1 G_{n-1} + b_2 G_{n-2} + \cdots + b_{n-1} G_1 + b_n = 0\) as follows. From (2.4)
\(F_2(b_1, b_2) = -(b_1 G_1 + 2b_2)\) and \(F_2(G_1, G_2) = -(b_1 G_1 + 2G_2)\). Adding and using (i) gives the result. We proceed in the same way for \(F_n\).

3. Another proof of (3.2) is to show that \(G = F^{-1} \circ S \circ F\) as follows. From (1.23)–(1.24), we have the map \(\phi : h \to u = 1 - \frac{h}{2}\)
\[
(b_1, b_2, \ldots, b_n, \ldots) \to \left( F_1(b_1), \frac{F_2(b_1, b_2)}{2}, \ldots, \frac{F_j(b_1, b_2, \ldots, b_j)}{j}, \ldots \right).
\]
Its inverse map gives the Schur polynomials. We calculate \(h\) from \(u\) with the relation \(h(u) = \exp(\frac{1-u(z)}{2})\). The map
\[
F : (b_1, b_2, \ldots, b_n, \ldots) \to (F_1(b_1), F_2(b_1, b_2), \ldots, F_j(b_1, \ldots, b_j), \ldots)
\]
is a bijection. The map \(S : 1 - \frac{h'}{h} \to 1 + \frac{h'}{h} = 1 - \frac{h'}{\bar{h}}\) with \(\bar{h} = \frac{1}{h}\) is also a bijection. Then the map \(\phi^{-1} \circ S \circ \phi\) is just \(h \to \bar{h} = \frac{1}{h}\). This gives (1.26). To calculate \(F_n^{-1}\), we have to solve the system in \((b_1, b_2, \ldots, b_n)\),
\[
F_1(b_1) = c_1, \quad F_2(b_1, b_2) = c_2, \quad F_n(b_1, b_2, \ldots, b_n) = c_n, \quad \ldots
\]

**Proof of (1.31)–(1.32).** For \(p \geq 1\), we consider the map \(\phi_p : h \to u = 1 - p\frac{h'}{h}\) which allows us to calculate \(h^p\). □

More identities similar to (3.1) and (3.2) can be found.

**Theorem 3.2.**

\[
F_n(-F_1(b_1), -F_2(b_1, b_2), \ldots, -F_k(b_1, b_2, \ldots, b_k), -F_n(b_1, b_2, \ldots, b_n)) = F_n(2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n) - F_n(b_1, b_2, b_3, \ldots, b_n),
\]

\(3.5\)

\[
F_n(F_1, F_2, \ldots, F_n) = F_n(0, -b_2, -2b_3, -3b_4, \ldots, -(n-1)b_n)
- F_n(b_1, b_2, \ldots, b_n),
\]

\(3.6\)

\[
G_n(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} b_k G_{n-k}(2b_1, 3b_2, 4b_3, \ldots, (j+1)b_j, \ldots),
\]

\(3.7\)

\[
G_n(F_1, F_2, \ldots, F_n) = \sum_{k=0}^{n} b_k G_{n-k}(0, -b_2, -2b_3, \ldots, -(j-1)b_j, \ldots).
\]

\(3.8\)

For \(p \in \mathbb{Z}, \ p \neq 0\),
\[
K_p^n(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} K_p^{-p}(b_1, b_2, \ldots, b_n)
\times K_k^p(2b_1, 3b_2, \ldots, (j+1)b_j, \ldots),
\]

\(3.9\)
\[ K^n_{k}(F_{1}, F_{2}, \ldots, F_{n}) = \sum_{k=0}^{n} K^{-p}_{n-k}(b_{1}, b_{2}, \ldots, b_{n}) \times K^{p}_{k}(0, -b_2, -2b_3, \ldots, -(j-1)b_j, \ldots). \] (3.10)

**Remark 3.2.** Consider the maps \( D^1 \) and \( D^{-1} \) from \( M \) to \( M \),
\[
D^1 : (b_1, b_2, \ldots, b_k, \ldots) \mapsto (2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n, \ldots),
\]
\[
D^{-1} : (b_1, b_2, \ldots, b_k, \ldots) \mapsto (0, b_2, 2b_3, \ldots, (n-1)b_n, \ldots),
\]
then (3.5)–(3.7) can be written as \( F \circ S \circ F = F \circ D^1 - F \) and \( F \circ F = F \circ S \circ D^{-1} \). Remark that \( K^{p}, F, G, \) and \( S \) are bijection while \( D^{-1} \) is not.

**Proof of Theorem 3.2.** To prove (3.5), we consider \( f(z) = zh(z) \).
\[
\frac{zf''}{f'} = - \sum_{k \geq 1} F_k(-F_1, -F_2, \ldots, -F_k)z^{k-1}. \tag{i}
\]
On the other hand
\[
\frac{zf''}{f'} = \frac{1}{z} + \frac{f''}{f'} - \frac{f'}{f} = \frac{1}{z} - \frac{f'}{f} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k)z^{k-1}.
\]
Using the expression of \( \frac{1}{z} - \frac{f'}{f} \), we deduce
\[
\frac{zf''}{f'} = - \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k)z^{k-1} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k)z^{k-1}. \tag{ii}
\]
The comparison of (i) and (ii) gives (3.5).

For (3.6), let \( g(z) = \frac{z}{h(z)} \). Then \( u(z) = \frac{g'(z)}{g(z)} = 1 + \sum_{n \geq 1} F_n(b_1, b_2, \ldots, b_n)z^n \) satisfies
\[
\frac{u'(z)}{u(z)} = - \sum_{k \geq 1} F_k(F_1(b_1), F_2(b_1, b_2), \ldots, F_k(b_1, b_2, \ldots, b_k))z^{k-1}. \tag{i}
\]
Thus we obtain \( F_k(F_1, F_2, \ldots, F_k) \) from this expansion. On the other hand, we calculate
\[
\frac{u'(z)}{u(z)} = \frac{d}{dz} \log(u(z)) = \frac{1}{z} \left( - \sum_{k \geq 1} F_kz^k + z \frac{g''(z)}{g'(z)} \right).
\]
Since \( g'(z) = \frac{h(z)-zh'(z)}{h(z)^2} \), we obtain
\[
z \frac{g''(z)}{g'(z)} = z \frac{(h-zh')'}{h-zh'} - 2z \frac{h'}{h} = z \frac{(h-zh')'}{h-zh'} + 2 \sum_{k \geq 1} F_kz^k.
\]
Using $h(z) - z h'(z) = 1 + \sum_{n \geq 2} (1 - n) b_n z^n$, we deduce

$$\frac{u'(z)}{u(z)} = \frac{1}{z} \left( \sum_{k \geq 1} F_k z^k - \sum_{k \geq 1} F_k (0, -b_2, -2b_3, \ldots, -(k-1)b_k) z^k \right).$$

(ii)

Then we compare the two identities (i) and (ii).

To prove (3.7), we consider

$$h(w) = h(z) + w h'(w)$$

and

$$\frac{1}{1 - \frac{wh'(w)}{h(w)}} = \sum_{n \geq 0} G_n(F_1(b_1), F_2(b_1, b_2), \ldots, F_n(b_1, b_2, \ldots, b_n)) w^n,$$

where

$$\frac{h(w)}{h(w) - wh'(w)} = (1 + b_1 w + b_2 w^2 + \ldots) \times \sum_{n \geq 0} G_n(0, -b_2, \ldots, -(n-1)b_n, \ldots) w^n$$

since

$$h(w) - wh'(w) = (1 + b_1 w + b_2 w^2 + \ldots + b_n w^n + \ldots) - (b_1 w + 2b_2 w^2 + \ldots + nb_n w^n + \ldots) = 1 - b_2 w^2 - 2b_3 w^3 - \ldots - (n-1)b_n w^n - \ldots.$$

To prove (3.7), we consider

$$\frac{h(w)}{h(w) + wh'(w)}.$$

To prove (3.9),

$$\left(1 + \frac{h'(z)}{h(z)} \right)^p = \left(1 - \sum_{k \geq 1} F_k^p \right)^p = \sum_{n \geq 0} K_n^p(-F_1, -F_2, \ldots, -F_n) z^n.$$

This is also equal to $\frac{(h(z) + \frac{z h'(z)}{h(z)})^p}{(h(z))^p}$. For (3.10), we take $1 - \frac{z h'(z)}{h(z)}$. ∎

**Remark 3.3.** With (3.7)–(3.10) we define differential operators on Faber polynomials. For (3.7)–(3.8), we have (see (1.16)) $G_{n-k} = -\frac{1}{n} \frac{\partial}{\partial b_k} F_n$, thus

$$G_n(-F_1, -F_2, \ldots, -F_n) = -\frac{1}{n+1} \left[ \sum_{k \geq 0} b_k \frac{\partial F_{n+1}}{\partial b_{k+1}} \right] (2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n).$$

(3.11)

For (3.9), we take for example $p = 2$. See Proposition 2.4. With (1.16), we deduce that

$$K_{n-k}^{-2} = \frac{1}{n+1} \frac{\partial}{\partial b_k} \left( \frac{\partial}{\partial b_1} F_{n+1} \right) = \frac{1}{n+2} \frac{\partial}{\partial b_k} \left( \frac{\partial}{\partial b_2} F_{n+2} \right)$$

and

$$K_n^2(-F_1, -F_2, \ldots, -F_n) = \frac{1}{n+1} U_2 \left( \frac{\partial}{\partial b_1} F_{n+1} \right) + K_{n-2}^2(b_1, \ldots, b_n)$$

$$= \frac{1}{n+2} U_2 \left( \frac{\partial}{\partial b_1} F_{n+2} \right) + \frac{1}{n+2} \frac{\partial^2 F_{n+2}}{\partial b_1^2},$$

