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MATRIX ULTRASPHERICAL POLYNOMIALS: THE 2×2 FUNDAMENTAL CASES

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Abstract. In this paper, we exhibit explicitly a sequence $\{P_w\}$ of 2×2 matrix valued orthogonal polynomials with respect to a weight $W_{p,n}$, for any pair of real numbers p and n such that $0 < p < n$. This weight reduces if and only if $p = n/2$, and the entries of P_w are expressed in terms of the Gegenbauer polynomials C_k^λ. Also the corresponding three-term recursion relations are given and we make some studies of the algebra $\mathcal{D}(W)$. The development of this work was motivated by results on spherical functions of fundamental type associated with the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$.

1. Introduction

The theory of special functions is closely connected with the theory of the harmonic analysis on homogeneous spaces. Among the classical (scalar valued) families of orthogonal polynomials with rich and deep connections to several branches of mathematics the Jacobi polynomials occupy a distinguished role.

On the two dimensional sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates the spherical functions are the Legendre polynomials $P_w(\cos\theta)$. Also the zonal spherical functions of the n-dimensional sphere S^n are given, in spherical coordinates, in terms of Gegenbauer (or ultraspherical) polynomials $C_w^\lambda(\cos\theta)$, with $\lambda = \frac{n-1}{2}$. More generally, the zonal spherical functions on a Riemannian symmetric space of rank one can always be expressed in terms of the classical Gauss hypergeometric functions. In the compact case we have Jacobi polynomials.

This fruitful connection between special functions and representation theory of Lie groups is also present in the matrix case: the matrix valued spherical functions of any K-type are closely related to matrix valued orthogonal polynomials. In this way several examples of matrix orthogonal polynomials with a differential operator have been obtained by focusing on a group representation approach. See for example [12], [14], [22], [23], [21], [10], [17], [25], [18].

The examples of matrix orthogonal polynomials introduced here are motivated by the spherical functions of fundamental K-type associated with the n-dimensional sphere $S^n \simeq G/K$, where $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$. These matrix valued spherical functions were studied in [30] and [32].

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Given an integer $n \geq 3$, we consider the irreducible representations π of K with fundamental highest weights, which are parameterized by the ℓ-tuples

$$\mathbf{m}_\pi = (\underbrace{1,\dots,1}_{p},\underbrace{0,\dots,0}_{\ell-p}) \in \mathbb{Z}^\ell, \qquad \ell = [n/2], \quad 0 < p < n/2 - 1.$$

For each $w \in \mathbb{N}_0$ and $\delta = 0, 1$, we have an irreducible spherical function $\Phi_{w,\delta}$, of type $\mathbf{m}_\pi$. In [31] these functions are studied in detail. The restriction of $\Phi_{w,\delta}$ to the subgroup A, corresponding to the Cartan decomposition $G = KAK$ gives rise a vector valued function $\mathbf{P}_{w,\delta}$, which is an eigenfunction of a certain second order differential operator with matrix coefficients, the restriction of the Casimir operator of G. The spherical functions $\{\Phi_{w,\delta}\}_{w,\delta}$ are orthogonal with respect to a certain natural inner product among these functions.

In [25], the spherical functions of any K-type were considered in the particular case of $n = 3$. See also [17] and [18] for the pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, which is closely related to the pair $(\mathrm{SO}(4), \mathrm{SO}(3))$.

In the present paper, we introduce the following sequences $\{P_w\}_{w\geq 0}$ of 2×2 matrix valued polynomials on $[-1, 1]$ whose entries are given in terms of the classical Gegenbauer polynomials:

$$P_w(x) = \begin{pmatrix} \frac{1}{n+1}\, C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w}\, C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1}\, C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w}\, C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix}$$

for real parameters p and n such that $0 < p < n$.

We shall prove that $\{P_w\}_{w\geq 0}$ are orthogonal with respect to the weight matrix

$$W(x) = W_{p,n} = (1 - x^2)^{\frac{n}{2}-1} \begin{pmatrix} p\,x^2 + n - p & -nx \\ -nx & (n-p)x^2 + p \end{pmatrix}, \quad x \in [-1, 1].$$

We will see that the weight reduces to scalar cases if and only if $p = n/2$. On the other hand, it is easy to verify that, by changing p by $n - p$, the weights are conjugated, namely

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) W_{p,n} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^* = W_{n-p,n}.$$

Now we discuss briefly the content of the paper. In Section 2 we recall the general notions of matrix valued orthogonal polynomials and some results from [15] about the differential operators having the matrix valued orthogonal polynomials P_w as eigenfunctions. In Section 3 we introduce some background about matrix valued spherical functions for a pair (G, K) and the particular case $(\mathrm{SO}(n+1), \mathrm{SO}(n))$.

In Section 4 we prove that our polynomials P_w satisfy $P_w D = \Lambda_w P_w$, where D is the (right-hand side) hypergeometric differential operator

$$D = \left(\frac{d^2}{dx^2}\right)(1 - x^2) - \left(\frac{d}{dx}\right)\left((n+2)x + 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

and the eigenvalue is the diagonal matrix

$$\Lambda_w(D) = \begin{pmatrix} -w(w+n+1) - p & 0 \\ 0 & -w(w+n+1) - n + p \end{pmatrix}.$$

In Section 5 we prove the three-term recursion relation satisfied by $\{P_w\}_{w\geq 0}$. We also consider the sequence of monic orthogonal polynomials $\{Q_w\}$ and exhibit its corresponding three-term recursion relation.

Section 6 is focused on the study of the algebra $\mathcal{D}(W)$. The first attempt to go beyond the issue of the existence of one non trivial element in $\mathcal{D}(W)$ and to study the full algebra is undertaken in [2]. In the example considered in [29], the conjecture set forth in [2] is proved and the structure of the algebra is studied in detail. In our case $\mathcal{D}(W)$ is a noncommutative algebra. We give a basis $\{D_1, D_2, D_3, D_4, I\}$ of the subspace of the differential operators in $\mathcal{D}(W)$ of order at most two. The differential operators D_1 and D_2 are symmetric operators, while D_3 and D_4 are not. We conjecture that, the full algebra $\mathcal{D}(W)$ is a polynomial algebra in these four differential operators of order two.

2. Background on matrix valued orthogonal polynomials

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [19] and [20]. In [3], the study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order differential operators was started. The first explicit examples of such polynomials are given in [12], [11], [13] and [5]. See also [6], [7], [8], [1], [2], [4] and the references given there.

