Higher-order recurrences for Bernoulli numbers✩

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Abstract

Euler’s well-known nonlinear relation for Bernoulli numbers, which can be written in symbolic notation as $(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n$, is extended to $(B_{k_1} + \ldots + B_{k_m})^n$ for $m \geq 2$ and arbitrary fixed integers $k_1, \ldots, k_m \geq 0$. In the general case we prove an existence theorem for Euler-type formulas, and for $m = 3$ we obtain explicit expressions. This extends the authors’ previous work for $m = 2$.

1. Introduction

The Bernoulli numbers $B_n$, $n = 0, 1, 2, \ldots$, can be defined by the generating function

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi.
$$

(1.1)

The first few values are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for all odd $n \geq 3$; we also have $(-1)^{n+1}B_{2n} > 0$ for all $n \geq 1$. These and many other properties can be found, for instance, in [1], [12], [14], or [18]; for a comprehensive bibliography, see [9].

One of the most remarkable identities for the Bernoulli numbers is Euler’s formula

$$
\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n \quad (n \geq 1),
$$

(1.2)

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which can be considered a convolution identity, or also a quadratic recurrence relation. This identity has been extended in various directions; see [3] for a summary with numerous references.

It will be convenient to use the symbolic notation (or “classical umbral calculus”; see, e.g., [10]) to write

\[(B_k + B_l)_n = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} B_{l+n-j}, \quad (1.3)\]

so that Euler’s formula takes the form

\[(B_0 + B_0)_n = -nB_{n-1} - (n - 1)B_n, \quad n \geq 1. \quad (1.4)\]

For the case \(k_1 = \cdots = k_m = 0\) a variant of the problem (with even positive indices \(i_j\) and even \(n\)) was settled by the second author [8], with analogous results for Euler numbers and Bernoulli and Euler polynomials. Further extensions and analogues were subsequently obtained by other authors; see [7,13,15–17].

It is the purpose of this paper to deal with the sums (1.4) in general. Since this can be considered a continuation of our previous paper [3], we will quote several auxiliary results from there.

Our main result, stated in Section 2, will be the existence of an Euler-type formula in the most general case. In Section 3 we show how the coefficients in this main result can be determined by computation. Furthermore, if the parameters \(k_1, \ldots, k_m\) are large enough (greater than \(m - 1\)), then we will be able to explicitly state the leading coefficient in the expansion; this will be done in Section 4. In Section 5 we indicate how to obtain formulas for all triples \((k_1, k_2, k_3)\) in the case \(m = 3\), and give explicit expressions when \(k_1 = k_2 = k_3\).

2. The existence result

Before we state the first and most general result of this paper, we introduce some notation. Let \(m \geq 2\) be an integer, and \(K := (k_1, \ldots, k_m)\) a vector of \(m\) nonnegative integers. Furthermore, we set \(s_m := k_1 + \cdots + k_m\).

**Theorem 1.** With notation as above, we have for all integers \(n \geq m - 1,\)

\[(B_{k_1} + \cdots + B_{k_m})_n = \sum_{v=-m+1}^{s_m} C^K_v (n) B_{n+v}, \quad (2.1)\]

where the polynomials \(C^K_v (x)\) have rational coefficients, depend only on the vector \(K\) (and not on \(n\)), are recursively computable, and \(\deg (C^K_v (x)) \leq m - 1\) for all \(v\). Furthermore, \(C^K_{-m+1} (n)\) vanishes unless \(k_1 = \cdots = k_m = 0\), in which case

\[C^K_{-m+1} (n) = (-1)^{n-1} \frac{n!}{(n-m+1)!} = (-1)^{m-1} n(n-1) \cdots (n - m + 2). \quad (2.2)\]

To begin the proof of this result we use the generating function

\[\frac{d^k}{dx^k} \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n+k} \frac{x^n}{n!}, \quad (2.3)\]
which follows directly from (1.1). By taking the Cauchy product of \( m \) instances of this generating function, with \( k \) replaced by \( k_1, \ldots, k_m \), respectively, we get with the definition (1.4),

\[
(B_{k_1} + \cdots + B_{k_m})^n = \left[ \frac{d^n}{dx^n} \prod_{j=1}^{m} \left( \frac{d^{k_j} x}{d^{k_j} e^x - 1} \right) \right]_{x=0}.
\] (2.4)

The right-hand side of (2.4) now motivates the following auxiliary result.