(3.12)

where $U_2$ is the differential operator

$$U_2 = \sum_{k \geq 1} K_k^2(2b_1, 3b_2, \ldots, (k+1)b_k) \frac{\partial}{\partial b_k}$$

$$= 4b_1 \frac{\partial}{\partial b_1} + (6b_2 + 4b_1^2) \frac{\partial}{\partial b_2} + (8b_3 + 12b_1b_2) \frac{\partial}{\partial b_3} + \cdots.$$
We have
\[ U_2 \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_1} U_2 = -4L_1 \]  
(3.14)
where \( L_1 \) is given by (1.13). In the same way, in (3.10), we put
\[ T_2 = \sum_{k \geq 1} K^2_k(0, -b_2, -2b_3, \ldots, -(k - 1)b_k) \frac{\partial}{\partial b_k} \]
\[ = -2b_2 \frac{\partial}{\partial b_2} - 4b_3 \frac{\partial}{\partial b_3} + (b_2^2 - 6b_4) \frac{\partial}{\partial b_4} + (4b_2b_3 - 8b_5) \frac{\partial}{\partial b_5} + \cdots. \]  
(3.15)
Then \( T_2[h(z)] = (h(z) - z h'(z))^2 - 1 \). We have
\[ K^2_n(F_1, F_2, \ldots, F_n) = \frac{1}{n + 1} T_2 \frac{\partial}{\partial b_1} F_{n+1} + K^{-2}_n(b_1, \ldots, b_n) \]
\[ = \frac{1}{n + 1} \frac{\partial}{\partial b_1} T_2 F_{n+1} + K^{-2}_n(b_1, b_2, \ldots, b_n) \]
\[ = \frac{1}{n + 2} T_2 \frac{\partial}{\partial b_2} F_{n+2} + \frac{1}{n + 2} \frac{\partial^2 F_{n+2}}{\partial b_1^2}. \]  
(3.16)
By (2.18), \( K^2_n(-F_1, -F_2, \ldots) - K^2_n(F_1, F_2, \ldots) = -4F_n = -\frac{4}{n+1} L_1(F_{n+1}), \) see (7.11). With (3.14), it gives \( \frac{\partial}{\partial b_1}(U_2 - T_2) F_n = 0 \), thus \( (U_2 - T_2) F_n \) does not depend on \( b_1 \) for \( n \geq 2 \). For \( p \geq 1 \),
\[ T_2 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} T_2 = -2(p - 1) \sum_{k \geq p} (k - p - 1)b_{k-p} \frac{\partial}{\partial b_k} \]
\[ = -2(p - 1) \sum_{n \geq 0} (n - 1)b_n \frac{\partial}{\partial b_{p+n}} = -2(p - 1)(L_p + 2W_p). \]  
(3.17)

**Corollary 3.3.** Let \( (P_k)_{k \geq 2} \) be the coefficients of the Schwarzian derivative as in (C1), and let
\[ \mathcal{H} = T_2 \frac{\partial}{\partial b_2} + \frac{\partial^2}{\partial b_1^2} \]  
(3.18)
then \( P_k(b_1, b_2, b_3, \ldots, b_k) + (k - 2)F_k(2b_1, 3b_2, \ldots, (k + 1)b_k) \) is equal to
\[ -\frac{1}{2(k + 2)} [\mathcal{H} F_{k+2}](2b_1, 3b_2, \ldots, (j + 1)b_j, \ldots). \]  
(3.19)

**Corollary 3.4.** Let \( T = \frac{\partial^2}{\partial b_1^2} + T_2 \frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4}). \) The condition (2.21) is equivalent to
\[ (TF_n)(G_1, G_2, \ldots, G_{n-2}) = 0 \quad \forall n \geq 4. \]  
(3.20)

**Proof.** The condition (2.21) is the same as
\[ K^2_n(-F_1, -F_2, \ldots) = b_n + b_1 b_{n-1} + b_{n-2} \quad \forall n \geq 1. \]  
(3.21)
From (1.26), we have \( K^p \circ S \circ F = K^p \circ F \circ G. \) Thus
\[ K^2_n(-F_1, -F_2, \ldots) = \frac{1}{n + 2} \left( T_2 \frac{\partial}{\partial b_2} F_{n+2} \right) \circ G + K^{-2}_n \circ G. \]  
(3.22)
Since \( G \circ G = \text{Identity} \) and \( G_1(b_1) = -b_1 \), we can write the right side in (3.21) as \((G_n - b_1G_{n-1} + G_{n-2})\) at the point \((G_1, G_2, \ldots, G_n)\). With (1.16), we have
\[
G_n - b_1G_{n-1} + G_{n-2} = -\frac{1}{n + 2} \left( \frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4} \right) F_{n+2}.
\]
Then (3.22) and (3.23) imply (3.20). The condition (3.20) is always satisfied for \( n = 1, 2, 3 \). For \( n = 4 \), it gives \( 3b_2 - 2b_1^2 = 1 \), for \( n = 5 \), \( 5b_3 - 3b_1b_2 - b_1 = 0 \). \( \square \)

4. The polynomials and their derivatives. Proof of the Main Theorem

4.1. The partial derivatives \((\frac{\partial}{\partial b_k})_{k \geq 1}\)

**Theorem 4.1.** For \( p \geq 1, n \geq 0 \),
\[
(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}.
\]
In particular, \( \frac{\partial F_p}{\partial b_p} = -p \), and \( \frac{\partial F_k}{\partial b_k} = -nG_{n-k} \) if \( k \leq n \). Let \((F^{-1}_n)_{n \geq 1}\) be the inverse Faber polynomials, then
\[
\frac{\partial}{\partial b_k} F^{-1}_p = -\frac{1}{k} F^{-1}_{p-k} \times 1_{k \leq p}.
\]

**Proof.** Let \( \psi(w) = w + b_1 + \frac{b_2}{w} + \frac{b_3}{w^2} + \cdots + \frac{b_p}{w^{p-1}} + \cdots \) and \( \psi_p(w) = \psi(w) - \frac{t}{w^p-1} \) \( p \geq 1 \).

We have \( w^{p} \psi'_p(w) = w^p \psi'(w) + (p - 1)t \),
\[
\frac{w \psi'_p(w)}{\psi_p(w)} = 1 + \sum_{n \geq 1} F_n(b_1, \ldots, b_{p-1}, b_p - t, b_{p+1}, \ldots) \times \frac{1}{w^n}.
\]

We differentiate this equation with respect to \( t \) and we make \( t = 0 \),
\[
\phi(w) = \frac{d}{dt} \bigg|_{t=0} \frac{w \psi'_p(w)}{\psi_p(w)} = \sum_{n \geq 1} \frac{\partial F_n}{\partial b_p} \times \frac{1}{w^n}.
\]
On the other hand
\[
\frac{w \psi'_p(w)}{\psi_p(w)} = \frac{w^p \psi'(w) + (p - 1)t}{w^p - 1 \psi(w) - t}.
\]

We calculate \( \phi \) with this expression
\[
\frac{d}{dt} \frac{w \psi'_p(w)}{\psi_p(w)} = w \left[ \frac{(p - 1)w^{p-2} \psi(w) + w^{p-1} \psi'(w)}{(w^{p-1} \psi(w) - t)^2} \right] = -w \frac{d}{dw} \frac{1}{w^p - 1 \psi(w) - t}.
\]
At \( t = 0 \),
\[
\phi(w) = -w \frac{d}{dw} \left( \frac{1}{w^{p-1} \psi(w)} \right) = -w \frac{d}{dw} \sum_{n \geq 0} G_n \times \frac{1}{w^{n+p}} = \sum_{n \geq 0} G_n(n + p) \frac{1}{w^{n+p}}.
\]
Comparing the two expressions of $\phi$ and since $F_n$ does not contain $b_p$ when $n < p$, we obtain the result. To calculate the derivatives of the map $F^{-1}$, we take

$$h(z) = 1 + \sum_{j \geq 1} F^{-1}_j(b_1, b_2, b_3, \ldots, b_j)z^j$$

since $F \circ F^{-1} = \text{Identity}$, we have $\frac{d}{dz} \log(h(z)) = -\sum_{k \geq 1} b_k z^{k-1}$. We differentiate with respect to $b_k$, for $k \geq 1$,

$$-z^{k-1} = \frac{\partial}{\partial b_k} \left( \frac{h'(z)}{h(z)} \right) = \frac{d}{dz} \left( \frac{\partial h(z)}{h(z)} \right).$$

We integrate this identity with respect to $z$,

$$-\frac{1}{k} z^k \times h(z) = \frac{\partial}{\partial b_k} h(z) = \sum_{j \geq 1} \left( \frac{\partial}{\partial b_k} F^{-1}_j \right) z^j.$$

Since $-\frac{1}{k} z^k \times h(z) = -\frac{1}{k} z^k (1 + \sum_{j \geq 1} F^{-1}_j z^j)$, we obtain the partial derivatives of $F^{-1}$.

**Corollary 4.2.** For $n \geq 0$, $p \geq 1$,

$$\frac{\partial}{\partial b_p} (G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) = -(n + p)b_n.$$  \hspace{1cm} (4.3)

**Proof.** Since $G \circ G = \text{Identity}$, it is a consequence of (4.1).

**Corollary 4.3.**

$$\frac{\partial G_n}{\partial b_k} = -K^{-2}_{n-k} \times 1_{n \geq k}.$$  \hspace{1cm} (4.4)

**Proof.** We differentiate $G = F^{-1} \circ S \circ F$.

$$\frac{\partial G_n}{\partial b_k} = \sum_{j \geq 1} \frac{\partial}{\partial b_j} (S \circ F) \times \left( -\frac{\partial F_j}{\partial b_k} \right) = \sum_{1 \leq j \leq n} \left( -\frac{1}{j} F^{-1}_{n-j} (S \circ F) \right) \times (jG_{j-k}).$$

After simplification by $j$, and since $F^{-1} \circ S \circ F = G$, we find

$$\frac{\partial G_n}{\partial b_k} = -\sum_{1 \leq j \leq n} G_{n-j} G_{j-k} = -K^{-2}_{n-k}.$$  \hspace{1cm} $\square$

The following operators up to a minus sign, $(Z_k)_{k \geq 0}$ were introduced in [2].

**Corollary 4.4.** With the recursion $F_{j+1} = -\sum_{0 \leq r \leq j} (r + 1)b_{r+1} G_{j-r}$, see (2.4), for $k \geq 0$, we deduce

$$Z_k = \sum_{r \geq 0} (r + 1)b_{r+1} \frac{\partial}{\partial b_{r+k+1}} \quad \text{and} \quad (j + k + 1)F_{j+1} = Z_k F_{j+k+1}.$$


**Proof.** From Theorem 4.1, \((j + k + 1)G_{j-r} = -\frac{\partial}{\partial b_{r+k+1}}F_{j+k+1}\). Thus, if \(k \geq 0\), with the recursion formula (2.4) where we replace \(G_{j-r}\), we find

\[(j + k + 1)F_{j+1} = \sum_{0 \leq r < j} (r + 1)b_{r+1}\frac{\partial}{\partial b_{r+k+1}}F_{j+k+1} = Z_k(F_{j+k+1}).\]

For \(k < 0\), then \(\frac{\partial}{\partial b_{r+k+1}}\) is defined only if \(r + k \geq 0\), i.e. \(r \geq -k\). We decompose the sum \(F_{j+1} = -\sum_{0 \leq r < -k} (r + 1)b_{r+1}G_{j-r} - \sum_{-k \leq r \leq j} (r + 1)b_{r+1}G_{j-r}\). □

4.2. **Proof of the main results**

**Proof of the Main Theorem.**

\[\frac{\partial}{\partial b_k} \left( \frac{1}{h(z)} \right) = -\frac{z^k}{h(z)^2} = -z^k - \sum_{n \geq 1} K_n^{-2} z^{n+k}.\]