Let $W = W(x)$ be a weight matrix of size N on the real line, that is a complex $N \times N$ matrix valued integrable function on the interval (a, b) such that $W(x)$ is positive definite almost everywhere and with finite moments of all orders. Let $\mathrm{Mat}_N(\mathbb{C})$ be the algebra of all $N \times N$ complex matrices and let $\mathrm{Mat}_N(\mathbb{C})[x]$ be the algebra over $\mathbb{C}$ of all polynomials in the indeterminate x with coefficients in $\mathrm{Mat}_N(\mathbb{C})$. We consider the following Hermitian sesquilinear form in the linear space $\mathrm{Mat}_N(\mathbb{C})[x]$:

$$\langle P, Q\rangle = \langle P, Q\rangle_W = \int_a^b P(x)W(x)Q(x)^* \, dx.$$

The following properties are satisfied, for all $P, Q, R \in \mathrm{Mat}_N(\mathbb{C})[x]$, $a, b \in \mathbb{C}$, $T \in \mathrm{Mat}_N(\mathbb{C})$

(1) $\langle aP + bQ, R\rangle = a\langle P, R\rangle + b\langle Q, R\rangle$,
(2) $\langle TP, R\rangle = T\langle P, R\rangle$,
(3) $\langle P, Q\rangle^* = \langle Q, P\rangle$,
(4) $\langle P, P\rangle \geq 0$. Moreover, if $\langle P, P\rangle = 0$, then $P = 0$.

Given a weight matrix W one can constructs sequences of matrix valued orthogonal polynomials, that is sequences $\{P_n\}_{n\geq 0}$, where P_n is a polynomial of degree n with nonsingular leading coefficient and $\langle P_n, P_m\rangle = 0$ for $n \neq m$.

We observe that there exists a unique sequence of monic orthogonal polynomials $\{Q_n\}_{n\geq 0}$ in $\mathrm{Mat}_N(\mathbb{C})$. Moreover, any other sequence of orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$ is of the form $P_n(x) = A_n Q_n(x)$, for some $A_n \in \mathrm{GL}_N(\mathbb{C})$.

By following a standard argument, given for instance in [19] or [20], one shows that the monic orthogonal polynomials $\{Q_n\}_{n\geq 0}$ satisfies a three-term recursion relation

$$xQ_n(x) = A_n Q_{n-1} + B_n Q_n(x) + Q_{n+1}(x), \qquad n \geq 0,$$

where $Q_{-1} = 0$ and A_n, B_n are matrices depending on n and not in x.

Two weights W and $\widetilde{W}$ are said to be *similar* if there exists a nonsingular matrix M, which does not depend on x, such that

$$\widetilde{W}(x) = MW(x)M^*, \quad \text{for all } x \in (a, b).$$

Notice that if $\{P_n\}_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to W, and $M \in \mathrm{GL}_N(\mathbb{C})$, then $\{P_n M^{-1}\}_{n\geq 0}$ is orthogonal with respect to $\widetilde{W} = MWM^*$. A weight matrix W reduces to a smaller size if there exists a nonsingular matrix M such that

$$MW(x)M^* = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}, \qquad \text{for all } x \in (a,b),$$

where W_1 and W_2 are weights of smaller size.

In the study of matrix valued orthogonal polynomials it is important the study of differential operators having them as eigenfunctions.

Let D be an right-hand side ordinary differential operator with matrix valued polynomial coefficients $F_i(x)$ of degree less than or equal to i of the form

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx}, \tag{1}$$

with the action of D on a polynomial function $P(x)$ given by

$$PD = \sum_{i=0}^{s} \partial^i(P)(x) F_i(x).$$

We say that the differential operator D is *symmetric* if $\langle PD, Q\rangle = \langle P, QD\rangle$, for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. It is matter of a careful integration by parts to see that the condition of symmetry for a differential operator of order two is equivalent to a set of three differential equations involving the weight W and the coefficients of the differential operator D.

Proposition 2.1 ([13] or [5])**.** *Let $W(x)$ be a weight matrix supported on (a,b). Let $D = \partial^2 F_2(x) + \partial F_1(x) + F_0$ as in (1). Then D is symmetric with respect to W if and only if*

$$\begin{cases} F_2 W = W F_2^* \\ 2(F_2W)' - F_1 W = W F_1^* \\ (F_2W)'' - (F_1W)' + F_0 W = W F_0^* \end{cases}$$

with the boundary conditions

$$\lim_{x\to a,b} F_2(x)W(x) = 0, \qquad \lim_{x\to a,b} \big(F_1(x)W(x) - WF_1^*(x)\big) = 0.$$

3. Spherical functions associated with the n-dimensional spheres

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.

Definition 3.1. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

i) $\Phi(e) = I$ ($I=$ identity transformation),
ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\,dk$, for all $x, y \in G$.

The reader can find a number of general results in [27] and [9]. For our purpose it is appropriate to recall the following facts.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called irreducible if V has no proper subspace invariant by $\Phi(g)$ for all $g \in G$.

If G is a connected Lie group, it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ that are invariant under all right translations by elements in K.

In the following proposition (V, π) will be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 3.2. *A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*

i) *Φ is analytic,*
ii) *$\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$,*
iii) *$[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.*

This result is a combination of results from [27], the reader can also see Proposition 2.3 in [25].

Spherical functions of type δ arise in a natural way upon considering representations of G (see Section 3 in [27]). If $g \mapsto \tau(g)$ is a continuous representation of G, say on a finite dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta(k^{-1})\tau(k)\,dk$$

is a projection of E onto $P_\delta E = E(\delta)$. If $P_\delta \neq 0$ the function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta \tau(g)a, \quad g \in G,\ a \in E(\delta),$$

is a spherical function of K-type δ.

If the representation τ is irreducible then the associated spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite dimensional irreducible representation of G.

The spherical functions that motivated this paper are those of fundamental K-type associated with the n-dimensional sphere $S^n \simeq G/K$, where $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$. These matrix valued spherical functions were studied in [30] and [32] and that, due to results obtained in [31], they also corresponds to spherical functions of the pair $(\mathrm{SO}(n+1), \mathrm{O}(n))$. See also [25] and [17] for the spherical functions of any K-type in the particular case of $n = 3$.

The fundamental representations π of K are parameterized by the ℓ-tuples

$$\mathbf{m}_\pi = (\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{\ell - p}) \in \mathbb{Z}^\ell,$$

with $\ell = [n/2]$, and $0 < p < n/2 - 1$.

Given a nonnegative integer w and $\delta = 0, 1$, we can consider $\Phi_{w,\delta}$, the irreducible spherical function of type π associated with the irreducible representation $\tau \in$

$\hat{\mathrm{SO}}(n+1)$ of highest weight of the form

$$\mathbf{m}_\tau = (w+1, \underbrace{1,\dots,1}_{p-1}, \delta, \underbrace{0,\dots,0}_{\ell'-p-1}) \in \mathbb{Z}^{\ell'}, \qquad \text{with } \ell' = \left[\tfrac{n+1}{2}\right].$$

From the representation theory of Lie groups we know that the spherical functions $\{\Phi_{w,\delta}\}_{w,\delta}$ are orthogonal with respect to a certain natural inner product among spherical functions. Therefore, after an accurate conjugation, one obtains vector-valued functions $\{\mathbf{P}_{w,\delta}\}_{w,\delta}$ that are orthogonal with respect to a matrix-weight W. The very well known fact that the spherical functions are eigenfunctions of the Casimir operator on G makes the function $\mathbf{P}_{w,\delta}$ into an eigenfunction of a certain differential operator D.