**Lemma 1.** Let \( m \geq 1 \) and \( k_1, \ldots, k_m \) be nonnegative integers.

(a) There exists a unique and recursively computable sequence of polynomials \( A_j(x) \in \mathbb{Q}[x] \) with \( \deg(A_j(x)) \leq m - 1, \ j = 0, 1, \ldots, s_m + m - 1 \), such that

\[
\prod_{j=1}^{m} \left( \frac{d^{k_j} x}{d^{k_j} e^x - 1} \right) = \sum_{j=0}^{s_m+m-1} A_j(x) \frac{d^j x}{d^j e^x - 1}.
\] (2.5)

(b) If we set \( A_j(x) = a_{j,m-1} x^{m-1} + a_{j,m-2} x^{m-2} + \cdots + a_{j,1} x + a_{j,0} \), then \( a_{j,i} = 0 \) whenever \( j - i > s_m \).

(c) We have \( a_{0,m-1} = 0 \) unless \( k_1 = \cdots = k_m = 0 \), in which case \( a_{0,m-1} = (-1)^{m-1} \).

For the proof of Lemma 1, and also for Section 5, we need an explicit result from [3] which we quote here as a lemma, in a somewhat simplified form. (The corresponding result in [3] includes \( k, l = 0 \).)

**Lemma 2.** Let \( k \) and \( l \) be positive integers, and set

\[
\left( \frac{d^k x}{d^k e^x - 1} \right) \left( \frac{d^l x}{d^l e^x - 1} \right) = \sum_{j=0}^{k+l+1} A_{j}^{k+l}(x) \frac{d^j x}{d^j e^x - 1}.
\] (2.6)

with \( A_j^{k,l}(x) = b_{j,k,l} x + b_{j,k,0} \). Then

\[
b_{j,k,l}^{k,l} = \begin{cases} 
(-1)^l \left( (-1)^k \binom{k}{j} + (-1)^l \binom{l}{j} \right) \frac{b_{k+j+l+1-j}}{(k+l+1)}, & 0 \leq j \leq k + l, \\
- \frac{k!}{(k+l+1)!}, & j = k + l + 1;
\end{cases}
\] (2.7)

\[
b_{j,k,0}^{k,l} = \begin{cases} 
(-1)^l \left( (-1)^k \binom{k}{j} + (-1)^l \binom{l}{j} \right) \frac{b_{k+j+l+1-j}}{(k+l+1)!}, & 0 \leq j \leq k + l - 1, \\
- \frac{k!}{(k+l)!}, & j = k + l, \\
0, & j = k + l + 1.
\end{cases}
\] (2.8)

The identity (2.7) also holds when \( k = 0 \) or \( l = 0 \).

**Proof of Lemma 1.** We prove this lemma by induction on \( m \). (a) For \( m = 1 \) the statement is trivial. For the remainder of the proof we indicate the dependence of \( A_j(x) \) on the \( k_1, \ldots, k_m \) by superscripts. The case \( m = 2 \) is immediate from Lemma 2. Now we suppose that (2.5) holds up to some \( m \), and we multiply both sides of (2.5) by

\[
\frac{d^{k_{m+1}} x}{d^{k_{m+1}} e^x - 1}.
\]

By using the result for \( m = 2 \) we get

\[
\left( \frac{d^{k_{m+1}} x}{d^{k_{m+1}} e^x - 1} \right) \left( \frac{d^m x}{d^m e^x - 1} \right) = \sum_{j=0}^{s_{m}+m-1} A_{j}^{m+1}(x) \frac{d^j x}{d^j e^x - 1}.
\]
\[ \prod_{j=1}^{m+1} \left( \frac{d^{kj}}{dx^j} x \right) = \sum_{j=0}^{s_m+m-1} A_j^{k_1,\ldots,k_m}(x) \left( \frac{d^j}{dx^j} e^x - 1 \right) \left( \frac{d^{km+1}}{dx^{km+1}} e^x - 1 \right) \]
\[ = \sum_{j=0}^{s_m+m-1} A_j^{k_1,\ldots,k_m}(x) \sum_{v=0}^{j+k_m+1} A_v^{j,k_m+1}(x) \frac{d^v}{dx^v} e^x \]
\[ = \sum_{v=0}^{s_m+1+m} \sum_{j=0}^{s_m+m-1} A_v^{j,k_m+1}(x) A_j^{k_1,\ldots,k_m}(x) \frac{d^v}{dx^v} e^x. \]

where by convention we take \( A_j^{k_1,\ldots,k_m}(x) \) to be the zero polynomial for \( j < 0 \). Now the inner summation on the right-hand side is the sum of products of polynomials with rational coefficients and of degrees at most 1 and at most \( m-1 \), respectively. Hence the inner sum is a polynomial of degree at most \( m \), with rational coefficients, and is recursively computable. The uniqueness of the polynomials \( A_j(x) \) also follows from this induction.