On the other hand, \(\frac{\partial}{\partial b_k} \left( \frac{1}{h(z)} \right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k} G_n z^n\). In these two last expressions, we identify the coefficients of equal power of \(z\). It gives (1.17). We have \((n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}\), then differentiating this expression with respect to \(k\), we obtain

\[(p + k + n)K_n^{-2} = \frac{\partial^2 F_{p+k+n}}{\partial b_k \partial b_p} \quad \forall p \geq 1, \forall k \geq 1.\]

We deduce higher order partial derivatives of \(F_j\) from

\[\frac{\partial}{\partial b_k} \left( \frac{1}{h(z)} \right) = -pK_{n-k}^{-1} \times 1_{n \geq k} \quad \text{for} \; n \geq 1, \; k \geq 1, \; p \neq 0, \; p \in \mathbb{Z}.\] (4.5)

\(K_0 = 1\) for any \(p\). The proof of (4.5) or equivalently \(\frac{\partial}{\partial b_k} = pK_{n-k}^{-1} \times 1_{n \geq k}\) is as follows,

\[\frac{\partial}{\partial b_k} \left( \frac{1}{h(z)^p} \right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k} (K_n^{-p}) z^n.\] (i)

On the other hand

\[\frac{\partial}{\partial b_k} \left( \frac{1}{h(z)^p} \right) = -p \frac{\partial}{\partial b_k} (h(z)) = -pz^k \frac{h(z)^{p+1}}{h(z)^{p+1}} = -pz^k \sum_{n \geq 1} K_n^{-1} z^n = -p \sum_{q \geq 1} K_q^{-1} z^{q+k}.\] (ii)

The identification of the coefficients of \(z^q\) in the two expressions (i) and (ii) gives (4.5). We see that one can calculate as derivatives of Faber polynomials all the \((K_n^{-p})\) for \(p \geq 1\). □

**Proof of (T2).** We wish to calculate \(K_n^{-p}\) for \(p \geq 2\). Let \(\tilde{h}(z) = \frac{1}{h(z)} = 1 + G_1 z + G_2 z^2 + \cdots + G_n z^n + \cdots\). Then \((T2)\) is obtained with the identification of coefficients of equal powers of \(z\) in \(\tilde{h}(z)^{-p} = 1 + \sum_{n \geq 1} K_n^{-p} (G_1, G_2, \ldots, G_n) z^n = h(z)^p\) with \(h(z)^p = 1 + \sum_{n \geq 1} K_n^{-p} (b_1, b_2, \ldots, b_n) z^n\). We can also give a proof with the composition of maps \(K_n^{-p} = F^{-1} \circ pI \circ F\) and

\[K_n^{-p} \circ G = F^{-1} \circ pS \circ F \circ F^{-1} \circ S \circ F = F^{-1} \circ pI \circ F = K_n^{-p}.\] □
Corollary 4.5. All the $K^p_n$, $n \geq 1$, $p \in \mathbb{Z}$ can be obtained as derivatives of Faber polynomials. For $p \geq 1$,

$$(-1)^p (p - 1)! (n + k_1 + k_2 + \cdots + k_p) K_n^{-p} (b_1, b_2, \ldots, b_n) = \frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_p}} (b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots).$$

(4.6)

Let

$$\phi (b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots) = \frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_p}} (b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots),$$

for $p \geq 1$, we have

$$(-1)^p (p - 1)! (n + k_1 + k_2 + \cdots + k_p) K_n^p (b_1, b_2, \ldots, b_n) = \phi (G_1 (b_1, b_2, \ldots), G_2 (b_1, b_2, \ldots), \ldots, G_q (b_1, b_2, \ldots), \ldots).$$

(4.7)

Corollary 4.6. For $p \geq 1$,

$$(-1)^p (p - 1)! (n + p) K_n^{-p} (b_1, b_2, \ldots, b_n) = \frac{\partial^p F_{n+p}}{\partial b_1^p} (b_1, b_2, \ldots, b_n, \ldots, b_{n+p}),$$

(4.8)

$$(-1)^p (p - 1)! (n + p) K_n^p (b_1, b_2, \ldots, b_n) = \frac{\partial^p F_{n+p}}{\partial b_1^p} (G_1 (b_1), G_2 (b_1, b_2), \ldots, G_{n+p} (b_1, b_2, \ldots, b_{n+p})).$$

(4.9)

In particular

$$K_n^{-(n+1)} = \frac{(-1)^{n+1}}{n! (2n + 1)} \frac{\partial^{n+1} F_{2n+1}}{\partial b_1^{n+1}} (b_1, b_2, \ldots, b_n, \ldots),$$

(4.10)

$$K_n^n = \frac{(-1)^n}{(n-1)! (2n + 1)} \left( \frac{\partial^n F_{2n+1}}{\partial b_1^n} \right) (G_1 (b_1), G_2 (b_1, b_2), \ldots, G_q (b_1, b_2, \ldots, b_q)).$$

(4.11)

Proof of the Main Corollary. Let $f(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_n z^{n+1} + \cdots$.

$$\frac{f''(z)}{f'(z)} = - \sum_{k \geq 1} F_k (2b_1, 3b_2, \ldots, (k+1)b_k) z^{k-1},$$

$$\left( \frac{f''(z)}{f'(z)} \right)' = - \sum_{k \geq 2} (k-1) F_k (2b_1, 3b_2, \ldots, (k+1)b_k) z^{k-2}. \quad (i)$$

On the other hand

$$- \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = - \frac{1}{2z^2} \left( \frac{z f''(z)}{f'(z)} \right)^2$$

$$= - \frac{1}{2z^2} \left( \sum_{k \geq 1} F_k (2b_1, 3b_2, \ldots, (k+1)b_k) z^k + 1 \right)^2.$$
\[= - \frac{1}{2z^2} \left( 1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, z^k) \right) \left( 1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, z^k) \right) - \frac{1}{2z^2} \]
\[= - \frac{1}{2z^2} \sum_{k \geq 2} K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \ldots, z^k) + \frac{1}{z^2} \sum_{k \geq 2} F_k(2b_1, 3b_2, \ldots, z^k). \quad (\text{ii}) \]

We add the two expressions (i) and (ii) to obtain the Main Corollary. \(\square\)

**Remark 4.1.** Let \(H_k(b_1, b_2, \ldots, b_k) = F_k(2b_1, 3b_2, 4b_3, \ldots, (k+1)b_k)\). With the expressions of the \((F_n)_{n \geq 1}\) in [3], we find \(H_1(b_1) = F_1(2b_1) = -2b_1\) and \(H_2(b_1, b_2) = F_2(2b_1, 3b_2) = 2(2b_1^2 - 3b_2)\)
\[H_3(b_1, b_2, b_3) = F_3(2b_1, 3b_2, 4b_3) = 2(-4b_1^3 + 9b_1b_2 - 6b_3),\]
\[H_4(b_1, b_2, b_3, b_4) = F_4(2b_1, 3b_2, 4b_3, 5b_4) = 2(8b_1^4 - 24b_1^2b_2 + 9b_2^2 + 16b_1b_3 - 10b_4).\]

We can calculate \(P_k\) with \(P_k = -(k - 1)H_k - \frac{1}{2} \sum_{j=1}^{k-1} H_{k-j}H_j\) or we can use (C1).

**Proof of (T3).** \(h(z)^p = \sum_{n \geq 0} K_n^p z^n = \left(1 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots\right)^p,\)
\[K_n^p = \frac{1}{2\pi i} \int_{\gamma} \frac{\left(1 + \phi_n(\xi) + b_{n+1}\xi^{n+1} + \cdots\right)^p}{\xi^{n+1}} d\xi = \frac{1}{2\pi i} \int_{\gamma} \frac{\left(1 + \phi_n(\xi)\right)^p}{\xi^{n+1}} d\xi\]
and we write Newton binomial formula \(\left(1 + \phi_n(\xi)\right)^p = 1 + p\phi_n(\xi) + \frac{p(p-1)}{2!} \phi_n(\xi)^2 + \cdots. \quad \square\)

**Proof of (T4).** If \(b_1 \neq 0\), then \(\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n = b_1z(1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1})\). Thus
\[(\phi_n(z))^k = b_1^kz^k \left(1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1}\right)^k = b_1^kz^k \sum_{j=1}^{n-1} K_j^k \left(b_2/b_1, \ldots, b_{j+1}/b_1\right)z^j.\]
The coefficient of \(z^n\) in this expression is obtained for \(j + k = n\). This gives (T4). We can obtain the exact expression of \(K_n^p\) as follows,
\[K_n^p = \sum_{1 \leq k_1 \leq n} C_{k_1}^p D_{n-k_1}^k = \sum_{1 \leq k_1 \leq n} C_{k_1}^p b_1^{k_1} \sum_{1 \leq k_2 \leq n-k_1} C_{k_2}^p \left(b_2/b_1\right)^{k_2} D_{n-k_1}^k \left(b_3/b_2, \ldots\right).\]
If \(b_1 \neq 0,\)
\[K_1^p = C_1^p b_1, \quad K_2^p = C_1^p b_2 + C_2^p b_1^2, \quad K_3^p = C_1^p b_3 + C_2^p b_1^2 K_1^p \left(b_2/b_1\right) + C_3^p b_1^3,\]
\[K_4^p = C_1^p b_4 + C_2^p b_1 b_2^2 K_2^p \left(b_3/b_1, b_4/b_1\right) + C_3^p b_1^3 K_1^p \left(b_3/b_1\right) + C_4^p b_1^4,\]
\[K_5^p = C_1^p b_5 + C_2^p b_1^2 K_3^p \left(b_2/b_1, b_3/b_1, b_4/b_1\right) + C_3^p b_1^3 K_2^p \left(b_3/b_1, b_4/b_1\right) + C_4^p b_1^4 K_1^p \left(b_3/b_1\right) + C_5^p b_1^5,\]
\[K_6^p = C_1^p b_6 + C_2^p b_1^2 K_4^p \left(b_2/b_1, b_3/b_1, b_4/b_1, b_5/b_1\right) + C_3^p b_1^3 K_3^p \left(b_2/b_1, b_3/b_1, b_4/b_1\right) + C_4^p b_1^4 K_2^p \left(b_2/b_1, b_3/b_1\right) + C_5^p b_1^5 K_1^p \left(b_2/b_1\right) + C_6^p b_1^6.\]
If \( b_1 = 0 \) and \( b_2 \neq 0 \),
\[
K_2^p = C_1^p b_2, \quad K_3^p = C_1^p b_3, \quad K_4^p = C_1^p b_4 + C_2^p b_2^2, \\
K_5^p = C_1^p b_5 + C_2^p b_2 K_1^p \left( \frac{b_3}{b_2} \right), \quad K_6^p = C_1^p b_6 + C_2^p b_2 K_2^p \left( \frac{b_3}{b_2}, \frac{b_4}{b_2} \right) + C_3^p b_2^3, \\
K_7^p = C_1^p b_7 + C_2^p b_2^2 K_3^p \left( \frac{b_3}{b_2}, \frac{b_4}{b_2}, \frac{b_5}{b_2} \right) + C_3^p b_2^3 K_1^p \left( \frac{b_3}{b_2} \right). \quad \ldots \quad \square 
\]