4. Matrix valued orthogonal polynomials associated with the n-dimensional spheres

Motivated by the results obtained in [30] we introduce the following family of polynomials, for $w \in \mathbb{N}_0$

$$P_w(x) = \begin{pmatrix} \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w} C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w} C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix} \tag{2}$$

with parameters $p, n \in \mathbb{R}$ such that $0 < p < n$. Here $C_n^\lambda(x)$ denotes the n-th Gegenbauer polynomial

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\, n+2\lambda \\ \lambda+1/2 \end{matrix}; \frac{1-x}{2}\right), \qquad x \in [-1,1],$$

as usual, we assume $C_n^\lambda(x) = 0$ if $n < 0$. We recall that C_n^λ is a polynomial of degree n, with leading coefficient $\frac{2^n(\lambda)_n}{n!}$.

Let us observe that the $\deg(P_w) = w$ and the leading coefficient of P_w is a nonsingular scalar matrix

$$\frac{2^w(\frac{n+1}{2})_w}{(n+1)\,w!}\,\mathrm{Id} = \frac{1}{w!} 2^{w-1}(\tfrac{n+3}{2})_{w-1}\,\mathrm{Id}, \tag{3}$$

where $(a)_w = a(a+1)\dots(a+w-1)$ denotes the Pochhammer's symbol.

We start by proving that the polynomials P_w given in (2) are eigenfunctions of the following differential operator D.

Theorem 4.1. *For each $w \in \mathbb{N}_0$, the matrix polynomial P_w is an eigenfunction of the differential operator*

$$D = \partial^2(1-x^2) - \partial\left((n+2)x + 2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right) - \left(\begin{smallmatrix} p & 0 \\ 0 & n-p \end{smallmatrix}\right),$$

with eigenvalue

$$\Lambda_w(D) = \begin{pmatrix} -w(w+n+1)-p & 0 \\ 0 & -w(w+n+1)-n+p \end{pmatrix}.$$

Proof. We need to verify that

$$P_w D = \Lambda_w P_w.$$

We will need to use the following properties of the Gegenbauer polynomials (for the first three see [16] page 40, and for the last one see [26], page 83, equation (4.7.27))

$$(1-x^2)\frac{d^2}{dx^2}C_m^\lambda(x) - (2\lambda+1)x\frac{d}{dx}C_m^\lambda(x) + m(m+2\lambda)C_m^\lambda(x) = 0, \tag{4}$$

$$\frac{d}{dx}C_m^\lambda(x) = 2\lambda\, C_{m-1}^{\lambda+1}(x), \tag{5}$$

$$2(m+\lambda)x\, C_m^\lambda(x) = (m+1)C_{m+1}^\lambda(x) + (m+2\lambda-1)C_{m-1}^\lambda(x), \tag{6}$$

$$\frac{(m+2\lambda-1)}{2(\lambda-1)}C_{m+1}^{\lambda-1}(x) = C_{m+1}^\lambda(x) - x\, C_m^\lambda(x). \tag{7}$$

Also we have (combining (6) and (7))

$$(m+\lambda)C_{m+1}^{\lambda-1}(x) = (\lambda-1)\Big(C_{m+1}^\lambda(x) - C_{m-1}^\lambda(x)\Big). \tag{8}$$

The entry $(1,1)$ of the matrix $P_wD - \Lambda_w P_w$ is

$$\begin{aligned}
&(1-x^2)(P_w)''_{11} - (n+2)x(P_w)'_{11} - 2(P_w)'_{12} + w(w+n+1)(P_w)_{11}\\
&\quad = (1-x^2)\left(\tfrac{1}{n+1}\, C_w^{\frac{n+1}{2}} + \tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)'' - (n+2)x\left(\tfrac{1}{n+1}\, C_w^{\frac{n+1}{2}} + \tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)'\\
&\qquad - \tfrac{2}{p+w}\left(C_{w-1}^{\frac{n+3}{2}}\right)' + w(w+n+1)\left(\tfrac{1}{n+1}\, C_w^{\frac{n+1}{2}} + \tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right).
\end{aligned}$$

From (4) we get

$$(1-x^2)\left(C_w^{\frac{n+1}{2}}\right)'' - (n+2)x\left(C_w^{\frac{n+1}{2}}\right)' + w(w+n+1)C_w^{\frac{n+1}{2}} = 0,$$

$$(1-x^2)\left(C_{w-2}^{\frac{n+3}{2}}\right)'' - (n+4)x\left(C_{w-2}^{\frac{n+3}{2}}\right)' + (w-2)(w+n+1)C_{w-2}^{\frac{n+1}{2}} = 0,$$

and from (5)

$$\left(C_{w-2}^{\frac{n+3}{2}}\right)' = (n+3)C_{w-2}^{\frac{n+5}{2}}.$$

Therefore the entry (1,1) of $P_wD - \Lambda_w P_w$, multiplied by $(p+w)/2$ is

$$-(n+3)C_{w-2}^{\frac{n+5}{2}} + x\, C_{w-3}^{\frac{n+5}{2}} + (w+n+1)C_{w-2}^{\frac{n+3}{2}} = 0.$$

This last identity follows from equation (7) with $\lambda = \frac{n+5}{2}$ and $m = w-3$.

From the previous verifications, by changing p by $n-p$, it follows that the entry $(2,2)$ of $P_wD - \Lambda_w P_w$ is zero.