(b) The case \( m = 1 \) is trivially true, while the statement for \( m = 2 \) follows from (2.8). Suppose now that the statement holds for some \( m \geq 2 \), and consider

\[ A_v^{k_1,\ldots,k_m}(x) = \sum_{j=v-k_m+1}^{s_m+m-1} A_v^{j,k_m+1}(x) A_j^{k_1,\ldots,k_m}(x), \tag{2.9} \]

for \( v = 0, 1, \ldots, s_{m+1} \). Obviously it suffices to prove the statement for each summand in (2.9). So fix \( j, 0 \leq j \leq s_m+m-1 \), write \( A_j^{k_1,\ldots,k_m}(x) \) as in Lemma 1(b) and set, to simplify notation, \( A_v^{j,k_m+1}(x) = b_{v,1} x + b_{v,0} \). Then

\[ A_v^{j,k_m+1}(x) A_j^{k_1,\ldots,k_m}(x) = \sum_{i=0}^{m} (b_{v,1} a_{j,i-1} + b_{v,0} a_{j,i}) x^i. \tag{2.10} \]

where by convention we assume \( a_{j,-1} = a_{j,m} = 0 \). Consider now the \( i \)th coefficient in (2.10). The first summand, namely \( b_{v,1} a_{j,i-1} \), vanishes by hypothesis if \( v - 1 > j + k_{m+1} \) (in which case \( b_{v,1} = 0 \)) or \( j - (i-1) > s_m \) (in which case \( a_{j,i-1} = 0 \)). If we now add these two inequalities, we get \( v - i > k_{m+1} + s_m = s_{m+1} \); this means that at least one of the original inequalities must hold if we assume that \( v - i < s_{m+1} \). Similarly, the second summand, namely \( b_{v,0} a_{j,i} \), vanishes if \( v > j + k_{m+1} \) or \( j - i > s_m \). Again, one of these two inequalities must hold if \( v - i > s_{m+1} \). This completes the proof of part (b).

(c) The case \( m = 1 \) is again trivial, and the statement for \( m = 2 \) follows from (2.7) which gives

\[ b_{0,1}^{k,l} \left[ (-1)^k + (-1)^l \right] B_{k+l+1} = \frac{B_{k+l+1}}{k+l+1}. \tag{2.11} \]

We now see that when \( k \) and \( l \) have different parities, then \( (-1)^k + (-1)^l = 0 \); if \( k, l \) have the same parity then \( B_{k+l+1} = 0 \) since odd-index Bernoulli numbers vanish, with the only exception \( B_1 = -1/2, \) so that \( b_{0,1}^{0,0} = -1 \).

Now consider (2.9) and (2.10), and let \( \tilde{a}_{0,m} \) be the coefficient of \( x^m \) in the polynomial \( A_0^{k_1,\ldots,k_{m+1}}(x) \). Then we have

\[ \tilde{a}_{0,m} = \sum_{j=0}^{s_m+m-1} b_{0,1}^{j,k_{m+1}} a_{j,m-1}. \]
Table 1
The polynomials $A_j(x)$ for $K = (1, 2, 3)$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{j,2}$</td>
<td>$\frac{1}{1320}$</td>
<td>0</td>
<td>$\frac{1}{770}$</td>
<td>0</td>
<td>$\frac{1}{1440}$</td>
<td>0</td>
<td>$\frac{1}{2720}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$a_{j,1}$</td>
<td>0</td>
<td>$\frac{1}{2170}$</td>
<td>$\frac{1}{1770}$</td>
<td>$\frac{1}{1720}$</td>
<td>0</td>
<td>$\frac{1}{77}$</td>
<td>$\frac{1}{1720}$</td>
<td>$\frac{1}{1770}$</td>
<td>0</td>
</tr>
<tr>
<td>$a_{j,0}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{1320}$</td>
<td>$\frac{1}{20}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{12}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence by (2.11) we have $\tilde{a}_{0,m} = 0$ unless $k_{m+1} = 0$ and $a_{0,m-1} \neq 0$. Therefore by induction we have $\tilde{a}_{0,m} = 0$ unless all $k_j = 0$, $j = 1, \ldots, m + 1$, in which case $\tilde{a}_{0,m} = (-1)^m$. This completes the proof. □