5. Identities related to the \((W_j)_{j \geq 1}\), the \((V_j^k)_{j \geq 1}\), \(k \in \mathbb{Z}\) and the \((V_j)_{j \geq 1}\)

It has been proved in [3] that
\[
W_j W_q = W_q W_j \quad \text{for } j \geq 1, \quad q \geq 1. \tag{5.1}
\]
For \( j \geq 1, m \geq 0 \),
\[
W_j F_m = m \delta_{j,m} \quad \text{and} \quad W_j G_m = G_{m-j} \tag{5.2}
\]
with the convention \( G_p = 0 \) if \( p < 0 \). Moreover \( K^p_0 = 1 \) and
\[
W_j (K^p_n) = 0 \quad \text{for } n < j, \quad W_j (K^p_n) = -p K^p_{n-j} \quad \text{for } n \geq j, \quad p \in \mathbb{Z}. \tag{5.3}
\]

**Proof of (5.1)–(5.3).** For (5.1), we remark that \( W_j W_p [h(z)] = z^{p+j} h(z) \). For the other identities, let \((b_1, \ldots, b_k, \ldots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots \), then \( \frac{\partial}{\partial b_j} [h(z)] = z^j \). It is enough to calculate \( W_j [h(z)] \), use (1.1), then calculate \( W_j [\frac{1}{h(z)}] \), use (1.3) by equating coefficients of similar powers of \( z \). This is done as follows. Let
\[
(b_1, b_2, \ldots, b_k, \ldots) \rightarrow \phi(b_1, b_2, \ldots, b_k, \ldots).
\]
Since \( W_j \) is a differential operator, \( W_j [\exp(\phi)] = \exp(\phi) \times W_j [\phi] \). With (1.1), we have \( W_j [h(z)] = h(z) \times (-\sum_{k=1}^{+\infty} \frac{W_j F_k}{k} z^k) \). Comparing this last expression with \( W_j [h(z)] = -z^j h(z) \), we deduce that \( z^j = \sum_{k=1}^{+\infty} \frac{W_j F_k}{k} z^k \). Equating the coefficients of \( z^k \) gives \( W_j (F_m) \). To obtain \( W_j [G_m] \), we calculate \( W_j [\frac{1}{h(z)}] = -\frac{W_j [h(z)]}{h(z)^2} \). With \( W_j [h(z)] = -z^j h(z) \), it gives \( W_j [\frac{1}{h(z)}] = \frac{z^j}{h(z)} \). With (1.2), it gives
\[
\frac{z^j}{h(z)} = \sum_{m \geq 1} W_j G_m z^m. \tag{i}
\]
In (i), we replace \( \frac{1}{h(z)} \) by (1.2), thus \( z^j (\sum_{m \geq 0} G_m z^m) = \sum_{m \geq 1} W_j [G_m] z^m \). In this identity, equating the coefficients of \( z^m \) gives \( W_j (G_m) \). In the same way,
\[
W_j [h(z)^p] = ph(z)^{p-1} W_j [h(z)] = -pz^j h(z)^p = -pz^j \left( 1 + \sum_{n \geq 1} K^p_n z^n \right).
\]
On the other hand, \( W_j [h(z)^p] = \sum_{s \geq 1} W_j [K^p_s] z^s \). Identifying the two expressions of \( W_j [h(z)^p] \), we obtain \(-pz^j (1 + \sum_{n \geq 1} K^p_n z^n) = \sum_{s \geq 1} W_j [K^p_s] z^s \). Equating the coefficients of equal powers of \( z^j \) gives \( W_j (K^p_n) \). \( \square \)
Theorem 5.1. The operators \( (V^k_j)_{j \geq 1}, k \in \mathbb{Z} \) satisfy (1.11),
\[
V^q_k V^p_s + (p + 1)V^p_{s+k} = \sum_{n \geq 0, j \geq 0} K^p_{n-1} J^{q+1}_j \frac{\partial^2}{\partial b_{n+s} \partial b_{k+j}} \tag{5.4}
\]
and \( V^k_j = \sum_{n \geq 0} K^k_n W_{j+n} \)
\[
V^k_j (F_p) = p K^k_p, \quad V^k_j (K^q_s) = -q K^q_{s-j}, \tag{5.5}
\]
\[
V^k_j [h(z)] = -z^j [h(z)]^{k+1}. \tag{5.6}
\]
The polynomials \( (P^k_n)_{n \geq 0}, P^k_0 = 1 \) (see (1.4)) satisfy
\[
V^k_j (P^p_{n+j}) = -(2k + j) P^p_{n-p+j+k} + \frac{2(k - p)(n - p + k + j)}{k + j} K^{k+j}_{n-p} \quad \text{if } k + j \neq 0,
\]
\[
V^{-j}_p (P^p_{n+j}) = j P^p_{n-p} + 2(j + p) F_{n-p},
\]
\[
V^k_j (P^p_{n+j}) = -(2k + j) P^p_{n+j} \tag{5.7}
\]
and the Neretin polynomials (C1), \( z^2 S(f)(z) = \sum_{k \geq 2} P_k z^k \)
\[
V^k_j (P_{j}) = -(k^3 - k) P^j_{j-k}. \tag{5.8}
\]

Proof of (5.7)–(5.8). Since \( V^p_j [h(z)] = -z^p h(z)^{p+1} \), we deduce for \( p \geq 1 \),
\[
V^p_j [f(z)] = -f(z)^{p+1}, \quad V^p_j \left( \frac{f'(z)}{f(z)} \right) = -f(z)^{p-1} f'(z) \tag{5.9}
\]
and \( V^p_j \left( \frac{f''(z)}{f(z)} \right) = -p(p + 1) f(z)^{p-1} f'(z) \). We obtain (5.7)–(5.8) by identification of coefficients. \( \square \)

In [3], the homogeneity operator \( L_0 = b_1 \frac{\partial}{\partial b_1} + 2b_2 \frac{\partial}{\partial b_2} + \cdots + kb_k \frac{\partial}{\partial b_k} + \cdots \) is expressed as \( L_0 = \sum_{j \geq 1} F_j W_j \).

Lemma 5.2. We have \( G_n = K^{-1}_n, b_n = K^1_n \),
\[
kG_k = \sum_{1 \leq j \leq k} F_j G_{k-j} \quad \text{and} \quad n K^p_n = -p \sum_{1 \leq j \leq n} F_j K^p_{n-j}. \tag{5.10}
\]

Proof of (5.10). From the recurrence relation for the polynomials \( (F_k)_{k \geq 0} \),
\[
L_0 = \sum_{k \geq 1} kb_k \frac{\partial}{\partial b_k} = -\sum_{j \geq 1} F_j \left( \sum_{k \geq j} b_{k-j} \frac{\partial}{\partial b_k} \right) = \sum_{j \geq 1} F_j W_j.
\]
\( K^p_n \) is homogeneous of degree \( n \), thus \( L_0 K^p_n = n K^p_n \). Since \( L_0 K^p_n = \sum_{j \geq 1} F_j W_j K^p_n = -\sum_{1 \leq j \leq n} F_j p K^p_{n-j} \), we obtain the recursion formula for \( K^p_n \). See (2.5), (2.7). \( \square \)

Theorem 5.3. The \( (\frac{\partial}{\partial b_j})_{j \geq 1} \) are given in terms of the \( (W_j)_{j \geq 1} \) with
\[
\frac{\partial}{\partial b_j} = -W_j - \sum_{k \geq 1} G_k W_{j+k}, \quad (5.11)
\]

\[
X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots \quad (5.12)
\]

satisfies

\[
X_0 = -\sum_{j \geq 1} G_j W_j. \quad (5.13)
\]

In particular,

\[
X_0(F_n) = -nG_n. \quad (5.14)
\]

**Proof.** For \((\frac{\partial}{\partial b_j})_{j \geq 1}\), it is enough to verify that

\[
\frac{\partial}{\partial b_j} [h(z)] = -W_j[h(z)] - \sum_{k \geq 1} G_k W_{j+k}[h(z)]. \quad (i)
\]

Since \(W_j[h(z)] = -z^j h(z)\), we have

\[
-W_j[h(z)] - \sum_{k \geq 1} G_k W_{j+k}[h(z)] = z^j (1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots) \times h(z).
\]

Thus (i) is the same as 

\[
z^j = z^j (1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots) \times h(z).
\]

It is the immediate consequence of (1.3). To prove (5.13), we see that \(X_0[h(z)] = 1 - h(z)\). We write \(X_0\) as \(X_0 = \sum_{j \geq 1} H_j W_j\). Applied to \(h(z)\), it gives

\[
X_0[h(z)] = 1 - h(z) = -\left[ \sum_{j \geq 1} H_j z^j \right] h(z).
\]

Thus

\[
\sum_{j \geq 1} H_j z^j = \frac{h(z) - 1}{h(z)} = 1 - \frac{1}{h(z)} = 1 - \sum_{n \geq 0} G_n z^n.
\]

By identification of the coefficient of \(z^j\), we find \(H_j = -G_j\). \(\Box\)

**Remark 5.1.** If we calculate \(W_{p+k} F_{n+p}\) with (5.2), we obtain with (5.11) another proof of (1.16),

\[
\frac{\partial}{\partial b_p} F_{n+p} = -W_p F_{n+p} - \sum_{k \geq 1} G_k W_{p+k} F_{n+p} = -(n+p)G_n.
\]

**Theorem 5.4.** For \(k, p \in \mathbb{Z}, j \geq 1\),

\[
V_j^k = \sum_{n \geq 1} K_{n-j}^{k-p} V_n^p \quad \text{and} \quad X_0 = \sum_{n \geq 1} \left[ K_n^{-p} - K_n^{-1-p} \right] V_n^p.
\]

We can also express the \((V_j^k)\) in terms of \((V_j)_{j \geq 1}\) with the inverse function \(f^{-1}(z)\).
6. The composition of differentials on coefficients

6.1. The inverse function

The inverse function is important in the study of coefficients regions, see [11, p. 104]. Also asymptotics of the derivatives of the Faber polynomials are calculated with inverse functions, see [10]. Let

\[ f(w) = wh(w) = w + b_1 w^2 + b_2 w^3 + \cdots + b_n w^{n+1} + \cdots. \]

We denote \( k(z) = f^{-1}(z) \) the inverse of \( f \), we have \((f \circ k)(z) = z\), letting \( w = k(z) \),

\[ \frac{w f'(w)}{f(w)} = 1 + w \frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k) w^k. \]