The entry $(1,2)$ of $P_wD - \Lambda_w P_w$ is

$$(1-x^2)(P_w)''_{12} - (n+2)x(P_w)'_{12} - 2(P_w)'_{11} + \big(w(w+n+1) - n + 2p\big)(P_w)_{12},$$

if we multiply it by $(p+w)$ we get

$$\begin{aligned}
&(1-x^2)\Big(C_{w-1}^{\frac{n+3}{2}}\Big)'' - (n+2)x\left(C_{w-1}^{\frac{n+3}{2}}\right)' + (w(w+n+1) - n + 2p)C_{w-1}^{\frac{n+3}{2}}\\
&\qquad - 2\tfrac{(p+w)}{n+1}\Big(C_w^{\frac{n+1}{2}}\Big)' - 2\Big(C_{w-2}^{\frac{n+3}{2}}\Big)'.
\end{aligned} \tag{9}$$

From (4) with $\lambda = \frac{n+3}{2}$ and $m = w-1$ we get

$$(1-x^2)\Big(C_{w-1}^{\frac{n+3}{2}}\Big)'' = (n+4)x\Big(C_{w-1}^{\frac{n+3}{2}}\Big)' - (w-1)(w+n+2)C_{w-1}^{\frac{n+3}{2}}.$$

From (5) we have $\frac{1}{n+1}\Big(C_{w}^{\frac{n+1}{2}}\Big)' = C_{w-1}^{\frac{n+3}{2}}$ and $\Big(C_{w-2}^{\frac{n+3}{2}}\Big)' = (n+3)C_{w-3}^{\frac{n+5}{2}}$. By replacing in (9) we get

$$2x\Big(C_{w-1}^{\frac{n+3}{2}}\Big)' - 2(w-1)C_{w-1}^{\frac{n+3}{2}} - 2(n+3)C_{w-3}^{\frac{n+5}{2}}. \tag{10}$$

Now from (5) and (7) we have the following identity

$$x\Big(C_{w-1}^{\frac{n+3}{2}}\Big)' = (n+3)C_{w-2}^{\frac{n+5}{2}} = (n+3)C_{w-1}^{\frac{n+5}{2}} - (w+n+2)C_{w-1}^{\frac{n+3}{2}}.$$

Thus (10) becomes

$$2(n+3)\Big(C_{w-1}^{\frac{n+5}{2}} - C_{w-3}^{\frac{n+5}{2}}\Big) - 2(2w+n+1)C_{w-1}^{\frac{n+3}{2}} = 0,$$

which follows from (8) with $\lambda = \frac{n+5}{2}$ and $m = w-2$.

To complete the proof of the theorem we need to verify that the entry $(2,1)$ of $P_wD - \Lambda_wP_w$ is zero. This leads us to made exactly the same computation by changing p by $n-p$. □

We introduce the weight matrix

$$W(x) = W_{p,n} = (1-x^2)^{\frac{n}{2}-1}\begin{pmatrix} p\,x^2+n-p & -nx \\ -nx & (n-p)x^2+p \end{pmatrix}, \quad x\in[-1,1]. \tag{11}$$

Proposition 4.2. *For $n \neq 2p$ the weight $W(x)$ does not reduce to a smaller size.*

Proof. Assume that there exists a nonsingular matrix M such that

$$MW(x)M^* = \begin{pmatrix} w_1(x) & 0 \\ 0 & w_2(x) \end{pmatrix}.$$

The entry $(1,2)$ of $MW(x)M^*$ is

$$x^2\big(p\,m_{11}m_{21}+(n-p)m_{12}m_{22}\big)-\big(m_{11}m_{22}+m_{12}m_{21}\big)n\,x+(n-p)m_{11}m_{21}+p\,m_{12}m_{22}.$$

From here we see that

$$m_{11}m_{22} + m_{12}m_{21} = 0, \tag{12}$$

$$p\,m_{11}m_{21} + (n-p)m_{12}m_{22} = 0, \tag{13}$$

$$(n-p)m_{11}m_{21} + p\,m_{12}m_{22} = 0. \tag{14}$$

By combining equations (13) and (14) we have that $(n-2p)m_{11}m_{21} = 0$. If $p \neq n-p$, by using (12) we obtain that $\det M = 0$, which is a contradiction. □

Remark 4.3. For $n = 2p$ the weight matrix W reduces to a scalar weights. In fact by taking $M = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ we have that

$$MW(x)M^* = 2p\,(1-x^2)^{\frac{n}{2}-1}\begin{pmatrix} (1-x)^2 & 0 \\ 0 & (1+x)^2 \end{pmatrix}.$$

Remark 4.4. We have that the weight matrices $W_{p,n}$ and $W_{n-p,p}$ are conjugated to each other. In fact, by taking $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we get

$$MW_{p,n}M^* = W_{n-p,n}.$$

From Proposition 2.1 and following straightforward computations, we can prove the following result.

Proposition 4.5. *The differential operator*

$$D = \partial^2 (1 - x^2) - \partial \left((n+2)x + 2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right) - \left(\begin{smallmatrix} p & 0 \\ 0 & n-p \end{smallmatrix}\right)$$

is symmetric with respect to the weight function $W(x)$.

Remark. Let us mention that the result in Proposition 4.5, for group parameters p and n, is also a direct consequence of the representation theory of Lie groups. This is because the Casimir operator is symmetric with respect to the L^2-inner product for matrix valued functions on G, and the differential operator D and the weight W are closely related with them.

In the scalar case, if D is a symmetric differential operator with respect to W and $\{P_n\}$ is a family of eigenfunctions of D with different eigenvalues, then the sequence $\{P_n\}$ is automatically orthogonal with respect to W. In the matrix case this is not always true because

$$\Lambda_n \langle P_n, P_m \rangle = \langle P_n D, P_m \rangle = \langle P_n, P_m D \rangle = \langle P_n, P_m \rangle \Lambda_m,$$

does not imply that $\langle P_n, P_m \rangle = 0$, for $n \neq m$.

Theorem 4.6. *The matrix polynomials $\{P_w\}_{w \geq 0}$ are orthogonal polynomials with respect to the matrix valued inner product*

$$\langle P, Q \rangle = \int_{-1}^{1} P(x) W(x) Q(x)^* \, dx.$$

Proof. We know that P_w is a polynomial of degree w and its leading coefficient is a nonsingular diagonal matrix (see (3)). We only have to verify that for $w \neq w'$, $\langle P_w, P_{w'} \rangle_W = 0$. Since P_w is an eigenfunction of the differential operator D, which is symmetric with respect to W, we have that

$$\Lambda_w \langle P_w, P_{w'} \rangle = \langle P_w D, P_{w'} \rangle = \langle P_w, P_{w'} D \rangle = \langle P_w, P_{w'} \rangle \Lambda_{w'}.$$

Therefore, for $i, j = 1, 2$ we have $\lambda_{w,i} \langle P_{w,i}, P_{w',j} \rangle = \lambda_{w',j} \langle P_{w,i}, P_{w',j} \rangle$, where $P_{w,i}$ is the i-th row of the matrix polynomial P_w,

$$\lambda_{w,1} = -w(w+n+1) - p, \quad \lambda_{w,2} = -w(w+n+1) - n + p$$

and

$$\langle P_{w,i}, P_{w',j} \rangle = \int_{-1}^{1} P_{w,i}(x) \, W(x) P_{w',j}^*(x) \, dx \in \mathbb{C}.$$

It is not difficult to verify that $\lambda_{w,i} \neq \lambda_{w',j}$, for $w \neq w'$ or $i \neq j$. Then we have

$$\langle P_{w,i}, P_{w',j} \rangle = 0 \quad \text{for } w \neq w' \text{ or } i \neq j. \tag{15}$$

Therefore $\langle P_w, P_{w'} \rangle = 0$, for $w \neq w'$, which concludes the proof of the theorem. □

Remark 4.7. When we consider the polynomials P_w given by the spherical functions on $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$, the parameters p and n are integers such that $0 < p < [n/2]$. The sequence defined by (2) has a larger set of parameters.

