A classical umbral view of the Riordan group
and related Sheffer sequences

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Abstract

The Riordan group is the set of infinite lower triangular invertible matrices with the group operation given by a matrix multiplication that combines both the usual Cauchy product and the composition of formal power series. It is related to a broad family of polynomial sequences in one variable called Sheffer sequences. Riordan arrays and Sheffer sequences have various applications in Combinatorics, Analysis, Probability, Physics, etc.

In this talk I will present an enlightening symbolic treatment of the Riordan group and related Sheffer sequences based on a renewed approach to umbral calculus initiated by Gian Carlo Rota in the 90’s and further developed by Di Nardo and Senato in the first decade of the present century.
Based on joint work with Ângela Mestre (CELC), Pasquale Petrullo (Università degli studi della Basilicata) and Maria Manuel Torres (CELC).
Contents

1. The exponential Riordan group
   - Definitions, examples and the fundamental theorem

2. Classical umbral calculus
   - Umbrae, generating functions, the dot operation and more

3. An umbral view of the Riordan group and Sheffer sequences
   - Definitions, fundamental theorem and Sheffer umbrae
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Let \( g \) and \( f \) be two formal exponential series; namely

\[
g(z) = 1 + g_1 z + g_2 \frac{z^2}{2!} + g_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad f(z) = f_1 z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots \quad (1)
\]

An (exponential) **Riordan array** is an infinite lower triangular matrix \( \mathcal{M} = (m_{n,k})_{n,k \geq 0} \), whose entries are generated by \( g \) and \( f \) as follows

\[
m_{n,k} = \left[ \frac{z^n}{n!} \right] \left( g(z) \frac{f(z)^k}{k!} \right) \quad \text{for} \quad n, k \geq 0.
\]

We shall write \( \mathcal{M} = (g(z), f(z)) = (g, f) \).
Some examples

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<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td><strong>Pascal array</strong></td>
<td>$e^z$, $z$</td>
<td>$g(z)$, $z$</td>
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<tr>
<td>$1$</td>
<td></td>
<td>$1$</td>
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<tr>
<td>1 1</td>
<td></td>
<td>$g_1$ 1</td>
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<tr>
<td>1 2 1</td>
<td></td>
<td>$g_2 2g_1$ 1</td>
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<tr>
<td>1 3 3 1</td>
<td></td>
<td>$g_3 3g_2 3g_1$ 1</td>
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<td>1 4 6 4 1</td>
<td></td>
<td>$g_4 4g_3 6g_2 4g_1$ 1</td>
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<tr>
<td>1 5 10 10 5 1</td>
<td></td>
<td>$g_5 5g_4 10g_3 10g_2 5g_1$ 1</td>
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<td>...</td>
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</tbody>
</table>

| **Appell array**        |             |                        |
| $1$                    |             | $1$                    |
| $g_1$ 1                |             | $g_1$ 1                |
| $g_2 2g_1$ 1          |             | $g_2 2g_1$ 1          |
| $g_3 3g_2 3g_1$ 1     |             | $g_3 3g_2 3g_1$ 1     |
| $g_4 4g_3 6g_2 4g_1$ 1|             | $g_4 4g_3 6g_2 4g_1$ 1|
| $g_5 5g_4 10g_3 10g_2 5g_1$ 1|   | $g_5 5g_4 10g_3 10g_2 5g_1$ 1|
| ...                    |             | ...                   |

| **Stirling array of 1st. kind** | $1$, $\log(1 + z)$ |                        |
| $1$                        |             | $1$                    |
| 0 1                       |             | $0 1$                  |
| 0 $-1$ 1                  |             | $0 1$                  |
| 0 2 $-3$ 1                |             | $0 1$                  |
| 0 $-6$ 11 $-6$ 1          |             | $0 1$                  |
| 0 24 $-50$ 35 $-10$ 1     |             | $0 1$                  |
| ...                       |             | ...                   |

| **Stirling array of 2nd. kind** | $1$, $e^z - 1$ |                        |
| $1$                          |             | $1$                    |
| 0 1                         |             | $0 1$                  |
| 0 1 1                       |             | $0 1$                  |
| 0 1 3 1                     |             | $0 1$                  |
| 0 1 7 6 1                   |             | $0 1$                  |
| 0 1 15 25 10 1              |             | $0 1$                  |
| ...                         |             | ...                   |
Entries of a general Riordan array for $0 \leq n, k \leq 3$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
g_1 & f_1 & 0 & 0 \\
g_2 & f_2 + 2f_1 g_1 & f_1^2 & 0 \\
g_3 & f_3 + 3f_2 g_1 + 3f_1 g_2 & 3f_1^2 g_1 + 3f_1 f_2 & f_1^3 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$
Let $A$ and $B$ be two exponential generating functions; that is

$$A(z) = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots$$

and

$$B(z) = b_0 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$$

and let $(g(z), f(z))$ be a Riordan array. Then

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \iff g(z) A(f(z)) = B(z).$$

Note that the composition $A(f(z))$ is well defined since $f$ has zero constant term.
### Example (FTRA in action)