Theorem 6.1. Let \( f(z) = z + b_1 z^2 + \cdots + b_n z^n + \cdots \). The inverse function of \( f \), \( f^{-1}(f(z)) = z \) is given in terms of the derivatives of the Faber polynomials of \( f(z) \) with

\[ f^{-1}(z) = \frac{1}{2i\pi} \int_{|\xi|=\rho} \frac{\xi f'(\xi)}{f(\xi) - z} \frac{d\xi}{\xi} = \sum_{n \geq 1} \left( \frac{1}{n} \sum_{n \geq 1} \frac{1}{(2n+1)} F_{2n+1}(b_1, b_2, \ldots, b_q, \ldots) \right) z^n. \] (6.1)

Let \( g(z) = z h(\frac{1}{z}) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots + \frac{b_{n+1}}{z^n} + \cdots \), then the inverse function of \( g \) is

\[ g^{-1}(z) = z - b_1 + \sum_{n \geq 1} \frac{1}{n} K_n b_1^n \frac{1}{z^n} \]

\[ = z - b_1 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \frac{1}{(n+1)!} \left[ \frac{\partial^{n+1}}{\partial b_1^n} F_{2n+1}(b_1, b_2, \ldots, b_q, \ldots) \right] \frac{1}{z^n}. \] (6.2)

\[ = z - b_1 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \frac{1}{(n+1)!} \left[ \frac{\partial^n}{\partial b_1^n} F_{2n+1} \right] \left( G_1(b_1), G_2(b_1, b_2), \ldots, G_q(b_1, b_2, \ldots) \right) \frac{1}{z^n}. \] (6.3)
Proof. \( \frac{zf'(z)}{f(z)} = - \sum_{k \geq 0} F_k z^k \) with \( F_0 = -1 \) and \( \frac{z^n}{f(z)^n} = \sum_{p \geq 0} K_p^n z^p \), we deduce that

\[
\frac{zf'(z)}{f(z)} = - \sum_{p \geq 0, k \geq 0} F_k K_p^{-n} z^{p+k-n}.
\]

The residue is obtained for \( p + k - n = -1 \) and is equal to \( - \sum_{0 \leq k \leq n-1} F_k K_{n-k-1}^{-n} \). This is the coefficient of \( z^n \) in the expression of \( f^{-1}(z) \). Thus the coefficient of \( z^{n+1} \) is

\[
K_{n}^{-(n+1)} - \sum_{1 \leq k \leq n} F_k K_{n-k}^{-(n+1)} = K_{n}^{-(n+1)} - \frac{n}{n+1} K_{n}^{-(n+1)}
\]

where we have used the recursion formula (2.7). For example the coefficients of \( z^4 \) is given when \( n = 3 \) by \( \frac{1}{2} K_3^{-4} = 5b_1^2b_2 - b_3 - 5b_1^3 \). The expressions of the coefficients of \( f^{-1}(z) \) in terms of the \( (K_n^k) \) were found in another way in [2, (1.2.8)–(1.2.9)]. For \( g^{-1}(z) \), we use [2, (1.2.8)] and (T2). \( \square \)

Proposition 6.2. We have

\[
\left[ h\left(f^{-1}(z)\right) \right]^p = 1 + \sum_{n \geq 1} \frac{1}{n-1} K_n^{1-n} z^n
\]

with \( f(z) = z h(z) \). Assume that \( p \geq 2 \), then

\[
\left[ h\left(f^{-1}(z)\right) \right]^p = 1 + \sum_{1 \leq n \leq p-1} \frac{p}{p-n} K_n^{p-n} z^n - F_p z^p - \sum_{n \geq p+1} \frac{p}{n-p} K_n^{p-n} z^n.
\]

Assume that \( p \geq 1 \), then

\[
\left[ h\left(f^{-1}(z)\right) \right]^{-p} = 1 + \sum_{n \geq 1} \frac{p}{n+p} K_n^{-(p+n)} z^n.
\]

The function \( \psi(z) = h\left(f^{-1}(z)\right) \) has been considered in [13]. The coefficients of \( [h\left(f^{-1}(z)\right)]^p \), \( p \in Z \) have been given in [2, (1.2.4) and (0.7)].

Proof. By a residue calculus,

\[
\left[ h\left(f^{-1}(z)\right) \right]^p = \sum_{n \geq 0} \left( \sum_{\lvert \xi \rvert = \rho} \int \frac{\xi f'\left(\xi\right)}{f\left(\xi\right)} \frac{1}{\xi^{n+1}} \frac{d\xi}{\xi^2} \right) z^n
\]

Since with (2.7), \( \sum_{1 \leq k \leq n} F_k K_{n-k}^{1-n} = \frac{n}{n-1} K_n^{1-n} \), the coefficient of \( z^n \) in the expansion of \( h\left(f^{-1}(z)\right) \) is given by \( K_n^{1-n} - \frac{n}{n-1} K_n^{1-n} = -\frac{1}{n-1} K_n^{1-n} \). Then we finish the proof as in Theorem 6.1. In the same way, we have

\[
\left[ h\left(f^{-1}(z)\right) \right]^p = \sum_{n \geq 0} \left( \sum_{\lvert \xi \rvert = \rho} \int \frac{\xi f'\left(\xi\right)}{f\left(\xi\right)} \frac{1}{\xi^{n+1}} \frac{d\xi}{\xi^2} \right) z^n.
\]
We have \( \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-p}} = -\sum_{k \geq 0, j \geq 0} F_k K_j^{p-n} \xi^{k+j+p-n} \), the coefficient of \( \xi^p \) in this expression is obtained when \( k + j = n \) and is equal to

\[
- \sum_{0 \leq k \leq n} F_k K_{n-k}^{p-n} = \sum_{1 \leq k \leq n} F_k K_{n-k}^{p-n} = -\frac{p}{n-p} K_n^{p-n} z^n
\]

when \( p \neq n \). If \( p = n \), we take the coefficient \( -F_p \) of \( \xi^p \) in \( \frac{\xi f'(\xi)}{f(\xi)} \).

**Remark 6.1.** Following [13], for any \( p \in \mathbb{Z}, p \neq 0 \),

\[
\frac{[h(f^{-1}(z))^p}{f'(f^{-1}(z))} = \sum_{n \geq 0} K_n^{p-(n+1)} z^n. \tag{6.7}
\]

**Proof.** We have

\[
\frac{[h(f^{-1}(z))^p}{f'(f^{-1}(z))} = \frac{1}{2i\pi} \int \frac{h(\xi)^p}{f(\xi)^n - z} \frac{d\xi}{\xi} = \frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{1 - \frac{z}{f(\xi)}} \frac{d\xi}{\xi} = \sum_{n \geq 0} \left( \frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{\xi^n h(\xi)^n} \frac{d\xi}{\xi} \right) z^n = \sum_{n \geq 0} \frac{1}{2i\pi} \int \frac{h(\xi)^{p-n-1}}{\xi^{n+1}} \frac{d\xi}{\xi^{n+1}} z^n.
\]

Since \( h(\xi)^{p-n-1} = \sum_{j \geq 0} K_j^{p-n-1} \xi^j \xi^{n+1} \), the residue is \( K_n^{p-(n+1)} \).

**Remark 6.2.** With the inverse function, we obtain also expressions of \( P_n^k \) (see Proposition 2.4),

\[
P_n^k(b_1, b_2, \ldots, b_n) = \sum_{0 \leq s \leq n} \frac{k + s}{k} K_s^k(b_1, b_2, \ldots, b_s)
\times K_{n-s}^s \left( \frac{1}{2} K_1(b_1), \frac{1}{3} K_2^{-3}(b_1, b_2), K_3^{-4}, \ldots, \frac{1}{p+1} K_p^{-(p+1)}, \ldots \right). \tag{6.8}
\]

**Proof.** From Proposition 2.3, \( \phi_k(\zeta) = \sum_{n \geq 0} \frac{k+n}{k} K_n^k \zeta^n \). We put \( \zeta = f^{-1}(z) \). From Theorem 6.1,

\[
f^{-1}(z)^n = z^n \sum_{j \geq 0} K_j^p \left( \frac{1}{2} K_1^{-2}(b_1), \frac{1}{3} K_2^{-3}(b_1, b_2), K_3^{-4}, \ldots, \frac{1}{p+1} K_p^{-(p+1)}, \ldots \right) z^j. \tag{6.9}
\]

This gives the first expression of \( P_n^k \).

**Remark 6.3.**

\[
(f^{-1})'(z) = 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \psi_k(z) \quad \text{with} \quad \psi_k(z) = \sum_{p \geq 0} \frac{(2k+p)!}{(k+p)!} D_{k+p}^p z^p. \tag{6.10}
\]

**Proof.** We use (T3).
\[
\frac{d}{dz} f^{-1}(z) = 1 + \sum_{n \geq 1} K_n^{-(n+1)} z^n = 1 + \sum_{1 \leq k \leq n, 1 \leq n} (-1)^k \frac{(n+k)!}{n! k!} D_{n-k}^k z^n
\]
\[
= 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \left( \sum_{n \geq k} \frac{(n+k)!}{n!} D_{n-k}^k z^n \right).
\]

**Remark 6.4.** For \( s \geq 1, p \in \mathbb{Z} \), then
\[
V_p^s \left[ f^{-1}(z) \right] = z^{1+s} \sum_{n \geq 0} K_n^{p-s-(n+1)} z^n = \frac{z^{1+s} [h(f^{-1}(z))]^{p-s}}{f'(f^{-1}(z))}
\]
\[
= \left( \frac{f(\zeta)^{1+s} h(\zeta)^{p-s}}{f'(\zeta)} \right)_{\zeta=f^{-1}(z)}.
\]

Let \( \phi_k(\xi) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\xi)^k = \sum_{n \geq 0} K_n^k \xi^n \), we have
\[
V_p^s[\phi_k(\xi)] = -\xi^s \sum_{n \geq 0} \frac{k(k+n+s)}{n+s} K_n^k \xi^n.
\] (6.11)

### 6.2. Composition of derivations and recurrence formulae

We know (see 1.25) that \( F_n(b_1, b_2, \ldots, b_n) + F_n(G_1, G_2, \ldots, G_n) = 0 \). In the following, we show how the differentiation of this identity yields the recursion formula (2.8) with \( p = -1 \) and \( r = -2 \), i.e. \( K_n^{-1} = \sum_{0 \leq j \leq n} K_j^{-2} K_{n-j}^1 \). Then we prove that it gives a partial differential equation satisfied by the \( (F_n)_{n \geq 1} \).