For the case $p, n \in \mathbb{N}$ and $0 < p < [n/2]$, the δ-th row of P_w is a scalar multiple of the polynomial $\mathbf{P}_{w,\delta}$ related to the the spherical function $\Phi_{w,\delta}$ (see last paragraph in Section 3). In fact, from Theorem 4.1 we have that the δ-th row of P_w is a polynomial eigenfunction of the differential operator D and from [30, Theorem 6.3] we know that such eigenfunction is unique up to scalar.

5. Three-term recursion relation

The main result of this section is the obtaining of a three-term recursion relation satisfied by the sequence of orthogonal polynomials studied in this paper. We give a proof of it by using some properties of the Gegenbauer polynomials.

We would like to point out that it would be also possible to obtain this kind of result, for integers parameters p and n, from the representation theory of Lie groups. See for example [22] and [24] for the cases of the complex projective plane and the complex hyperbolic plane.

Theorem 5.1. *The orthogonal polynomials $\{P_w\}_{w\geq 0}$ satisfy the three-term recursion relation*

$$x\,P_w(x) = A_w P_{w-1}(x) + B_w P_w(x) + C_w P_{w+1}(x),$$

where

$$A_w = \begin{pmatrix} \frac{(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(2w+n+1)} & 0 \\ 0 & \frac{(n+w)(p+w+1)(n-p+w-1)}{(p+w)(n-p+w)(2w+n+1)} \end{pmatrix},$$

$$B_w = \begin{pmatrix} 0 & \frac{-p}{(p+w)(p+w+1)} \\ \frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0 \end{pmatrix}, \qquad C_w = \frac{w+1}{2w+n+1} I.$$

Proof. We recall that the three-term recursion relation for Gegenbauer polynomials $C_m^\lambda(x)$ is

$$2(m+\lambda)xC_m^\lambda(x) = (m+1)C_{m+1}^\lambda(x) + (m+2\lambda-1)C_{m-1}^\lambda(x). \tag{16}$$

Let $\lambda = \frac{n+3}{2}$. To verify the $(1,1)$-entry of the equation in the statement of the theorem we need to prove that

$$\begin{aligned} &x\Big(\tfrac{1}{n+1}C_w^{\lambda-1}(x) + \tfrac{1}{p+w}C_{w-2}^\lambda(x)\Big) \\ &\quad = \tfrac{(n+w)(p+w-1)(n-p+w+1)}{(2w+n+1)(p+w)(n-p+w)}\Big(\tfrac{1}{n+1}C_{w-1}^{\lambda-1}(x) + \tfrac{1}{p+w}C_{w-3}^\lambda(x)\Big) \\ &\qquad - \tfrac{p}{(p+w)(p+w+1)(n-p+w)}C_{w-1}^\lambda(x) \\ &\qquad + \tfrac{w+1}{2w+n+1}\Big(\tfrac{1}{n+1}C_{w+1}^{\lambda-1}(x) + \tfrac{1}{p+w+1}C_{w-1}^\lambda(x)\Big). \end{aligned} \tag{17}$$

From (16) we have

$$\begin{aligned} (2w+n+1)x\,C_w^{\frac{n+1}{2}}(x) &= (w+1)C_{w+1}^{\frac{n+1}{2}}(x) + (w+n)C_{w-1}^{\frac{n+1}{2}}(x), \\ (2w+n-1)x\,C_{w-2}^{\frac{n+3}{2}}(x) &= (w-1)C_{w-1}^{\frac{n+3}{2}}(x) + (w+n)C_{w-3}^{\frac{n+3}{2}}(x). \end{aligned}$$

By replacing these identities in (17), it is enough to verify that

$$\begin{aligned} &\tfrac{(w+n)}{(n+1)(2w+n+1)}\Big(-1 + \tfrac{(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)}\Big)C_{w-1}^{\lambda-1}(x) \\ &\quad + \Big(-\tfrac{p}{(p+w)(p+w+1)(n-p+w)} + \tfrac{w+1}{(2w+n+1)(p+w+1)} - \tfrac{w-1}{(p+w)(2w+n-1)}\Big)C_{w-1}^\lambda(x) \\ &\quad + \tfrac{(n+w)}{p+w}\Big((\tfrac{n-p+w-1}{(2w+n+1)(n-p+w)} - \tfrac{1}{2w+n-1}\Big)C_{w-3}^\lambda(x) = 0. \end{aligned} \tag{18}$$

Thus, by using the relation (8) among Gegenbauer polynomials

$$C_m^{\lambda-1}(x) = \tfrac{\lambda-1}{m+\lambda-1}\left(C_m^\lambda(x) - C_{m-2}^\lambda(x)\right), \tag{19}$$

with $\lambda=\frac{n+3}{2}$ and $m=w-1$, the identity in (18) follows after some straightforward computations.

Now we will verify that the $(1,2)$-entry of the equation in the statement of the theorem holds. We need to verify

(20)
$$\frac{1}{p+w}\,x\,C^{\lambda}_{w-1}(x)=\frac{(n+w)(n-p+w+1)}{(p+w)(2w+n+1)(n-p+w)}C^{\lambda}_{w-2}(x)$$
$$-\frac{p}{(p+w)(p+w+1)}\left(\frac{1}{n+1}C^{\lambda-1}_{w}(x)+\frac{1}{n-p+w}C^{\lambda}_{w-2}(x)\right)+\frac{w+1}{(2w+n+1)(p+w+1)}C^{\lambda}_{w}(x)$$

with $\lambda=\frac{n+3}{2}$.
From (19) we have $\frac{1}{n+1}\,C^{\lambda-1}_{w}=\frac{1}{2w+n+1}\big(C^{\lambda}_{w}-C^{\lambda}_{w-1}\big)$. Thus, the right-hand side of (20) is

$$\left(\frac{(n+w)(n-p+w+1)}{(p+w)(2w+n+1)(n-p+w)}+\frac{p\,\big((n-p+w)-(2w+n+1)\big)}{(p+w)(p+w+1)(2w+n+1)(n-p+w)}\right)C^{\lambda}_{w-2}(x)$$
$$+\frac{\big(-p+(p+w)(w+1)\big)}{(p+w)(2w+n+1)(p+w+1)}C^{\lambda}_{w}(x)=\frac{n+w+1}{(p+w)(2w+n+1)}C^{\lambda}_{w-2}(x)+\frac{w}{(p+w)(2w+n+1)}C^{\lambda}_{w}(x)$$

From the recursion relation (16) with $\lambda=\frac{n+3}{2}$ and $m=w-1$, we obtain

$$\frac{n+w+1}{(p+w)(2w+n+1)}C^{\lambda}_{w-2}(x)+\frac{w}{(p+w)(2w+n+1)}C^{\lambda}_{w}(x)=\frac{1}{p+w}\,x\,C^{\lambda}_{w-1}(x),$$

which proves (20).