Stirling numbers of 2nd. kind $S(n, j)$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
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<tr>
<td>0</td>
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<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
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</tr>
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<td>0</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
<td></td>
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<tr>
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</tr>
</tbody>
</table>

$$
\frac{1}{1-e^z} = B(z) = \sum_{n=0}^{\infty} B_n e^n = \sum_{n=0}^{\infty} S(n, j) e^j
$$

Bell numbers $B_n$

$$
\sum_{j=0}^{n} S(n, j) = B_n .
$$

$$(1, e^z - 1) \cdot e^z = 1 \cdot e^{[e^z-1]} = e^{[e^z-1]} = B(z) .$$
The (exponential) Riordan group

Let \((g, f)\) and \((h, l)\) be given as in (1). Consider the \textit{multiplication}

\[
\left( g(z), f(z) \right) \left( h(z), l(z) \right) = \left( g(z) h(f(z)), l(f(z)) \right).
\]  

(2)

The Riordan array \((1, z)\) is the \textit{identity} element with respect to (2). Since \(g_0 \neq 0\), it follows that \(g\) has multiplicative inverse \(g^{-1}\). Also, if \(f_1 \neq 0\), then \(f\) has compositional inverse \(f^{-1}\); that is, \(f(f^{-1}(z)) = f^{-1}(f(z)) = z\). In this case, a Riordan array \((g, f)\) is invertible with respect to (2) and its \textit{inverse} is given by

\[
(g(z), f(z))^{-1} = \left( \frac{1}{g(f^{-1}(z))}, f^{-1}(z) \right).
\]

The \textit{Riordan group \(R\)} is the set of all invertible Riordan arrays, together with multiplication (2) as the group operation.
Some distinguished Riordan subgroups

1. **The Appell subgroup**: \( \{(g(z), z)\} \).

2. **The Associated subgroup**: \( \{(1, f(z))\} \).

3. **The Bell subgroup**: \( \{(g(z), zg(z))\} \).

4. **The Stochastic subgroup**:

\[
\begin{align*}
\left\{(g(z), rz) \in \text{Rio} \right| (g(z), f(z)) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \right\}.
\end{align*}
\]
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The *magic trick*

The Bell numbers $B_n$ are the coefficients in the Taylor series expansion of $e^{[e^z - 1]}$; namely

$$e^{[e^z - 1]} = 1 + \sum_{n=1}^{\infty} B_n \frac{z^n}{n!} \simeq 1 + \sum_{n=1}^{\infty} B_n \frac{z^n}{n!} = e^{Bz}$$

The symbol $\simeq$ stresses out the purely formal character of this manipulation.
The classical umbral calculus basic data

1. a commutative integral domain \( R \) with identity 1. \( R = \mathbb{C}[x, y] \).

2. a set \( A = \{\alpha, \beta, \gamma, \ldots\} \) of umbrae, called *alphabet*.

3. a linear functional \( E : R[A] \rightarrow R \) called *evaluation* such that
   - \( E[1] = 1 \) and \( p \in R[A] \) is called umbral polynomial.
   - \( E[x^n y^m \alpha^i \beta^j \cdots \gamma^k] = x^n y^m E[\alpha^i]E[\beta^j] \cdots E[\gamma^k] \) (*uncorrelation*)

4. two special umbrae: \( \varepsilon \) (*augmentation*) and \( \upsilon \) (*unity*) such that
   \[ E[\varepsilon^n] = \delta_{0,n} \quad \text{and} \quad E[\upsilon^n] = 1, \]
   for all \( n \geq 0 \).
**Similarity and generating functions (g.f.)**

1. **α represents** a sequence \((a_n)_{n \geq 1}\) if \(E[\alpha^n] = a_n\) for all \(n \geq 1\). We say that \(a_n\) is the \(n\)-th **moment** of \(\alpha\). Assume \(a_0 = 1\).

2. **Umbral equivalence:** \(\alpha \simeq \gamma \iff E[\alpha] = E[\gamma]\).

3. **Similarity:** \(\alpha \equiv \gamma \iff E[\alpha^n] = E[\gamma^n], \forall n \geq 0\).

4. The **generating function** of \(\alpha\) is the exponential formal series

\[ e^{\alpha z} := 1 + \sum_{n \geq 1} \alpha^n \frac{z^n}{n!} \in R[A][[z]], \]

so that \(E[e^{\alpha z}] = 1 + \sum_{n \geq 1} a_n \frac{z^n}{n!} =: f_\alpha(z) \in R[[z]]\).