The differentiation of (1.25) with respect to \( b_k \) gives
\[
\frac{\partial}{\partial b_k} F_n(b_1, b_2, \ldots, b_n) + \sum_{j=1}^{n} \left( \frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \ldots, G_n) \times \frac{\partial G_j}{\partial b_k} (b_1, b_2, \ldots) = 0.
\] (6.12)

We know from (1.16) that \( \frac{\partial F_n}{\partial b_j} (b_1, b_2, \ldots, b_n) = -nG_{n-j} (b_1, b_2, \ldots, b_n) \). This expression calculated at the point \( (G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) \) gives
\[
\left( \frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \ldots, G_n) = -nG_{n-j} (G_1, G_2, \ldots, G_n) = -nb_{n-j}.
\] (6.13)

We replace in (6.12), we obtain
\[
\frac{\partial F_n}{\partial b_k} (b_1, b_2, \ldots, b_n) = \sum_{j=1}^{n-1} nb_{n-j} \frac{\partial G_j}{\partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\] (6.14)

or equivalently
\[
\frac{\partial F_n}{\partial b_k} (b_1, b_2, \ldots, b_n) = \sum_{j=1}^{n-1} nb_j \frac{\partial G_{n-j}}{\partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\]

On the other hand, \( -nG_j = \frac{\partial F_n}{\partial b_{n-j}} \). We replace in (6.14), we obtain
\[
\frac{\partial F_n}{\partial b_k} (b_1, b_2, \ldots, b_n) = \sum_{j=k}^{n-1} b_{n-j} \frac{\partial^2 F_n}{\partial b_{n-j} \partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\] (6.15)
We go back to the expressions of the partial derivatives of $F_n$ in terms of the $K^p_n$ to see that (6.15) is the same as $K^{-1}_n = \sum_{0 \leq j \leq n} K^{-2}_j K^1_{n-j}$.

**Lemma 6.3.**

$$-K^{-2}_{n-k} = \frac{\partial G_n}{\partial b_k} \quad \forall n, k, n \leq k,$$

and

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}} \quad r, s \geq 1, \quad n \geq 1.$$

**Proof.** From (T1). \[ \square \]

**Theorem 6.4.** Let $X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_n \frac{\partial}{\partial b_n} - \cdots$. Then the identity

$$K^{-1}_n = \sum_{0 \leq j \leq n} K^{-2}_j K^1_{n-j}$$

(6.16)

is the same as

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r, s \geq 1, \quad n \geq 1.$$ (6.17)

**Proof.** From (6.15), $\frac{\partial F_n}{\partial b_k} + \sum_{j \geq 1} b_j \frac{\partial^2 F_n}{\partial b_j \partial b_k} = n \frac{\partial G_n}{\partial b_k}$. The left side of this equation is

$$\frac{\partial}{\partial b_k} \left( \sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} F_n \right) = -\frac{\partial}{\partial b_k} (X_0 F_n).$$

On the other hand, $n \frac{\partial G_n}{\partial b_k}$ is given by Lemma 6.3. This proves the theorem. \[ \square \]

**7. First order differential operators on $\mathcal{M}$**

We have seen that the operators $(W_j)_{j \geq 1}, X_0, \frac{\partial}{\partial b_j}$ allow to pass from polynomials $(F_k)_{k \geq 1}$ to polynomials $(G_m)_{m \geq 1}$. In particular, we found $(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}$ and $W_j G_m = G_{m-j}$. Operators $(Z_k)$ in [2] are of this type. In [9], family of vector fields related to the Virasoro algebra have been considered. We found that the operators $(V_k)_{k \geq 1}$ transforms the Neretin polynomials $P_j$ into $-(k^3 - k) P_{j-k}$. In the following, we construct first order differential operators on the manifold $\mathcal{M}$ which permit to pass from one polynomial to the other.

**7.1. The operators $(X_k)_{k \in \mathbb{Z}}$**

The operator $X_0 = -\sum_{j \geq 1} G_j W_j = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_n \frac{\partial}{\partial b_n} - \cdots$ has appeared to be a natural operator on $\mathcal{M}$. We have

$$X_0 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} X_0 = b_p \frac{\partial}{\partial b_p}.$$ (7.1)

On the other hand, $\frac{\partial}{\partial b_k} = -\sum_{j \geq 1} G_j W_{j+k}$. 
Definition 7.1. For $k \geq 1$, we put

$$X_k = \frac{\partial}{\partial b_k} = - \sum_{j \geq 1} G_j W_{j+k} \quad \text{and} \quad X_{-k} = - \sum_{j \geq 1} G_{j+k} W_j. \quad (7.2)$$

Since $W_j = \sum_{p \geq 0} G_p \frac{\partial}{\partial G_{j+p}}$, we deduce

$$X_{-k} = - \sum_{j \geq 1, p \geq 0} G_{j+k} G_p \frac{\partial}{\partial G_{j+p}} = - \left[ \sum_{0 \leq i \leq r-1} G_{r-i+k} G_i \right] \frac{\partial}{\partial G_r}. \quad (7.3)$$

Proposition 7.2. For $n, k \geq 1$, we have $X_0(F_n) = -nG_n$ and

$$X_k(F_n) = -nG_{n-k} \times 1_{k \leq n} \quad \text{and} \quad X_{-k}(F_n) = -nG_{n+k}. \quad (7.4)$$

Proof. From $W_j F_p = p \delta_{j,p}$. □

Remark 7.1. In terms of the coordinates $(b_k)_{k \geq 1}$,

$$X_{-1} = -(b^2_2 - b^2_1) \frac{\partial}{\partial b_1} - (b_3 - b_1 b_2) \frac{\partial}{\partial b_2} - \cdots - (b_{n+1} - b_1 b_n) \frac{\partial}{\partial b_n} - \cdots,$$

$$X_{-2} = -(b^3_3 - 2b_1 b_2 + b_3) \frac{\partial}{\partial b_1} - \cdots - (b^2_2 b_n - b_2 b_n - b_1 b_{n+1} + b_{n+2}) \frac{\partial}{\partial b_n} - \cdots.$$

In terms of the coordinates $(G_k)_{k \geq 1}$,

$$X_0 = -G_1 \frac{\partial}{\partial G_1} - (G_2 + G^2_1) \frac{\partial}{\partial G_2} + (G_n - K^2_n(G_1, G_2, \ldots)) \frac{\partial}{\partial G_n} + \cdots,$$

$$X_{-1} = -G_2 \frac{\partial}{\partial G_1} - (G_2 G_1 + G_3) \frac{\partial}{\partial G_2} - (G^2_2 + G_3 G_1 + G_4) \frac{\partial}{\partial G_3}$$

$$- (G_5 + 2G_2 G_3 + G_1 G_4) \frac{\partial}{\partial G_4} - (G_6 + 2G_4 G_2 + G_1 G_5 + G^2_3) \frac{\partial}{\partial G_5} - \cdots$$

$$= \sum_{n \geq 1} \left( G_{n+1} + G_n G_1 - K^2_{n+1}(G_1, G_2, \ldots, G_n, G_{n+1}) \right) \frac{\partial}{\partial G_n}.$$

For $k \geq 2$, $X_{-k} = \sum_{n \geq 1} H_n \frac{\partial}{\partial G_n}$ with

$$H_n = G_{n+k} + G_{n+k-1} G_1 + G_{n+k-2} G_2 + \cdots + G_n G_k - K^2_{n+k}(G_1, G_2, \ldots, G_{n+k}).$$

From our main theorem, we see that the coefficient $H_n$ is a sum of partial derivatives of Faber polynomials.

Lemma 7.3. The condition $X_{-k}(F_n) = -nG_{n+k}$ for $n \geq 1$ and $k \geq 0$ determines the operators $X_{-k}$ in a unique way. Consider differential operators $(\tilde{X}_{-k})$, $k \geq 0$, of the form

$$\tilde{X}_{-k} = B^1_k \frac{\partial}{\partial b_1} + B^2_k \frac{\partial}{\partial b_2} + \cdots + B^n_k \frac{\partial}{\partial b_n} + \cdots \quad \text{for} \quad k \geq 0,$$

where the $B^n_k$ are homogeneous polynomials in the variables $(b_1, b_2, \ldots, b_n, \ldots)$ of degree $n + k$ and such that $\tilde{X}_{-k}(F_n) = -nG_{n+k}$ for $n \geq 1$, $k \geq 0$, then $\tilde{X}_{-k} = X_{-k}$. 
Moreover, $X_0[h(z)] = -h(z) + 1$, $X_{-1}[h(z)] = -\frac{h(z)}{z} + \frac{1}{z} + b_1 h(z)$,

$$X_{-2}[h(z)] = -\frac{h(z)}{z^2} + \frac{1}{z^2} - \frac{G_1 h(z)}{z} - G_2 h(z),$$

$$X_{-3}[h(z)] = -\frac{h(z)}{z^3} + \frac{1}{z^3} - \frac{G_1 h(z)}{z^2} - \frac{G_2 h(z)}{z} - G_3 h(z),$$

$$\ldots$$

$$X_{-j}[h(z)] = \frac{1}{z^j} - \sum_{0 \leq k \leq j} \frac{G_k}{z^{j-k}} \times h(z).$$

**Proof.** Let $h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$. For $X_0$, the condition $X_0[F_n] = -nG_n$ for $n \geq 1$ implies that

$$X_0 \frac{h'}{h} = -\sum_{k \geq 1} X_0(F_k)z^{k-1} = \sum_{k \geq 1} kG_k z^{k-1} = \frac{d}{dz} \left( \frac{1}{h(z)} \right).$$

Exchanging the order of derivation $X_0$ and $\frac{d}{dz}$, we have $\frac{d}{dz} X_0(h) = \frac{d}{dz} \left( \frac{1}{h} \right)$. Integrating with respect to $z$ gives $X_0(h) = \frac{1}{z} + \alpha$ where $\alpha$ is a constant. If we take $\alpha = -1$, then $X_0(h) = 1 - h$. To express $X_0$ in the $(b_k)$ coordinates, we have

$$X_0[h(z)] = -b_1 z - b_2 z^2 - \cdots - b_n z^n - \cdots = -b_1 \frac{\partial}{\partial b_1} h(z) - b_2 \frac{\partial}{\partial b_2} h(z) - \cdots.$$

To get $X_0$ in terms of the $(G_k)_{k \geq 1}$ coordinates, we consider $\tilde{h}(z) = \frac{1}{h(z)}$. We have

$$X_0[\tilde{h}(z)] = X_0 \left[ \frac{1}{h(z)} \right] = \tilde{h}(z) - \tilde{h}(z)^2 = \sum_{n \geq 1} [G_n - K_n^2(G_1, \ldots, G_k, \ldots)] z^n.$$

Since $z^n = \frac{\partial}{\partial G_n}[\tilde{h}(z)]$, we obtain the result. For $X_{-1}$, the method is the same. From $X_{-1}(F_n) = -nG_{n+1}$ for $n \geq 1$, we deduce

$$X_{-1} \frac{h'}{h} = -\sum_{k \geq 1} X_{-1}(F_k)z^{k-1} = \sum_{k \geq 1} kG_{k+1} z^{k-1}$$

$$= \frac{1}{z} \sum_{k \geq 1} kG_k z^{k-1} - \frac{1}{z^2} \sum_{k \geq 1} G_k z^k = \frac{1}{z} \frac{d}{dz} \left( \frac{1}{h(z)} \right) - \frac{1}{z^2} \left( \frac{1}{h(z)} \right) + \frac{1}{z^2}.$$