For the entries $(2,2)$ and $(2,1)$ we proceed in a similar way, by observing that we need to do the same computations that in the cases $(1,1)$ and $(1,2)$ respectively, changing p by $n-p$. This concludes the proof of the theorem. □

The monic sequence of matrix orthogonal polynomials is given by

$$Q_w=\frac{w!(n+1)}{2^w\left(\frac{n+1}{2}\right)_w}P_w,\qquad w\geq 0. \tag{21}$$

From Theorem 5.1 we easily obtain the corresponding recursion relation for the monic sequence of orthogonal polynomials.

Corollary 5.2. *The monic sequence of orthogonal polynomials $\{Q_w\}$ satisfies the following three-term recursion relation*

$$x\,Q_w(x)=\widetilde{A}_wQ_{w-1}(x)+\widetilde{B}_wQ_w(x)+Q_{w+1}(x),$$

where

$$\widetilde{A}_w=\begin{pmatrix}\frac{w(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(n+2w-1)(n+2w+1)} & 0\\ 0 & \frac{w(n+w)(p+w+1)(n-p+w-1)}{(p+w)(n-p+w)(n+2w-1)(n+2w+1)}\end{pmatrix},$$

$$\widetilde{B}_w=\begin{pmatrix}0 & \frac{-p}{(p+w)(p+w+1)}\\ \frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0\end{pmatrix}.$$

We conclude this section with the first polynomials of the sequence of monic polynomials $\{Q_w\}_w$. We recall that our original polynomials P_w are a multiple of Q_w, see (21).

$$Q_0 = \mathrm{Id}, \quad Q_1 = \begin{pmatrix} x & \frac{1}{p+1} \\ \frac{1}{n-p+1} & x \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} x^2 - \frac{p}{(n+3)(p+2)} & \frac{2}{p+2}x \\ \frac{2}{n-p+2}x & x^2 - \frac{p}{(n+3)(n-p+2)} \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} x^3 - \frac{3(p+1)}{(n+5)(p+3)}x & \frac{3}{p+3}x^2 - \frac{3}{(n+5)(p+3)} \\ \frac{3}{n-p+3}x^2 - \frac{3}{(n+5)(n-p+3)} & x^3 - \frac{3(n-p+1)}{(n+5)(n-p+3)}x \end{pmatrix}.$$

Remark 5.3. Observe that from (15) and (21) we have that $\langle Q_w, Q_w \rangle$ is always a diagonal matrix. Moreover one can verify that

$$\langle Q_w, Q_w \rangle = \|Q_w\|^2 = \frac{\pi 2^{[w/2]}\Gamma(n/2+1+[w/2])}{w!(n+2w+1)\Gamma((n+3)/2)} \prod_{k=1}^{[(w-1)/2]} (n+2k+1) \times \begin{pmatrix} \frac{p\,(n-p+w+1)}{p+w} & 0 \\ 0 & \frac{(n-p)(p+w+1)}{n-p+w} \end{pmatrix}.$$

6. The algebra $\mathcal{D}(W)$

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [19] and [20]. The study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order differential operators started in [3].

We consider right-hand side differential operators

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx}, \tag{22}$$

with the action of D on the polynomial $P(x)$ given by

$$(PD)(x) = \sum_{i=0}^{s} \partial^i(P)(x) F_i(x).$$

We consider the following subalgebra of the algebra of all right-hand side differential operators with coefficients in $\mathrm{Mat}_N(\mathbb{C})[x]$,

$$\mathcal{D} = \{D = \textstyle\sum_{i=0}^{s} \partial^i F_i \,:\, F_i \in \mathrm{Mat}_N(\mathbb{C})[x], \deg F_i \le i\}.$$

Proposition 6.1 ([15], Propositions 2.6 and 2.7). *Let $W = W(x)$ be a weight matrix of size N and let $\{Q_n\}_{n\ge 0}$ be the sequence of monic orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$. If D is a right-hand side ordinary differential operator of order s, as in* (22)*, such that*

$$Q_n D = \Lambda_n Q_n, \qquad \text{for all } n \ge 0,$$

with $\Lambda_n \in \mathrm{Mat}_N(\mathbb{C})$, then $F_i = F_i(x) = \sum_{j=0}^{i} x^j F_j^i$, $F_j^i \in \mathrm{Mat}_N(\mathbb{C})$, is a polynomial and $\deg(F_i) \le i$. Moreover D is determined by the sequence $\{\Lambda_n\}_{n\ge 0}$ and

$$\Lambda_n = \sum_{i=0}^{s} [n]_i F_i^i, \qquad \text{for all } n \ge 0, \tag{23}$$

where $[n]_i = n(n-1)\cdots(n-i+1)$, $[n]_0 = 1$.

Given a matrix-weight W the algebra

$$\mathcal{D}(W) = \{D \in \mathcal{D} \,:\, P_n D = \Gamma_n(D) P_n,\, \Gamma_n(D) \in \mathrm{Mat}_N(\mathbb{C}),\ \text{for all } n \geq 0\}$$

is introduced in [15], where $\{P_n\}_{n\geq 0}$ is any sequence of matrix valued orthogonal polynomials with respect to W.

We observe that the definition of $\mathcal{D}(W)$ depends only on the weight matrix W and not on the particular sequence of orthogonal polynomials, since two sequences $\{P_w\}$ and $\{Q_w\}$ of matrix orthogonal polynomials with respect to the weight W are related by $P_w = A_w Q_w$, for some invertible matrices A_w for $w \geq 0$ (see [15, Corollary 2.5]).

Proposition 6.2 ([15], Proposition 2.8). *The mapping $D \mapsto \Gamma_n(D)$ is a representation of $\mathcal{D}(W)$ in $\mathbb{C}^N$ for each $n \geq 0$. Moreover the sequence of representations $\{\Gamma_n\}_{n\geq 0}$ separates the elements of $\mathcal{D}(W)$.*

We remark that the result in Proposition 6.2 says that the map

$$D \mapsto (\Gamma_0(D), \Gamma_1(D), \Gamma_2(D), \ldots\ldots)$$

is an injective morphism of $\mathcal{D}(W)$ into $\mathrm{Mat}_N(\mathbb{C})^{\mathbb{N}_0}$, the direct product of infinite copies, indexed by $\mathbb{N}_0$, of the algebra $\mathrm{Mat}_N(\mathbb{C})$. In particular, if $D_1, D_2 \in \mathcal{D}(W)$ then

$$D_1 = D_2 \quad \text{if and only if} \quad \Gamma_n(D_1) = \Gamma_n(D_2) \ \text{ for all } n \geq 0. \tag{24}$$

For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$, the adjoint of D in $\mathcal{D}(W)$, such that

$$\langle PD, Q\rangle = \langle P, QD^*\rangle,$$

for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. See Theorem 4.3 and Corollary 4.5 in [15].