Remark. An alternative proof of the uniqueness of the polynomials $A_j(x)$ rests on the linear independence of the power series $(x/(e^x - 1))^r$ over the field $\mathbb{Q}(x)$. Indeed, if we have a linear relation

$$\sum_{j=0}^n f_j(x) \frac{x^j}{(e^x - 1)^j} = 0$$

with $f_n(x) \neq 0$, then there exists a positive integer $k$ with $f_n(2\pi ki) \neq 0$. But this means that the left-hand side of the above equation has a pole of order $n$, which is a contradiction.

We also remark that part (b) in Lemma 1 explains the lower left triangle of zeros in Table 1, and part (c) accounts for the zero in the upper right-hand corner of the table. (In general there will not be a larger triangle of zeros in that corner.)

Proof of Theorem 1, continued. With (2.4) and (2.5) we get

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{j=0}^{s_m+m-1} \sum_{i=0}^{m-1} a_{j,i} \left[ d^n \left( \frac{x^j}{d^i x^i} e^x - 1 \right) \right]_{x=0}.$$  (2.12)

Now by Leibniz’s rule for higher derivatives of a product we have

$$\left[ d^n \left( \frac{x^j}{d^i x^i} e^x - 1 \right) \right]_{x=0} = \sum_{k=0}^{n} \binom{n}{k} \left[ \frac{d^k}{d^i x^i} e^x \right] \left( \frac{d^{i+n-k}}{d^i x^i} e^x - 1 \right)_{x=0} = \binom{n}{i} \frac{n!}{(n-i)!} B_{j+n-i},$$

where we have used (2.3) in the last step. Thus, with (2.12) and upon changing the order of summation, we get

$$(B_{k_1} + \cdots + B_{k_m})^n = \sum_{j=0}^{s_m+m-1} \sum_{i=0}^{m-1} \frac{n!}{(n-i)!} B_{j+n-i} a_{j,i} \left( \sum_{\nu=-m+1}^{s_m+m-1} \sum_{i=0}^{m-1} a_{\nu+i,i} \frac{n!}{(n-i)!} B_{n+i} \right).$$  (2.13)

with the convention that $a_{j,i} = 0$ for $j < 0$. Now the inner sum on the right-hand side is clearly a polynomial in $n$ of degree at most $m - 1$, with rational coefficients that are computable by Lemma 1. We denote this polynomial by $C^K_\nu(n)$, that is,
\[ C^{K}_{\nu}(n) = \sum_{i=0}^{m-1} a_{\nu+i,i} \frac{n!}{(n-i)!}. \] (2.14)

Since by the last part of Lemma 1 we have \( a_{\nu+i,i} = 0 \) whenever \( \nu > s_{m} \), the sum in (2.1) goes only up to \( s_{m} \).

Finally, by (2.14) we have \( C^{K}_{-m+1}(n) = a_{0,m-1}n!/(n-m+1)! \), and so the last assertion of Theorem 1 follows from Lemma 1(c). This completes the proof. \( \square \)

3. Connections with Stirling numbers

In this section we use some basic properties of Stirling numbers of the second kind to derive a simpler and practically feasible recurrence relation for the polynomials \( A_{j}(x) \) in (2.5). Some useful properties and references for the Stirling numbers of the second kind, \( S(n,k) \), can be found in [3]. As we did in [4] and (in a different notation) in [3], we define the linear polynomial

\[ T(n,j) := (j-1)!S(n+1,j)x - nS(n,j) \quad (j \geq 1). \] (3.1)

The main connection with Bernoulli numbers is then given by the following expansion, which was proved in [3] and is also used in [2].