Exchanging the order of derivation $X_{-1}$ and $\frac{d}{dz}$, we have

$$\frac{d}{dz} X_{-1}(h) = \frac{d}{dz} \left( \frac{1}{zh(z)} - \frac{1}{z} \right).$$

Integrating with respect to $z$ gives $X_{-1}(h) = \frac{1}{zh(z)} - \frac{1}{z^2} + \text{constant}$. Taking the constant equal to $b_1$ gives $X_{-1}$. In the same way, $X_{-j} F_n = -nG_{n+j}$, for $n \geq 1$ implies that

$$X_{-j} \left( \frac{h'}{h} \right) = -\sum_{k \geq 1} W_{-j} F_k z^{k-1} = \sum_{k \geq 1} kG_{k+j} z^{k-1}$$

$$= \frac{1}{z^j} \sum_{k \geq 1} (k + j) G_{k+j} z^{k+j-1} - \frac{j}{z^{j+1}} \sum_{k \geq 1} G_{k+j} z^{k+j}.$$
\[
\frac{d}{dz} \left( \frac{1}{z^j} h(z) - \frac{1}{z^j} \sum_{0 \leq k \leq j} G_k z^k \right). \quad \square
\]

7.2. The operators \((M_k)_{k \in \mathbb{Z}}\)

We have \(L_0 = \sum_{k \geq 1} k b_k \frac{\partial}{\partial b_k} = \sum_{j \geq 1} F_j W_j\).

**Definition 7.2.** For \(k \geq 1\), let
\[
M_k = \sum_{j \geq 1} F_j W_{j+k} \quad \text{and} \quad M_{-k} = \sum_{j \geq 1} F_{j+k} W_j. \quad (7.5)
\]

**Proposition 7.4.** For \(k \geq 1\), \(M_{-k} F_n = n F_{n+k}\), \(M_k (F_n) = n F_{n-k} \times 1_{n \geq k}\). Moreover \(M_k M_p - M_p M_k = (k - p) M_{k+p}\) for \(p, k \in \mathbb{Z}\).

**Proof.** From \(W_j (F_n) = n \delta_{n,j}\). For the last identity, we verify that \(M_k M_p F_n - M_p M_k F_n = (k - p) M_{k+p} F_n\). \(\square\)

**Lemma 7.5.** In terms of the coordinates \((b_k)_{k \geq 1}\), for \(k \geq 1\),
\[
M_k = \sum_{j \geq 1} F_j W_{j+k} = b_1 \frac{\partial}{\partial b_{k+1}} + 2 b_2 \frac{\partial}{\partial b_{k+2}} + \cdots + p b_p \frac{\partial}{\partial b_{k+p}} + \cdots. \quad (7.6)
\]

In particular if \(h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_p z^p + \cdots\), we have
\[
M_k [h(z)] = z^{k+1} h'(z) \quad \text{for} \ k \geq 1. \quad (7.7)
\]

**Proof.** We verify the identity on \(h(z)\). From \(W_j [h(z)] = -z^j h(z)\). Thus
\[
M_k [h(z)] = -z^k \left( \sum_{j \geq 1} F_j z^j \right) \times h(z) = z^{k+1} h'(z).
\]

Since \((b_1 \frac{\partial}{\partial b_{k+1}} + 2 b_2 \frac{\partial}{\partial b_{k+2}} + \cdots + p b_p \frac{\partial}{\partial b_{k+p}} + \cdots) h(z) = z^{k+1} h'(z)\), we obtain (7.6). For \(k \geq 0\), the operators \(M_{-k} = \sum_{j \geq 1} F_{j+k} W_j\) are given by \(M_0 = L_0\),

\[
M_{-1} = (2 b_2 - b_1^2) \frac{\partial}{\partial b_1} + (3 b_3 - b_1 b_2) \frac{\partial}{\partial b_2} + \cdots + (n+1) b_{n+1} - b_1 b_n \frac{\partial}{\partial b_n} + \cdots,
\]

\[
M_{-2} = \sum_{j \geq 1} [((j+2) b_{j+2} - b_{j+1} b_1 + b_j (b_1^2 - 2 b_2)) \frac{\partial}{\partial b_j}]
\]

\[
\cdots
\]

\[
M_{-k} = \sum_{j \geq 1} (b_j F_k + b_{j+1} F_{k-1} + \cdots + b_{j+k-1} F_1 + (j + k) b_{j+k}) \frac{\partial}{\partial b_j}. \quad \square
\]

**Remark 7.2.** On \(\mathcal{M}\), define the differential operators
\[
L_k = M_k - W_k \quad \text{for} \ k \geq 1. \quad (7.8)
\]
With the convention $F_0 = -1$,

$$L_k = \sum_{j \geq 0} F_j W_{j+k} = F_0 W_k + F_1 W_{k+1} + F_2 W_{k+2} + \cdots$$ (7.9)

then $L_k = M_k - W_k, k \geq 1$, is the Kirillov operator

$$L_k = \frac{\partial}{\partial b_k} + \sum_{n \geq 1} (n + 1)b_n \frac{\partial}{\partial b_{n+k}}. \quad (7.10)$$

For $f(z) = z h(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_p z^{p+1} + \cdots$, we have $L_k[f(z)] = z^{1+k} f'(z)$ and

$$L_k(F_n) = n F_{n-k} \times 1_{n \geq k}. \quad (7.11)$$

7.3. The operators $(V_j)_{j \geq 1}$ and $(V^k_j)_{j \geq 1}$

We do not stay anymore in the class of polynomials $(F_n), (G_n)$. For $j \geq 1$ and $k \in \mathbb{Z}$, see (1.9)–(1.10),

$$V_j = -\sum_{n \geq 0} K_{j+1}^{n} \frac{\partial}{\partial b_{n+j}} \quad \text{and} \quad V^k_j = -\sum_{n \geq 0} K^k_{j+1}^{n} \frac{\partial}{\partial b_{n+j}}. \quad (7.12)$$

The polynomials $(P^k_n)_{n \geq 0}$, see Proposition 2.4 and [1, (A.1.2)], are given by

$$\frac{zf'(z)}{f(z)}h(z)^k = 1 + \sum_{n \geq 1} P^k_n f(z)^n \quad (7.13)$$

where $f(z) = zh(z)$. We have the recursion formulas, for $q \in \mathbb{Z},$

$$(n + 1)b_n = \sum_{0 \leq j \leq n} P^q_j K_{n-j}^{j+1-q}, \quad \frac{n + 1}{k + q} K^k_{n-j} = \sum_{j=0}^{n} P^q_j K^j_{n-j}, \quad (7.14)$$

$$-F_n = \sum_{0 \leq j \leq n} P^q_j K_{n-j}^{j-q}. \quad (7.15)$$

With (7.14), we replace $(n + 1)b_n$ in (7.10). It gives for $L_k, k \geq 1$ (with $b_0 = 1$),

$$L_k = \sum_{0 \leq n, 0 \leq j \leq n} P^q_j K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} = \sum_{0 \leq j} P^q_j \left[ \sum_{n \geq j} K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} \right] = -\sum_{j \geq 0} P^q_j V_{j+k}^{j-q}. \quad (7.16)$$

For $q = -k$ and $k \geq 1$, we obtain

$$L_k = -\sum_{j \geq 0} P^{-k}_j V_{j+k} = -\sum_{j \geq 1} P^{-k}_{j-k} \times 1_{j \geq k} V_j. \quad (7.16)$$

Definition 7.3. For any $k \in \mathbb{Z}$, with the convention $P^k_n = 0$ if $n < 0$, we put

$$L_k = -\sum_{j \geq 1} P^{-k}_{j-k} V_j. \quad (7.17)$$
7.4. The Kirillov operators \((L_{-p})_{p \geq 1}\)

It has been proved in [1, (A.4.5)] that the vector fields \((L_{-p})_{p \geq 0}\) obtained by Kirillov in [7] are such that for \(f(z) = zh(z)\), it holds
\[
L_{-p}[f(z)] = \sum_{j \geq 0} P_{1+j+p}^p f(z)^{j+2}.
\]  
(7.18)

**Proposition 7.6.** Let \(L_{-p}, p \geq 1\) the operator defined by (7.18), then \(L_{-p}\) is given by (7.17), we have
\[
L_{-p} = - \sum_{j \geq 1} P_{j+p}^p \nabla_j.
\]  
(7.19)

**Proof.** From (5.9), \(V_j[f(z)] = -f(z)^{j+1}\). \(\square\)

**Remark 7.3.** We have
\[
L_{-p} = \sum_{r \geq 1} A_r^p \frac{\partial}{\partial b_r} \quad \text{with} \quad A_r^p = \sum_{1 \leq j \leq r} P_{j+p}^p K_{r-j}^{j+1},
\]  
(7.20)

\[
L_{-p} = - \sum_{r \geq 1} B_r^p W_r \quad \text{with} \quad B_r^p = \sum_{1 \leq j \leq r} P_{j+p}^p K_{r-j}^j.
\]  
(7.21)

**Proof.** From (7.18), \(L_{-p} = \sum_{j \geq 0, n \geq 0} P_{1+j+p}^p K_{n}^{j+2} \frac{\partial}{\partial b_{n+j+1}}\). Thus
\[
L_{-p} = \sum_{r \geq 0} A_{r+1}^p \frac{\partial}{\partial b_{r+1}} \quad \text{with} \quad A_{r+1}^p = \sum_{0 \leq j \leq r} P_{1+j+p}^p K_{r-j}^{j+2}.
\]  
This proves (7.20). We obtain (7.21) with (5.11),
\[
L_{-p} = - \sum_{j \geq 0, n \geq 0, k \geq 0} P_{1+j+p}^p K_{n}^{j+2} G_k W_{n+j+k+1}
\]
\[
= - \sum_{j \geq 0, s \geq 0} P_{1+j+p}^p K_{s}^{j+1} W_{j+s+1}. \quad \square
\]

**Remark 7.4.** We have \(L_{-k} = M_{-k} - Y_{-k}\) with
\[
Y_{-k} = \sum_{r \geq 1} J_r^k W_r \quad \text{and} \quad J_r^k = \sum_{s=0}^k P_s^k K_{r+k-s}^{s-k}.
\]  
(7.22)

In particular \(L_{-1} = M_{-1} - X_{-1}\).