We shall write \(e^{\alpha z} \simeq f_\alpha(z)\). We have \(\alpha \equiv \gamma \iff e^{\alpha z} \simeq e^{\gamma z}\).
### Some distinguished umbrae

<table>
<thead>
<tr>
<th>name</th>
<th>$\alpha$</th>
<th>$e^{\alpha z} \simeq f_\alpha(z)$</th>
<th>$E[\alpha^n] = a_n \ (n \geq 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>augmentation</td>
<td>$\varepsilon$</td>
<td>1</td>
<td>$\delta_{0,n}$</td>
</tr>
<tr>
<td>Bernoulli</td>
<td>$\iota$</td>
<td>$\frac{z}{e^z-1}$</td>
<td>$b_n \ (n$-th Bernoulli number)</td>
</tr>
<tr>
<td>unity</td>
<td>$\nu$</td>
<td>$e^z$</td>
<td>1</td>
</tr>
<tr>
<td>singleton</td>
<td>$\chi$</td>
<td>$1+z$</td>
<td>$\delta_{0,n}, \ n = 0, 1$</td>
</tr>
<tr>
<td>Bell</td>
<td>$\beta$</td>
<td>$e^{[e^z-1]}$</td>
<td>$B_n \ (n$-th Bell number)</td>
</tr>
</tbody>
</table>
Let \((a_n)_{n \geq 1}\) and \((l_n)_{n \geq 1}\) be two arbitrary sequences in \(R\) represented by umbrae \(\alpha\) and \(\lambda\) respectively. Then, the umbra \(\alpha + \lambda\) has moments

\[
E[(\alpha + \lambda)^n] = \sum_{j=0}^{n} \binom{n}{j} E[\alpha^j \lambda^{n-j}] = \sum_{j=0}^{n} \binom{n}{j} a_j l_{n-j}.
\]

Thus, the umbra \(\alpha + \lambda\) represents the sequence

\[
\sum_{j=0}^{n} \binom{n}{j} a_j l_{n-j}.
\]
The handling of sequences of binomial type

Let \((a_n)_{n\geq 1}\) and \((l_n)_{n\geq 1}\) be two arbitrary sequences in \(R\) represented by umbrae \(\alpha\) and \(\lambda\) respectively. Then, the umbra \(\alpha + \lambda\) has moments

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\]

uncorrelation

Thus, the umbra \(\alpha + \lambda\) represents the sequence \(\sum_{j=0}^{n} \binom{n}{j} a_j l_{n-j}\).

Suppose now that \((l_n) = (a_n)\). Does \(\alpha + \alpha\) represent the sequence

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\sum_{j=0}^{n} \binom{n}{j} a_j a_{n-j}.
\]
The handling of sequences of binomial type

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\]

Thus, the umbra \(\alpha + \lambda\) represents the sequence \(\sum_{j=0}^{n} \binom{n}{j} a_j l_{n-j}\).

Suppose now that \((l_n) = (a_n)\). Does \(\alpha + \alpha\) represent the sequence \(\sum_{j=0}^{n} \binom{n}{j} a_j a_{n-j}\)? No, since \(E[(\alpha + \alpha)^n] = E[(2\alpha)^n] = 2^n a_n\).
Auxiliary umbrae

Key feature: Each sequence \((a_n)_{n \geq 1}\) in \(R\) can be represented by infinitely many similar (auxiliary) umbrae. This fact is called saturation. The alphabet \(A\) will contain all possible auxiliary umbrae.
The dot operations

Let $k$ be a nonnegative integer and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be $k$ uncorrelated umbra similar to $\alpha$ (with moments $a_i$). The dot-product $k \cdot \alpha$ is an auxiliary umbra defined to satisfy $k \cdot \alpha \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Two umbrae $\alpha$ and $\lambda$ are said to be inverse to each other (with respect to addition) when $\alpha + \lambda \equiv \varepsilon$. We shall write $\lambda \equiv -1 \cdot \alpha$.

We define $-k \cdot \alpha \equiv -1 \cdot \alpha_1 + \cdots + -1 \cdot \alpha_k$. Also, we set $0 \cdot \alpha \equiv \varepsilon$.

The dot-power $\alpha^k$ is an auxiliary umbra such that $\alpha^k \equiv \alpha_1 \alpha_2 \cdots \alpha_k$. We assume $\alpha^0 \equiv \nu$. We have $E[(\alpha^k)^n] = a_n^k$ for all $n \geq 0$. 