Lemma 3. For any \( m \geq 0 \) we have

\[ \frac{d^{m}}{dx^{m}} \frac{x}{e^{x} - 1} = (-1)^{m} \sum_{j=1}^{m+1} \frac{T(m,j)}{(e^{x} - 1)^{j}}. \] (3.2)

While the proof of Lemma 1 allows us, in principle, to compute the polynomials \( A_{j}(x) \), and thus also the \( C^{K}_{\nu}(n) \), this would be rather cumbersome in practice. The main significance of Lemma 1 lies in the fact that it shows us that the \( A_{j}(x) \) are polynomials over \( \mathbb{Q} \) of degree at most \( m-1 \). This is used in the following result which will lead to easier computations.

Theorem 2. Let \( m \geq 1 \) and \( k_{1}, \ldots, k_{m} \) be nonnegative integers. Then for each \( r = 1, 2, \ldots, s_{m} + m \) we have polynomials \( A_{j}(x) \) of degree at most \( m-1 \), with

\[ \sum_{i_1 + \cdots + i_m = r} \prod_{j=1}^{m} T(k_j, i_j) = \sum_{j=r-1}^{s_{m}+m-1} (-1)^{s_{m}+m-j} A_{j}(x) T(j, r), \] (3.3)

and the \( A_{j}(x) \) are the same as in Lemma 1.

Proof. With (3.2) we get

\[ \prod_{j=1}^{m} \left( \frac{d^{k_j}}{dx^{k_j}} \frac{x}{e^{x} - 1} \right) = (-1)^{s_{m}} \prod_{j=1}^{m} \left( \sum_{i=1}^{k_j+1} \frac{T(k_j, i)}{(e^{x} - 1)^{i}} \right) \]

\[ = (-1)^{s_{m}} \sum_{r=1}^{s_{m}+m} \left( \sum_{i_1 + \cdots + i_m = r} \prod_{j=1}^{m} T(k_j, i_j) \right) \frac{1}{(e^{x} - 1)^{r}}, \] (3.4)
where the inner sum on the right is empty, and thus zero, for \( r < m \). On the other hand, we have from (2.5) and (3.2),

\[
\prod_{j=1}^{m} \left( \frac{dk_j}{d x^j} \frac{x}{e^x - 1} \right) = \sum_{j=0}^{s_m+m-1} A_j(x)(-1)^j \sum_{r=1}^{j+1} T(j, r) \left( e^x - 1 \right)^{-r} = \sum_{r=1}^{s_m+m} \left( \sum_{j=r-1}^{s_m+m-1} (-1)^j A_j(x) T(j, r) \right) \frac{1}{(e^x - 1)^r}. \tag{3.5}
\]

Since the functions \((e^x - 1)^{-r}, r = 1, 2, \ldots, s_m + m\), are linearly independent over \( \mathbb{Q}(x) \) (see the Remark following the proof of Lemma 1), we immediately get (3.3) from comparing the right-hand sides of (3.4) and (3.5). \( \square \)

If we set \( r = s_m + m \) in (3.3), the only nonzero term on the left corresponds to \( i_j = k_j + 1 \) for \( j = 1, 2, \ldots, m \). Then we use the fact that \( S(n, n) = 1 \) and \( S(n, k) = 0 \) for \( k > n \), then with (3.1) we get

\[
A_{s_m+m-1}(x) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + m - 1)!} x^{m-1}. \tag{3.6}
\]

We can now use this as the beginning of a recurrence relation for the \( A_j(x) \); just rewrite (3.3) as

\[
(-1)^{s_m-r-1}(r-1)! x A_{r+1}(x) = \sum_{i_1, \ldots, i_m \in \mathbb{N}} \prod_{j=1}^{m} T(k_j, i_j) + \sum_{j=r}^{s_m+m-1} (-1)^{s_m-j-1} A_j(x) T(j, r). \tag{3.7}
\]

This can be used as a "downwards" recursion, successively for \( r = s_m + m - 1, s_m + m - 2, \ldots, 1 \). Computations are facilitated through the fact that major computer algebra systems, such as Maple or Mathematica, have the Stirling numbers (of both kinds) as built-in functions.

As an example we take \( k_1 = 1, k_2 = 2, k_3 = 3 \). If we set, as in Lemma 1(b), \( A_j(x) = a_{j,2} x^2 + a_{j,1,1} x + a_{j,0} \) \((j = 0, 1, \ldots, 8)\), then (3.6) leads to the column for \( j = 8 \) in Table 1 (note that \( 1!2!3!/(6 + 3 - 1)! = 1/3360 \)), and (3.7) gives all the successive columns.