**Proof.** From (7.15), \(M_{-k} = \sum_{r \geq 1} F_{j+k} W_j = - \sum_{r \geq 1} \left[ \sum_{0 \leq s \leq j+k} P_s^q K_{k+j-s}^{s-q} \right] W_j\)
\[
\sum_{0 \leq s \leq j+k} P_s^q K_{k+j-s}^{s-q} = \sum_{0 \leq s \leq k} P_s^q K_{k+j-s}^{s-q} + \sum_{1 \leq s \leq j} P_s^q K_{k+j-s}^{k+s-q}.
\]

With \(k = q\), the second sum is \(J_j^k\) as in (7.21). The first sum gives \(Y_{-k}\). \(\square\)
Remark 7.5. With (1.11) and (5.7), we find for any $p \in \mathbb{Z}$, $j \geq 1$,
\[ L_{-p}V_j - V_j L_{-p} = \sum_{1 \leq s \leq j} (V_j (P_{s+p}^n)) V_s. \] (7.23)

8. Second order differential operators

Let $\triangle_0 = \sum_{p \geq 1, q \geq 1} F_{p+q}(W_{p+q} + W_p W_q)$. 

Proposition 8.1. Let $L_0 = \sum_{j \geq 1} F_j W_j$ be the homogeneity operator, then 
\[ \triangle_0 F_n = n(n - 1)F_n \quad \text{and} \quad (\triangle_0 + L_0) F_n = n^2 F_n. \]

Proof. Because of (1.6), $W_p W_q F_n = 0$. On the other hand 
\[ \sum_{p \geq 1, q \geq 1} F_{p+q} W_{p+k} F_n = n \times \left( \sum_{p \geq 1, q \geq 1} \delta_{p+q,n} \right) F_n. \]
Then we remark that $\left( \sum_{p \geq 1, q \geq 1} \delta_{p+q,n} \right) = n - 1$. □

Definition 8.1. We consider $Q_j = \sum_{p \geq 1, q \geq 1, p+q=j} W_p W_q$ for $j \geq 2$. 

Because of (1.8), $\triangle_0$ and $Q_j$ are second order differential operators on the manifold $\mathcal{M}$. With the expression (1.8) of $W_{p+q} + W_p W_q$, we have 
\[ \frac{\partial}{\partial b_j} (W_p W_q + W_{p+q}) - (W_p W_q + W_{p+q}) \frac{\partial}{\partial b_j} = W_p \frac{\partial}{\partial b_{q+j}} + W_q \frac{\partial}{\partial b_{j+p}}. \]

Since $W_p$ and $W_q$ commute, we have $Q_2 = W_1^2 = K_2^0(W_1, 0)$, $Q_3 = 2 W_1 W_2 = K_2^2(W_1, W_2, 0)$, $Q_4 = 2 W_1 W_3 + W_2^2 = K_2^2(W_1, W_2, W_3, 0)$, ..., $Q_n = K_2^n(W_1, W_2, \ldots, W_{n-1}, 0)$ and $Q_j W_p = W_p Q_j$ for $j \geq 2$ and $p \geq 1$.

Since $W_{j-k} W_k G_n = W_{j-p} W_p G_n$, for $k \leq j$, $p \leq j$, we have $Q_2 G_j = G_{j-2}$, $Q_3 G_j = 2G_{j-3}$, ..., $Q_n G_j = (n-1) G_{j-n}$.

The operator $\triangle_0$ decomposes into $\triangle_0 = \triangle_1 + \triangle_2$ with 
\[ \triangle_1 = \sum_{j \geq 2} F_j Q_j, \]
\[ \triangle_2 = \sum_{j \geq 2} \left( \sum_{p \geq 1, q \geq 1, p+q=j} 1 \right) F_j W_j = \sum_{j \geq 2} (j - 1) F_j W_j. \]

Since $W_j F_n = n \delta_{j,n}$, we have $Q_j F_n = 0$, $j \geq 2$ and since $Q_j (G_n) = (j - 1) G_{n-j}$, we find 
\[ \triangle_1 F_n = 0, \]
\[ \triangle_2 F_n = n(n - 1) F_n, \]
and 
\[ \triangle_1 G_n = \triangle_2 G_n = \sum_{j \geq 2} (j - 1) F_j G_{n-j}. \]

We deduce that
\[ \Delta_1 Q_j = Q_j \Delta_1, \quad W_j \Delta_1 - \Delta_1 W_j = j Q_j, \]
\[ \Delta_2 Q_j = Q_j \Delta_2, \quad W_j \Delta_2 - \Delta_2 W_j = j(j - 1) W_j, \]
and \[ \Delta_2 \Delta_1 - \Delta_1 \Delta_2 = \sum_{k \geq 2} k(k - 1) F_k Q_k. \]

**Lemma 8.2.** Let \( X_0 = - \sum_{j \geq 1} G_j W_j \) and \( L_0 = \sum_{j \geq 1} F_j W_j \), then
\[ X_0 G_n = G_n - K_n^{2}(G_1, G_2, \ldots, G_n), \quad X_0 F_n = - n G_n, \tag{8.1} \]
\[ X_0 L_0 = L_0 X_0, \tag{8.2} \]
\[ L_0 \Delta_2 = \Delta_2 L_0 \quad \text{and} \quad L_0 \Delta_1 - \Delta_1 L_0 = \sum_{k \geq 2} k F_k Q_k. \tag{8.3} \]

**Proof.** (8.1) results from the expression of \( X_0 \) in the \( (G_n)_{n \geq 1} \) coordinates. Because of (1.6), we have
\[ L_0 X_0 = \sum_{q \geq 1} \left[ \sum_{j \geq 1} F_j G_{q-j} \right] W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q \]
and
\[ X_0 L_0 = \sum_{q \geq 1} q G_q W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q. \]
Since \( \sum_{j \geq 1} F_j G_{q-j} = q G_q \), see (2.5), it proves that \( X_0 L_0 = L_0 X_0 \). The identities (8.3) are consequence of (1.6). \( \Box \)

9. The conformal map from the exterior of the unit disk onto the exterior of \([-2, +2]\)

Let \( \psi(w) = w + \frac{1}{w} \) be the conformal map from the exterior of the unit disk onto the exterior of \([-2, 2]\). The Faber polynomials \( F_n(z) \) of \([-2, 2]\) are given by
\[ \frac{w^2 - 1}{w^2 - wz + 1} = \sum_{n=0}^{\infty} F_n(z) w^{-n}. \]
They satisfy the differential equation
\[ (z^2 - 4) F''_n(z) + z F'_n(z) = n^2 F_n(z). \tag{9.1} \]
In the following, we consider Faber polynomials \( F_n(b_1, b_2, 0, 0, \ldots, 0) \). All the \( b_j \) are zero except \( b_1 \) and \( b_2 \). We have \( F_1(b_1) = -b_1 \), \( F_2(b_1, b_2) = b_1^2 - 2 b_2 \), \( F_3(b_1, b_2, 0) = b_3^3 + 3 b_1 b_2 \), \( F_4(b_1, b_2, 0, 0) = b_4^4 - 4 b_1^2 b_2 + 2 b_2^2 \), . . .

**Theorem 9.1.** Faber polynomials associated to \( \psi(w) = w + b_1 + \frac{b_2}{w} \) verify
\[ ((z - b_1)^2 - 4 b_2) F''_n(z) + (z - b_1) F'_n(z) = n^2 F_n(z). \tag{9.2} \]
In particular, if \( b_1 = 0 \) and \( b_2 = 1 \), we obtain (9.1).

To prove the theorem, we need the following lemmas.

**Lemma 9.2.** Let \( L = \frac{\partial^2}{\partial^2 b_1} + \frac{\partial}{\partial b_2} + \sum_{k \geq 1} b_k \frac{\partial^2}{\partial b_2 \partial b_k} \), then \( LF_n = 0. \)

**Proof.** From (6.15). \( \Box \)
Lemma 9.3. Consider $\Delta_0 = \sum_{p \geq 1, q \geq 1} F_{p+q}(W_{p+q} + W_p W_q)$ and let $\phi(b_1, b_2)$ be a function defined on $\mathcal{M}$, which depends only of $b_1, b_2$, then

$$\Delta_0 \phi = \left[ (b_1^2 - 2b_2) \frac{\partial^2}{\partial b_1^2} + 2b_1b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_1^2} \right] \phi.$$

Proof. Let $\phi$ depend only on the variables $b_1$ and $b_2$. If $p > 2$ or $q > 2$, we have $[W_{p+q} + W_p W_q] \phi = 0$. If $p = 2$, $q = 2$, then $(W_4 + W_2^2) \phi = W_2^2 \phi = \frac{\partial^2}{\partial b_2^2} \phi$. If $p = 2$, $q = 1$ or $p = 1$, $q = 2$, $(W_3 + W_2 W_1) \phi = \frac{\partial}{\partial b_2} (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2}) \phi$. If $p = 1$, $q = 1$, then $(W_2 + W_1^2) \phi = [-\frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2})]^2 \phi$.

We calculate $\Delta_0 \phi = [F_2(W_2 + W_1^2) + 2F_3(W_3 + W_2 W_1) + F_4(W_4 + W_2^2)] \phi$. This gives

$$\Delta_0 \phi = \left[ F_2 \left( -\frac{\partial}{\partial b_2} + \left( \frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2} \right)^2 \right) + 2F_3 \left( \frac{\partial}{\partial b_2} \left( \frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2} \right) + F_4 \frac{\partial^2}{\partial b_2^2} \right) \right] \phi$$

or equivalently

$$\Delta_0 \phi = \left[ F_2 \left( \frac{\partial^2}{\partial b_1^2} + 2b_1 \frac{\partial^2}{\partial b_1 \partial b_2} + b_2^2 \frac{\partial^2}{\partial b_2^2} \right) + 2F_3 \left( \frac{\partial^2}{\partial b_1 \partial b_2} + b_1 \frac{\partial^2}{\partial b_2^2} \right) + F_4 \frac{\partial^2}{\partial b_2^2} \right] \phi.$$

Replacing $F_2$, $F_3$, $F_4$, we find Lemma 9.3. □

Proof of the theorem. From Lemma 9.2, we know that

$$\left( 2b_1b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_2^2} \right) F_n = \left( -2b_2 \frac{\partial}{\partial b_2} - 2b_2 \frac{\partial^2}{\partial b_2^2} \right) F_n.$$

We replace the right hand side in the expression of $\Delta_0$ and we find

$$\left[ (b_1^2 - 4b_2) \frac{\partial^2}{\partial b_1^2} - 2b_2 \frac{\partial}{\partial b_2} \right] F_n = n(n - 1) F_n. \quad (9.3)$$

Since $F_n$ is homogeneous, $b_1 \frac{\partial}{\partial b_1} F_n + 2b_2 \frac{\partial}{\partial b_2} F_n = n F_n$. Replacing $-2b_2 \frac{\partial}{\partial b_2} F_n$ in (9.2), we find

$$(b_1^2 - 4b_2) \frac{\partial^2}{\partial b_1^2} + b_1 \frac{\partial F_n}{\partial b_1} = n^2 F_n. \quad □$$

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References