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For any $k \in \mathbb{Z}$, we can write

$$e^{(k \cdot \alpha)z} = \nu + \sum_{n=1}^{\infty} (k \cdot \alpha)^n \frac{z^n}{n!} \sim f_{k \cdot \alpha}(z) = [f_{\alpha}(z)]^k = e^{k \log f_{\alpha}(z)}.$$

In general, for any umbra $\gamma$ (with moments $g_i$), the auxiliary umbra $\gamma \cdot \alpha$ is defined to satisfy

$$e^{(\gamma \cdot \alpha)z} \sim f_{\gamma \cdot \alpha}(z) := f_{\gamma}(\log f_{\alpha}(z)) = 1 + g_1 \log f_{\alpha}(z) + g_2 \frac{[\log f_{\alpha}(z)]^2}{2!} + \cdots$$

Similarly, for $k \geq 0$, we have

$$e^{(\alpha \cdot k)z} \sim f_{\alpha \cdot k}(z) := 1 + \sum_{n=1}^{\infty} a_n^k \frac{z^n}{n!}.$$
The role played by the Bell umbra $\beta$ \textsuperscript{[di Nardo & Senato 2001]}

Recall that the g.f. of the Bell umbra is $e^{\beta z} \simeq e^{[e^z-1]}$. Hence, the umbra $\beta \cdot \gamma$ has g.f.

$$
e^{(\beta \cdot \gamma) z} \simeq f_{\beta}(\log f_{\gamma}(z)) = e^{[e^{\log f_{\gamma}(z)} - 1]} = e^{[f_{\gamma}(z) - 1]}.$$

The composition umbra of $\alpha$ and $\gamma$ is the umbra $\alpha \cdot \beta \cdot \gamma$ with g.f.

$$
e^{(\alpha \cdot \beta \cdot \gamma) z} \simeq f_{\alpha}(\log f_{\beta}(\log f_{\gamma}(z))) = f_{\alpha}(f_{\gamma}(z) - 1).$$

If $\gamma \cdot \beta \cdot \alpha \equiv \chi \equiv \alpha \cdot \beta \cdot \gamma$, we say that $\gamma$ is the compositional inverse of $\alpha$ and vice versa. We shall write $\gamma = \alpha^{<-1>}$.

Note that $E[\alpha] \neq 0$ for $\alpha^{<-1>}$ to exist.
Useful identities and dictionary

1. \[ \alpha \cdot \varepsilon \equiv \varepsilon \equiv \varepsilon \cdot \alpha. \]
2. \[ \beta \cdot \chi \equiv \upsilon \equiv \chi \cdot \beta. \]
3. \[ \alpha \eta \equiv \eta \alpha. \]
4. \[ r\alpha \equiv \alpha \cdot (r\upsilon). \] (In general, note that \[ r \cdot \alpha \equiv r\upsilon \cdot \alpha \not\equiv \alpha \cdot r\upsilon \equiv r\alpha \])

<table>
<thead>
<tr>
<th>Umbrae</th>
<th>formal power series</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha + \eta )</td>
<td>Cauchy product</td>
</tr>
<tr>
<td>( \alpha \eta )</td>
<td>Hadamard product</td>
</tr>
<tr>
<td>( \alpha \dot{+} \eta )</td>
<td>usual addition</td>
</tr>
<tr>
<td>( \alpha \cdot \beta \cdot \gamma )</td>
<td>formal composition</td>
</tr>
</tbody>
</table>
(\(A, +, \cdot\)) is a left distributive algebra

1. \((A, +)\) is an abelian group.

\[
\alpha + \eta = \eta + \alpha
\]
\[
(\alpha + \eta) + \gamma = \alpha + (\eta + \gamma)
\]
and
\[
\alpha + \varepsilon = \alpha \equiv \varepsilon + \alpha
\]
\[
\alpha + (-1 \cdot \alpha) = \varepsilon \equiv (-1 \cdot \alpha) + \alpha
\]

2. \((A, \cdot)\) is a monoid.

\[
\alpha \cdot (\eta \cdot \gamma) = (\alpha \cdot \eta) \cdot \gamma
\]
and
\[
\alpha \cdot \upsilon = \alpha \equiv \upsilon \cdot \alpha
\]

3. The scalar product.

\[
1 \alpha = \alpha
\]
\[
r(\alpha + \eta) = r\alpha + r\eta
\]
and
\[
r(s\alpha) = (rs)\alpha
\]
\[
(r + s)\alpha = r\alpha + s\alpha
\]

4. The left distributive laws.

\[
(\alpha + \eta) \cdot \gamma = \alpha \cdot \gamma + \eta \cdot \gamma
\]
and
\[
\alpha \cdot (r\eta) = r(\alpha \cdot \eta)
\]
\[
\gamma \cdot (\alpha + \eta) \neq \gamma \cdot \alpha + \gamma \cdot \eta
\]
and
\[
\alpha \cdot (r\eta) \neq (r\alpha) \cdot \eta
\]
Let $r \in R$. The \textit{constant} umbra $\varsigma_r$ has moments $E[\varsigma_r^n] = r$ for all $n \geq 1$. We have

1. $\varsigma_0 \equiv \varepsilon$ and $\varsigma_1 \equiv \nu$.