The map $D \mapsto D^*$ is a *-operation in the algebra $\mathcal{D}(W)$. Moreover it is showed that $\mathcal{S}(W)$, the set of all symmetric operators in $\mathcal{D}(W)$, is a real form of the space $\mathcal{D}(W)$, i.e.

$$\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W),$$

as real vector spaces. In particular to determine the algebra $\mathcal{D}(W)$ it is equivalent to determine all symmetric operators $\mathcal{S}(W)$.

In particular we have

Corollary 6.3. *A differential operator $D \in \mathcal{D}(W)$ is a symmetric operator if and only if*

$$\Lambda_w(D)\langle Q_w, Q_w\rangle = \langle Q_w, Q_w\rangle \Lambda_w(D)^*$$

for all $w \geq 0$.

Also it is worth to recall the following important result from [15].

Proposition 6.4 (Proposition 2.10). *If $D \in \mathcal{D}$ is symmetric then $D \in \mathcal{D}(W)$.*

Starting with [12], [10] and [5] one has a growing collection of weight matrices W for which the algebra $\mathcal{D}(W)$ is not trivial, i.e. does not consist only of scalar multiples of the identity operator. The first attempt to go beyond the issue of the existence of one non trivial element in $\mathcal{D}(W)$ and to study the full algebra is undertaken in [2]. In the example considered in [29], the conjecture set forth in [2] is proved and the structure of the algebra is studied in detail.

In this section we discuss some properties of the structure of this algebra for our weight matrix

$$W(x) = (1-x^2)^{\frac{n}{2}-1} \begin{pmatrix} p\,x^2+n-p & -nx \\ -nx & (n-p)x^2+p \end{pmatrix}, \qquad 2p \neq n.$$

We observe that in this particular case, our polynomials $\{P_w\}_w$,

$$P_w(x) = \begin{pmatrix} \frac{1}{n+1}\,C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w}\,C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w}\,C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w}\,C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1}\,C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w}\,C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix}$$

and the monic orthogonal polynomials $\{Q_w\}_w$,

$$Q_w = \frac{w!(n+1)}{2^w\left(\frac{n+1}{2}\right)_w} P_w,$$

have the same sequence of eigenvalues, since they are related by a scalar multiple.

First of all we have that the space of differential operators of *order zero* in $\mathcal{D}(W)$ consists of scalar multiplies of the identity operator. In fact, a differential operator of order zero having the sequence of monic orthogonal polynomials $\{Q_w\}_w$ as eigenfunctions, is a constant matrix L such that

$$Q_w L = \Lambda_w\, Q_w, \qquad \text{for all } w \geq 0.$$

From (23) we have that $\Lambda_w = L$ for every w. When $w = 1$, we obtain that the entries of L satisfy $L_{11} = L_{22}$ and $(p+1)L_{12} = (n-p+1)L_{21}$. Thus, looking at the case $w = 2$ we get $(n-2p)L_{12} = 0$. Therefore we obtain that any operator of order zero L in $\mathcal{D}(W)$ is a multiple of the identity matrix.

Now we will study differential operators of order at most two in the algebra $\mathcal{D}(W)$. Let $\{Q_w\}$ the monic sequence of orthogonal polynomials with respect to W and D a differential operator in $\mathcal{D}$. From Proposition 6.1 we have

$$D = \partial(A_2x^2 + A_1x + A_0) + \partial(B_1x + B_0) + C \in \mathcal{D}(W)$$

if and only if

$$Q_w D = \big(w(w-1)A_2 + wB_1 + C\big)Q_w \qquad \text{for all } w \geq 0. \tag{25}$$

Here $A_2, A_1, A_0, B_1, B_0, C$ are 2×2 complex matrices. Let us denote $Q_{w,j}$ the coefficients of the polynomial Q_w, i.e: $Q_w = \sum_{j=0}^{w} Q_{w,j}\,x^j$, with $Q_{w,w} = I$. Therefore $D \in \mathcal{D}(W)$ if and only if

$$\begin{aligned} j(j-1)Q_{w,j}A_2 + j(j+1)Q_{w,j+1}A_1 + (j+1)(j+2)Q_{w,j+2}A_0 + jQ_{w,j}B_1 \\ + (j+1)Q_{w,j+1}B_0 + Q_{w,j}C - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,j} = 0 \end{aligned}$$

for all $w \geq 0$ and $j = 0, \ldots, w$. For $j = w-1$ and $j = 0$ we obtain

$$\begin{aligned} (w-1)(w-2)Q_{w,w-1}A_2 + w(w-1)A_1 + (w-1)Q_{w,w-1}B_1 + wB_0 \\ + Q_{w,w-1}C - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,w-1} = 0 \end{aligned} \tag{26}$$

and

$$2Q_{w,2}A_0 + Q_{w,1}B_0 + Q_{w,0}\,C - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,0} = 0. \tag{27}$$

Now from (26) considering $w=1$ and $w=2$, and from (27) considering $w=2$, we obtain

$$\begin{aligned} B_0 &= (B_1+C)Q_{1,0} - Q_{1,0}C,\\ 2A_1 &= (2A_2+2B_1+C)Q_{2,1} - Q_{2,1}B_1 - 2B_0 - Q_{2,1}C,\\ 2A_0 &= (2A_2+2B_1+C)Q_{2,0} - Q_{2,1}B_0 - Q_{2,0}C. \end{aligned}$$

From the expression of Q_1 and Q_2, given at the end of Section 5, we know that

$$Q_{1,0} = \begin{pmatrix} 0 & \frac{1}{p+1} \\ \frac{1}{n-p+1} & \end{pmatrix}, Q_{2,1} = \begin{pmatrix} 0 & \frac{2}{p+2} \\ \frac{2}{n-p+2} & 0 \end{pmatrix}, Q_{2,0} = \frac{-p}{(n+3)} \begin{pmatrix} \frac{1}{(p+2)} & 0 \\ 0 & \frac{1}{(n-p+2)} \end{pmatrix}.$$

By using (21) and (2) it is easy to see that

$$Q_{w,w-1} = \begin{pmatrix} 0 & \frac{w}{p+w} \\ \frac{w}{n-p+w} & \end{pmatrix}, \qquad \text{for all } w \geq 1.$$