In (2.14) we saw how the polynomials \( C^k_v(n) \) are related to the \( A_j(x) \). Here it reduces to

\[
C^{(1,2,3)}(n) = a_{v+2,2} n(n - 1) + a_{v+1,1} n + a_{v,0}, \quad v = -2, -1, \ldots, 6. \tag{3.8}
\]

Thus the highest term in (2.1) for \( K = (1, 2, 3) \) is

\[
\left( \frac{1}{3360} n(n - 1) + \frac{1}{210} n + \frac{1}{60} \right) B_{n+6} = \frac{(n+8)(n+7)}{3360} B_{n+6}.
\]

For the other terms, and for other parameter vectors \( K \), see Corollary 1 below.

Corollary 1. For all \( n \geq 2 \) we have

\[
(B_1 + B_1 + B_1)^n = \frac{(n+5)(n+4)}{120} B_{n+3} + \frac{n+3}{4} B_{n+2} - \frac{n^2 - n - 24}{24} B_{n+1} - \frac{n-1}{4} B_n + \frac{n(n-2)}{30} B_{n-1};
\]
is the following result, proved in [4]; see also [6].

\[ (B_1 + B_1 + B_2)^n = \frac{(n+6)(n+5)}{360} B_{n+4} + \frac{n+4}{12} B_{n+3} - \frac{n^2 + n - 24}{72} B_{n+2} \]

\[ - \frac{n}{12} B_{n+1} + \frac{(n+1)(n-1)}{90} B_n; \]

\[ (B_1 + B_2 + B_2)^n = \frac{(n+7)(n+6)}{1260} B_{n+5} + \frac{n+5}{60} B_{n+4} - \frac{(n+5)(n+4)}{360} B_{n+3} \]

\[ - \frac{1}{6} B_{n+2} - \frac{n(n-13)}{360} B_{n+1} - \frac{n-1}{60} B_n + \frac{n(n-2)}{210} B_{n-1}; \]

\[ (B_1 + B_1 + B_3)^n = \frac{(n+7)(n+6)}{840} B_{n+5} + \frac{n+5}{20} B_{n+4} - \frac{n^2 - n + 50}{120} B_{n+3} \]

\[ - \frac{n-3}{12} B_{n+2} + \frac{n(n-3)}{60} B_{n+1} + \frac{n-1}{30} B_n - \frac{n(n-2)}{105} B_{n-1}; \]

\[ (B_2 + B_2 + B_2)^n = \frac{(n+8)(n+7)}{5040} B_{n+6} + \frac{n+5}{60} B_{n+4} - \frac{n^2 - n - 32}{240} B_{n+2} \]

\[ + \frac{(5n+26)(n-1)}{1260} B_n; \]

\[ (B_1 + B_2 + B_3)^n = \frac{(n+8)(n+7)}{3360} B_{n+6} + \frac{n+6}{120} B_{n+5} - \frac{(n+5)}{720} B_{n+4} - \frac{1}{12} B_{n+3} \]

\[ + \frac{(6n-19)(n+18)}{6840} B_{n+2} - \frac{1}{120} B_{n+1} + \frac{(n-1)(n-20)}{2520} B_n; \]

\[ (B_1 + B_1 + B_4)^n = \frac{(n+8)(n+7)}{1680} B_{n+6} + \frac{n+6}{30} B_{n+5} - \frac{n^2 - n - 75}{180} B_{n+4} - \frac{n-4}{12} B_{n+3} \]

\[ + \frac{11n^2 - 31n + 48}{720} B_{n+2} + \frac{n}{20} B_{n+1} - \frac{(13n-8)(n-1)}{1260} B_n. \]

**Proof.** Use (3.6) and (3.7) to create the equivalent of Table 1 for each parameter vector \( K \). Then use (3.8) to compute the coefficients \( C_{\nu}^K(n) \) in (2.1). \( \square \)

4. The leading coefficient

In this short section we use a certain convolution formula for Stirling numbers of the second kind, proved elsewhere, to find an explicit expression for the leading coefficient \( C_{\nu}^K(n) \) in the expansion (2.1).