2. $E[(\varsigma_r \alpha)^n] = r a_n$ while $E[(r \alpha)^n] = r^n a_n$.

3. $\varsigma_r \varsigma_s \equiv \varsigma_s \varsigma_r \equiv \varsigma_{rs}$ for any $r, s \in R$.

4. $\varsigma_r \cdot \alpha \equiv \varsigma_r \alpha \equiv \alpha \varsigma_r$ while $\alpha \cdot \varsigma_r \not\equiv \alpha \varsigma_r$.

5. $e^{(\varsigma_r \alpha)z} \sim 1 + r(f_\alpha(z) - 1)$ while $e^{(r \alpha)z} \sim f_\alpha(rz)$.

In fact, we have $\varsigma_r \equiv \chi \cdot r \cdot \beta \cdot \alpha$. 

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Classical umbral view of Riordan gp. and related Sheffer seq.  
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Let $\alpha$ be an umbra with moments $a_i$. The \textit{derivative} umbra $\alpha_D$ of $\alpha$ is the umbra whose powers satisfy $\alpha^n_D \simeq n\alpha^{n-1}$ for $n \geq 1$. In particular $E[\alpha_D] = 1$ and this implies that $\alpha_D$ has compositional inverse $\alpha_{D}^{-1}$.

The g.f. of $\alpha_D$ satisfy $e^{\alpha_D z} \simeq 1 + z e^{\alpha z}$.

The \textit{primitive} umbra $\alpha_P$ of $\alpha$ is the umbra whose powers satisfy $\alpha^n_P \simeq \frac{\alpha^{n+1}}{a_1(n+1)}$ for $n \geq 0$. Therefore $a_1 \neq 0$, so that only umbrae with compositional inverse have primitive umbra.

The g.f. of $\alpha_P$ satisfy $e^{\alpha z} \simeq 1 + a_1 z e^{\alpha_P z}$.
Straightforward identities

**Lemma**

Let $\alpha \in A$ be any umbra and let $r \in R$ be a nonzero scalar. Then

1. $(\alpha_D)_P \equiv \alpha$. In addition, if $a_1 = E[\alpha] \neq 0$ then $(\alpha_P)_D \equiv \varsigma_{1/a_1} \alpha$.

2. $(\varsigma_r \alpha)_P \equiv \alpha_P$, $(r \alpha)_P \equiv r \alpha_P$, $(\varsigma_r \alpha)_D \equiv \varsigma_r \alpha_D$ and $(r \alpha)_D \equiv \varsigma_{1/r} (r \alpha_D)$.

**Theorem (A.M.P.T. 2010)**

Let $\alpha$ and $\gamma$ be two umbrae with first moments $a_1, g_1 \neq 0$. Then

$$(\alpha \cdot \beta \cdot \gamma)_P \equiv \gamma_P + \alpha_P \cdot \beta \cdot \gamma.$$
Applications of Corollary

Taking $\gamma = \alpha^{<-1>}$ yields:

1. $\chi \equiv \alpha \cdot \beta \cdot \alpha^{<-1>} \equiv ((\alpha^{<-1>})_p + \alpha_p \cdot \beta \cdot \alpha^{<-1>})_D$,
2. $\varepsilon \equiv \chi_p \equiv (\alpha^{<-1>})_p + \alpha_p \cdot \beta \cdot \alpha^{<-1>}$,
3. $(\alpha^{<-1>})_p \equiv -1 \cdot \alpha_p \cdot \beta \cdot \alpha^{<-1>}$,
4. $\alpha_p \equiv -1 \cdot (\alpha^{<-1>})_p \cdot \beta \cdot \alpha$.

If $E[\alpha] = 1$ then

1. $\alpha^{<-1>} \equiv (-1 \cdot \alpha_p \cdot \beta \cdot \alpha^{<-1>})_D$,
2. $\alpha \equiv (-1 \cdot (\alpha^{<-1>})_p \cdot \beta \cdot \alpha)_D$. 
Lagrange’s inversion formula I

Let \( f(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!} \) and suppose that \( a_0 = 0 \) and \( a_1 \neq 0 \). Then \( f^{-1} \) is well defined. One version of Lagrange’s inversion formula states that for any integer \( n \geq 1 \), it holds

\[
\left[ \frac{z^n}{n!} \right] f^{-1}(z) = \left[ \frac{z^{n-1}}{(n-1)!} \right] \left( \frac{f(z)}{z} \right)^{-n}.
\]

Let \( \alpha \) be an umbra such that \( e^{\alpha z} \simeq 1 + f(z) \). Then \( f(z) \simeq a_1 z e^{\alpha \cdot P z} \) and \( e^{\alpha^{-1} z} \simeq (a_1 z e^{\alpha \cdot P z})^{-1} \).

Proposition (di Nardo & Niederhausen & Senato 2009)

Let \( \alpha \) be an umbra with compositional inverse \( \alpha^{-1} \) (hence \( a_1 = E[\alpha] \neq 0 \)). For any \( n \geq 1 \), we have \( (\alpha^{-1})^n \simeq \frac{1}{a_1^n} (-n \cdot \alpha \cdot P)^{n-1} \) or equivalently, \( \alpha \cdot n (\alpha^{-1})^n \simeq (-n \cdot \alpha \cdot P)^{n-1} \). In particular, \( (\alpha^{-1})^n \simeq (-n \cdot \alpha)^{n-1} \).
Lagrange’s inversion formula II

Another version of Lagrange’s inversion formula states that for any integer $n \geq 1$ and $f(z)$ as before, it holds

$$\left[ \frac{z^n}{n!} \right] \Phi \left( f^{< -1 >} (z) \right) = \left[ \frac{z^{n-1}}{(n-1)!} \right] D\Phi(z) \left( \frac{f(z)}{z} \right)^{-n},$$

where $\Phi(z)$ is any formal exponential series and $D$ is the usual differential operator on formal power series.

**Theorem (A.M.P.T. 2010)**

Let $\alpha$ be any umbra and $\gamma$ an umbra with compositional inverse $\gamma^{<-1>}$ (hence $g_1 = E[\gamma] \neq 0$). For any $n \geq 1$ we have

$$(\alpha \cdot \beta \cdot \gamma^{<-1>})^n \simeq \frac{1}{g_1^n} \alpha(\alpha - n \cdot \gamma_P)^{n-1}.$$
Abel’s identity

The *adjoint* umbra of $\gamma$ is $\gamma^* = \beta \cdot \gamma^{<-1>}.$

**Theorem (A.M.P.T. 2010)**

*Let $\gamma \in A$ be any umbra with $g_1 = E[\gamma] \neq 0.$ For any umbrae $\alpha$ and $\delta$ we have*

$$
(\alpha + \delta)^n \simeq \sum_{k=0}^{n} \binom{n}{k} \gamma^k (\alpha + k \cdot \gamma \cdot \gamma^*)^{n-k} (\delta \cdot \gamma^*)^k.
$$
Contents

1. The exponential Riordan group
   - Definitions, examples and the fundamental theorem

2. Classical umbral calculus
   - Umbrae, generating functions, the dot operation and more

3. An umbral view of the Riordan group and Sheffer sequences
   - Definitions, fundamental theorem and Sheffer umbrae
The Riordan group revisited

Definition

Given two umbrae $\alpha$ and $\gamma$, we say that $(\alpha, \gamma)$ represent the Riordan array $(g, f)$ if

$$e^{\alpha z} \simeq g(z) \quad \text{and} \quad e^{\gamma z} \simeq 1 + f(z).$$

Thus, the Riordan group is given by

$$\mathcal{R}_{\text{io}} = \left\{ (\alpha, \gamma) \in A \times A : E[\gamma] \neq 0 \right\}.$$

The group operation reads as

$$(\alpha, \gamma)(\zeta, \eta) \equiv (\alpha + \zeta \cdot \beta \cdot \gamma, \eta \cdot \beta \cdot \gamma).$$

The inverse of $(\alpha, \gamma)$ is

$$(\alpha, \gamma)^{-1} = (-1 \cdot \alpha \cdot \beta \cdot \gamma^{<1>}, \gamma^{<1>})$$

and the identity is $$(\varepsilon, \chi).$$
Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $\gamma \rightsquigarrow \gamma^{<-1>}$)

Let $\mathcal{M} = (\alpha, \gamma) \in \text{Riad}$. Then

$$m_{n,k} \simeq \binom{n}{k} \gamma^k (\alpha + k \cdot \gamma_p)^{n-k} \quad \text{for} \quad n, k \geq 0.$$
Fundamental theorem of Riordan arrays  (FTRA)

Let \((\alpha, \gamma) \in \text{Rio}\) and \(\delta \in A\). The group \(\text{Rio}\) acts over \(A\) by

\[
(\alpha, \gamma) \cdot \delta = \alpha + \delta \cdot \beta \cdot \gamma.
\]

Given any two umbrae \(\delta, \eta \in A\), the FTRA is equivalent to saying that there exists an invertible Riordan array \((\alpha, \gamma) \in \text{Rio}\) such that

\[
(\alpha, \gamma) \cdot \delta = \eta.
\]

That is, the \(\text{Rio}\)-action is transitive.