If we were to set up Table 1 for \( K = (2, 2, 2) \), we would find that \( a_{8,2} = 1/5040 \), \( a_{7,1} = 1/315 \), and \( a_{6,0} = 1/90 \), with all the other coefficients for \( j = 6, 7, 8 \) vanishing. This is actually true in general: If \( k_1, \ldots, k_m \) are sufficiently large then in addition to (3.6) there are explicit formulas for \( A_{s_m}(x) \), \( A_{s_m+1}(x) \), \ldots, \( A_{s_m+m-2}(x) \) as monomials.

**Theorem 3.** Let \( m \geq 2 \) and \( k_j \geq m - 1 \) for \( j = 1, \ldots, m \). Then we have

\[ A_{s_m+m-v}(x) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + m - v)!} \frac{(m-1)}{(v-1)!} x^{m-v} \]  

(4.1)

for \( v = 1, 2, \ldots, m \).

The identity (3.6) is obviously a special case of (4.1). The main ingredient in the proof of Theorem 3 is the following result, proved in [4]; see also [6].
Lemma 4. Let $m \geq 2$ and $k_j \geq m - 1$ for $j = 1, \ldots, m$. Then for all integers $r \geq s_m + 1$ we have
\[
\sum_{i_1 + \cdots + i_m = r} \prod_{j=1}^{m} \frac{T(k_j, i_j)}{k_j!} = \sum_{v=1}^{s_m + m - 1} (-1)^{v-1} \binom{m-1}{v-1} \frac{T(s_m + m - v, r)}{(s_m + m - v)!} \chi^{m-v}.
\] (4.2)

Proof of Theorem 3. Using the uniqueness of $A_j(x)$ and changing the order of summation on the right-hand side of (3.3), we see that (4.1) follows immediately from (3.3) and (4.2).

If we use the notation of Lemma 1(b) and set $i := m - v$, then we get from (4.1) for $i = 0, 1, \ldots, m - 1$,
\[
a_{s_m + i, i} = (-1)^{m-i} \frac{k_1! \cdots k_m!}{(s_m + i)!} \binom{m-1}{i}.
\] (4.3)
This, substituted into (2.14), gives
\[
C^K_{s_m}(n) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + n)!} \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{(s_m + i)! (n-i)!} = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + n)!} \sum_{i=0}^{m-1} \binom{m-1}{i} \binom{s_m + n}{n-i}.
\]
The sum on the right-hand side has the explicit evaluation \((\frac{s_m + m - 1 + n}{n})\). This follows from a variant of the well-known "Vandermonde convolution"; see, e.g., identity (3.4) in [11], or (5.27) in [12, p. 170]. Hence we have
\[
C^K_{s_m}(n) = (-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + n)!} \frac{(s_m + m - 1 + n)!}{(s_m + m - 1)!},
\]
which proves the following result.

Corollary 2. Let $m \geq 2$ and $k_j \geq m - 1$ for $j = 1, \ldots, m$. Then the leading term in the expansion (2.1) is
\[
(-1)^{m-1} \frac{k_1! \cdots k_m!}{(s_m + m - 1)!} \frac{(s_m + m - 1 + n)!}{(s_m + m - 1)!} B_{n+s_m}.
\] (4.4)

We see that (4.4) is consistent with Corollary 1 for $K = (2, 2, 2)$, and with the list of specific expansions for $m = 2$ and $k_1 \geq 1, k_2 \geq 1$ given in [3, Corollary 2.4]. However, it appears that (4.4) remains true for all the other special cases listed in Corollary 1. Thus, while computations show that (4.1) is not valid for $v = m$ unless $k_j \geq m - 1$ for all $j$, it appears that (4.3) remains valid for $i = 0$. We will not consider this possible improvement here.

5. The case $m = 3$

The proof of Lemma 1 indicates that in general the determination of the polynomial $A(x)$ in (2.5) through iterating (2.9) would be very cumbersome, and we cannot expect reasonable closed expressions. However, it is still possible to find explicit expressions for $m = 3$, generalizing those listed in Corollary 1.
We rewrite (2.9) for \( \nu = 0, 1, \ldots, k_1 + k_2 + k_3 + 2 \):

\[
A_{\nu}^{k_1,k_2,k_3}(x) = \sum_{j=\nu-k_3-1}^{k_1+k_2+1} A_{\nu}^{j,k_3}(x) A_j^{k_1,k_2}(x)
\]

\[
= \sum_{j=\nu-k_3-1}^{k_1+k_2+1} (a_{\nu}^{j,k_3,x} + b_{\nu}^{j,k_3})(a_j^{k_1,k_2,x} + b_j^{k_1,k_2}).
\]  

(5.1)

We can now combine this with Lemma 2 and (2.14) to obtain an expression for (2.1) for any triple \( K = (k_1, k_2, k_3) \). For instance, in this way we obtain the following special formulas which supplement Corollary 1.