By replacing \(\delta\) with \(\delta \cdot \beta \cdot \gamma\) in Abel's identity and using the umbral characterization for the entries of an invertible Riordan array, we obtain

\[
(\alpha + \delta \cdot \beta \cdot \gamma)^n \simeq \sum_{k=0}^{n} m_{n,k} \delta^k \quad \text{for all } n \geq 0.
\]
Some important Riordan subgroups

1. **The Appell subgroup**: \( \{(\alpha, \chi)\} \).

\[
(\alpha, \chi) (\zeta, \chi) \equiv (\alpha + \zeta, \chi) \quad \text{and} \quad (\alpha, \chi)^{-1} \equiv (-1 \cdot \alpha, \chi).
\]

2. **The Associated subgroup**: \( \{(\epsilon, \gamma)\} \).

\[
(\epsilon, \gamma) (\epsilon, \eta) \equiv (\epsilon, \eta \cdot \gamma) \quad \text{and} \quad (\epsilon, \gamma)^{-1} \equiv (\epsilon, \gamma^{<-1>}).
\]

3. **The Bell subgroup**: \( \{ (\alpha, \alpha_D) \} \).

\[
(\alpha, \alpha_D) (\zeta, \zeta_D) \equiv (\alpha + \zeta \cdot \beta \cdot \alpha_D, \zeta_D \cdot \beta \cdot \alpha_D)
\]

and

\[
(\alpha, \alpha_D)^{-1} \equiv (-1 \cdot \alpha \cdot \beta \cdot \alpha_D^{<-1>}, \alpha_D^{<-1>}).
\]

Note that

\[
\zeta_D \cdot \beta \cdot \alpha_D \equiv (\alpha + \zeta \cdot \beta \cdot \alpha_D)_D \quad \text{and} \quad \alpha_D^{<-1>} \equiv (-1 \cdot \alpha \cdot \beta \cdot \alpha_D^{<-1>})_D.
\]
4. The **extended Bell subgroup**: $\{ (\alpha, \gamma_D) \}$.

\[
(\alpha, \gamma_D)(\zeta, \eta_D) \equiv (\alpha + \zeta \cdot \beta \cdot \gamma_D, \eta_D \cdot \beta \cdot \gamma_D)
\]

and

\[
(\alpha, \gamma_D)^{-1} \equiv (-1 \cdot \alpha \cdot \beta \cdot \gamma_D^{<-1>}, \gamma_D^{<-1>})
\]

Note that $\eta_D \cdot \beta \cdot \gamma_D \equiv (\gamma + \eta \cdot \beta \cdot \gamma_D)_D$ and $\gamma_D^{<-1>} \equiv (-1 \cdot \gamma \cdot \beta \cdot \gamma_D^{<-1>})_D$.

This subgroup clearly contains the Bell subgroup.
5. The **Stabilizer subgroups**: Given any \( \delta \in A \), the stabilizer \( Stab(\delta) \) of \( \delta \) (with respect to the \( \text{Rio} \)-action) is

\[
Stab(\delta) = \left\{ (\alpha, \gamma) \in \text{Rio} : \alpha + \delta \cdot \beta \cdot \gamma \equiv \delta \right\}.
\]

Since \((\alpha + \delta \cdot \beta \cdot \gamma)^n \approx \sum_{k=0}^{n} m_{n,k} \delta^k\), the identity \(\alpha + \delta \cdot \beta \cdot \gamma \equiv \delta\) is equiv. to

\[
\sum_{k=0}^{n-1} m_{n,k} \delta^k + (m_{n,n} - 1) \delta^n = 0, \quad \text{for all} \quad n \geq 1.
\]

In particular, we have

\[
Stab(\varepsilon) = \left\{ (\alpha, \gamma) \in \text{Rio} : \alpha \equiv \varepsilon \right\} = \text{Associated subgroup}.
\]

\[
Stab(\nu) = \left\{ (\alpha, \gamma) \in \text{Rio} : \alpha + \beta \cdot \gamma \equiv \nu \right\} = \text{Stochastic subgroup}.
\]

\[
Stab(\chi) = \left\{ (\alpha, \gamma) \in \text{Rio} : \alpha + \gamma \equiv \chi \right\}.
\]
### Entries for some Riordan subgroups

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>$m_{n,k} \simeq \binom{n}{k} \gamma^k \left( \alpha + k \cdot \gamma \right)^{n-k} \quad n, k \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appell $(\alpha, \chi)$</td>
<td>$\binom{n}{k} \alpha^{n-k}$</td>
</tr>
<tr>
<td>Associated $(\varepsilon, \gamma)$</td>
<td>$\binom{n}{k} \gamma^k \left( k \cdot \gamma \right)^{n-k}$</td>
</tr>
<tr>
<td>Bell $(\alpha, \alpha_D)$</td>
<td>$\binom{n}{k} \left( (k + 1) \cdot \alpha \right)^{n-k}$</td>
</tr>
<tr>
<td>extended Bell $(\alpha, \gamma_D)$</td>
<td>$\binom{n}{k} \left( \alpha + k \cdot \gamma \right)^{n-k}$</td>
</tr>
</tbody>
</table>
Sheffer umbrae


Let \((g(z), f(z)) \in \mathfrak{Rio}\). A polynomial sequence \(s_n(x)\) is said to be Sheffer for \((g(z), f(z))\) if they satisfy

\[
\sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} = \frac{1}{g(f^{-1}(z))} e^{x f^{-1}(z)}.
\]

Representing \((g(z), f(z))\) by the pair of umbrae \((\alpha, \gamma)\), the Sheffer umbra \(\sigma_x^{(\alpha, \gamma)}\) for \((\alpha, \gamma)\) is defined as

\[
\sigma_x^{(\alpha, \gamma)} \equiv -1 \cdot \alpha \cdot \beta \cdot \gamma^{-1} + x \cdot \nu \cdot \beta \cdot \gamma^{-1} \equiv (-1 \cdot \alpha + x \cdot \nu) \cdot \gamma^*.
\]

By construction, the g.f. of \(\sigma_x^{(\alpha, \gamma)}\) is

\[
e^{\sigma_x^{(\alpha, \gamma)} z} \simeq \sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!},
\]

so that its moments \((\sigma_x^{(\alpha, \gamma)})^n \simeq s_n(x)\) form a Sheffer sequence.
Characterization of Sheffer sequences

Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $-1 \cdot \alpha \cdot \gamma^{*} \leadsto \alpha$)

The umbral expression for Sheffer polynomials $s_{n}(x)$ is given by

$$s_{n}(x) \sim \sum_{k=0}^{n} \left[ \binom{n}{k} \gamma^{<-1>}^{k} \left( -1 \cdot \alpha \cdot \gamma^{*} + k \cdot (\gamma^{<-1>})_{P} \right)^{n-k} \right] x^{k}$$

$$\sim \sum_{k=0}^{n} m_{n,k}^{-1} x^{k},$$

where $m_{n,k}^{-1}$ is the $(n, k)$-th entry of $(\alpha, \gamma)^{-1} = (-1 \cdot \alpha \cdot \gamma^{*}, \gamma^{<-1>})$. 
### Some distinguished Sheffer sequences

<table>
<thead>
<tr>
<th>Name</th>
<th>$s_n(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appell $(\alpha, \chi)$</td>
<td>$\sum_{k=0}^{n} \binom{n}{k} (-1 \cdot \alpha)^{n-k} x^k$</td>
</tr>
<tr>
<td>Associated $(\varepsilon, \gamma)$</td>
<td>$\sum_{k=0}^{n} \binom{n}{k} (\gamma_{&lt;-1&gt;})^k (\varepsilon \cdot (\gamma_{&lt;-1&gt;})_{\mathcal{P}})^{n-k} x^k$</td>
</tr>
<tr>
<td>Bell $(\alpha, \alpha_{\mathcal{D}})$</td>
<td>$\sum_{k=0}^{n} \binom{n}{k} ((k + 1) \cdot (\alpha_{\mathcal{D}}<em>{&lt;-1&gt;})</em>{\mathcal{P}})^{n-k} x^k$</td>
</tr>
<tr>
<td>extended Bell $(\alpha, \gamma_{\mathcal{D}})$</td>
<td>$\sum_{k=0}^{n} \binom{n}{k} (-1 \cdot \alpha \cdot \gamma^*<em>{\mathcal{D}} + k \cdot (\gamma</em>{\mathcal{D}}<em>{&lt;-1&gt;})</em>{\mathcal{P}})^{n-k} x^k$</td>
</tr>
</tbody>
</table>
Definition

Let \((\alpha, \gamma) \in \mathfrak{Ri}o\) and let \(x, y \in A\). The bi-parameterized Sheffer umbra corresponding to \((\alpha, \gamma)\) is given by

\[
\sigma_{x,y}^{(\alpha,\gamma)} \equiv (y \cdot \alpha + x \cdot \nu) \cdot \gamma^*
\]

Note that \(\sigma_{x,-1}^{(\alpha,\gamma)} \equiv \sigma_x^{(\alpha,\gamma)}\) (di Nardo and Senato Sheffer umbra), \(\sigma_{x,0}^{(\alpha,\gamma)} \equiv x \cdot \gamma^* \equiv \sigma_x^{(\varepsilon,\gamma)}\) (Associated umbra with respect to \(\gamma\)), etc.
Thank you!