**Corollary 3.** For all \( n \geq 2 \) we have

\[
(B_0 + B_0 + B_0)^n = \frac{(n-1)(n-2)}{2} B_n + 3 \frac{n(n-2)}{2} B_{n-1} + n(n-1) B_{n-2},
\]

\[
(B_0 + B_0 + B_1)^n = \frac{n(n-1)}{6} B_{n+1} + \frac{(n-1)(n+1)}{2} B_n + \frac{n(n+1)}{3} B_{n-1},
\]

\[
(B_0 + B_1 + B_1)^n = \frac{n(n+3)}{24} B_{n+2} + \frac{n(n+8)}{12} B_{n+1} - \frac{n^2 - 19n - 6}{24} B_n - \frac{n(n-2)}{12} B_{n-1},
\]

\[
(B_0 + B_0 + B_2)^n = \frac{n(n-1)}{12} B_{n+2} + \frac{n(n-1)}{3} B_{n+1} + \frac{(5n-2)(n-1)}{12} B_n + \frac{n(n-2)}{6} B_{n-1}.
\]

While in general the method just outlined is not a very satisfactory result, in the special case \( k_1 = k_2 = k_3 \) an explicit general formula can be obtained. We set \( k := k_1 = k_2 = k_3 \), so that \( K = (k,k,k) \).

**Theorem 4.** For all \( k \geq 1 \) and \( n \geq 2 \) we have with \( K = (k,k,k) \),

\[
(B_k + B_k + B_k)^n = \sum_{j=-1}^{3k} C_j^K(n) B_{n+j},
\]  

(5.2)

where

\[
C_{3k}^K(n) = \frac{k!}{(3k+2)!} (n + 3k + 1)(n + 3k + 2),
\]

(5.3)

\[
C_j^K(n) = 0, \quad 2k + 1 \leq j \leq 3k - 1,
\]

(5.4)

\[
C_j^K(n) = \frac{3(-1)^j k!^2}{(j+2)!((2k-j)!)} (n + (j+1))((2k-j)n - (j+2)k) \frac{B_{3k-j}}{3k-j}
\]

\[+ \frac{3}{k+1} \frac{(k+1)}{(j+1)} \sum_{i=j}^{k} \frac{(k-j)}{(i-j)} \left[ n(n-1)(k-i) \frac{i-j}{j+2} \right. 
\]

\[\left. - nk(k-j) + k^2(j+1) \right] \frac{B_{2k-i} B_{k-j+i}}{(2k-i)(k-j+i)}, \quad -1 \leq j \leq 2k,
\]

(5.5)

where for \( j \geq k + 1 \) the summation on the right is considered to be 0.
The proof rests on (5.1) and Lemma 2, and Theorem 1 in [5] is also used. We skip the details which are long and tedious.

Finally, we also list two more specific expansions, directly obtained from Theorem 4. They supplement Corollaries 1 and 3.

**Corollary 4.** For all \( n \geq 2 \) we have

\[
(B_3 + B_3 + B_3)^n = \frac{(n + 11)(n + 10)}{184800} B_{n+9} + \frac{(n + 6)(n - 21)}{5600} B_{n+5} - \frac{(n + 4)(3n - 15)}{1680} B_{n+3} + \frac{n(n - 1)}{300} B_{n+1} - \frac{n(n - 2)}{1155} B_{n-1};
\]

\[
(B_4 + B_4 + B_4)^n = \frac{(n + 14)(n + 13)}{6306300} B_{n+12} + \frac{n + 9}{6300} B_{n+8} + \frac{(n + 7)(n - 16)}{5880} B_{n+6} - \frac{n^2 - n - 32}{600} B_{n+4} + \frac{63n^2 + 65n - 768}{13860} B_{n+2} - \frac{(n - 1)(437n + 1646)}{143325} B_n.
\]

We have used the computer algebra system Maple to check and verify the expansions in Corollaries 1, 3 and 4.

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**References**