To determine the matrices $A_2=(a_{ij})$, $B_1=(b_{ij})$ and $C=(c_{ij})$, we first combine the entries in the diagonal of the matrix (26) to obtain

$$\begin{aligned} 2(n+2)a_{21} &= \frac{\big((n+p+2)b_{21}-2c_{21}\big)}{p+1} + \frac{(p+2)(p+w)(2c_{12}-(n-p)b_{12})}{(n-p+1)(n-p+2)(n-p+w)},\\ 2(n+2)a_{12} &= \frac{\big((2n-p+2)b_{12}-2c_{12}\big)}{n-p+1} + \frac{(n-p+2)(n-p+w)(2c_{21}-p\,b_{21})}{(p+1)(p+2)(p+w)}. \end{aligned}$$

Since these identities are valid for any $w\geq 3$ we conclude that, if $n\neq 2p$ then

$$2c_{12} = (n-p)b_{12} \qquad \text{and} \qquad 2c_{21} = p\,b_{21}.$$

Therefore

$$b_{21} = 2(p+1)a_{21} \qquad \text{and} \qquad b_{12} = 2(n-p+1)a_{12}.$$

By combining the non diagonal entries of (26) we have

$$(n-2p+1)\big((n+2)a_{11}-b_{11}\big) = (n-2p-1)\big((n+2)a_{22}-b_{22}\big)$$

and

$$c_{11}-c_{22} = (p+1)(p+2)a_{22} - p(p+1)a_{11} + p\,b_{11} - (p+1)b_{22}.$$

Equation (27) with $w=3$ says that

$$2Q_{3,2}A_0 + Q_{3,1}B_0 + Q_{3,0}\,C - \big(6A_2+3B_1+C\big)Q_{3,0} = 0.$$

Now, by using the expression of $Q_3 = x^3 + Q_{3,2}x^2 + Q_{3,1}x + Q_{3,0}$ given at the end of Section 5, it is not difficult to see that

$$b_{11} = (n+2)a_{11}.$$

Thus

$$\begin{aligned} b_{22} &= (n+2)a_{22},\\ c_{11}-c_{22} &= p(n-p+1)a_{11} - (p+1)(n-p)a_{22}. \end{aligned}$$

Therefore, we have obtained that for any differential operator of the form

$$D = \partial(A_2x^2 + A_1x + A_0) + \partial(B_1x + B_0) + C \in \mathcal{D}(W)$$

the matrices $A_2, A_1, A_0, B_1, B_0, C$ are given by

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{12}-a_{21} & a_{11}-a_{22} \\ a_{22}-a_{11} & a_{21}-a_{12} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix},$$
$$B_1 = \begin{pmatrix} (n+2)a_{11} & 2(n-p+1)a_{12} \\ 2(p+1)a_{21} & (n+2)a_{22} \end{pmatrix},$$
$$B_2 = \begin{pmatrix} -p\,a_{21}+(n-p+2)a_{12} & (n-p+2)a_{11}-(n-p)a_{22} \\ -p\,a_{11}+(p+2)a_{22} & (p+2)a_{21}-(n-p)a_{12} \end{pmatrix},$$
$$C = \begin{pmatrix} p\,(n-p+1)a_{11}+c & (n-p)(n-p+1)a_{12} \\ p\,(p+1)a_{21} & (p+1)(n-p)a_{22}+c \end{pmatrix}.$$

for some $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$.

Theorem 6.5. *The differential operators of order at most two in $\mathcal{D}(W)$ are of the form*

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0,$$

where

$$\begin{aligned} F_2(x) =& x^2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + x \begin{pmatrix} a_{12}-a_{21} & a_{11}-a_{22} \\ a_{22}-a_{11} & a_{21}-a_{12} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}, \\ F_1(x) =& x \begin{pmatrix} (n+2)a_{11} & 2(n-p+1)a_{12} \\ 2(p+1)a_{21} & (n+2)a_{22} \end{pmatrix} \\ &+ \begin{pmatrix} -p\,a_{21}+(n-p+2)a_{12} & (n-p+2)a_{11}-(n-p)a_{22} \\ -p\,a_{11}+(p+2)a_{22} & (p+2)a_{21}-(n-p)a_{12} \end{pmatrix}, \\ F_0 =& \begin{pmatrix} p\,(n-p+1)a_{11}+c & (n-p)(n-p+1)a_{12} \\ p\,(p+1)a_{21} & (p+1)(n-p)a_{22}+c \end{pmatrix}. \end{aligned} \tag{28}$$

with a_{11}, a_{12}, a_{21}, a_{22}, c arbitrary complex numbers.

Proof. We have already proved that any differential operator in $\mathcal{D}(W)$ is of this form for some constant $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$. Let $\mathcal{D}_2$ be the complex vector space of the differential operators in $\mathcal{D}(W)$ of order at most two. Then we have that

$$\dim \mathcal{D}_2 \leq 5.$$

From Proposition 2.1 it is not difficult to see that a differential operator D of order two, with coefficients given by (28), is a symmetric operator if and only if

$$a_{11}, a_{22}, c \in \mathbb{R} \qquad \text{and} \quad p\,a_{21} = (n-p)\,\overline{a}_{12}. \tag{29}$$

From Proposition 6.4 any symmetric operator $D \in \mathcal{D}$ belongs to the algebra $\mathcal{D}(W)$. Thus there exists (at least) five linearly independent symmetric operators in $\mathcal{D}(W)$. Therefore

$$\dim \mathcal{D}_2 = 5.$$

This concludes the proof of the theorem. □

Corollary 6.6. *There are no operators of order one in the algebra $\mathcal{D}(W)$.*

The elements of the sequence $\{Q_w\}_w$ are eigenfunctions of the operators in $\mathcal{D}(W)$ and they satisfy

$$Q_w D_j = \Lambda_w(D_j) Q_w, \qquad \text{for } j = 1,2,3,4,\ w \geq 0.$$

We explicitly state the eigenvalues Λ_w using formula (23): for a differential operator $D=\partial^2 F_2+\partial F_1+F_0$ we have

$$\Lambda_w(D)=w(w-1)F_2^2+wF_1^1+F_0^0,$$

with F_i^i (i=1,2,3) the leading coefficient of the polynomial coefficient F_i of the differential operator D. Therefore we get

Corollary 6.7. *Let $D\in\mathcal{D}(W)$, defined as in Theorem 6.5. The matrix monic orthogonal polynomials $\{Q_w\}_w$ satisfy*

$$Q_wD=\Lambda_w(D)Q_w,\qquad \text{for } w\geq 0,$$

where the eigenvalue $\Lambda_w(D)$ is given by

$$\Lambda_w(D)=\begin{pmatrix}(w+p)(w+n-p+1)a_{11}+c & (w+n-p)(w+n-p+1)a_{12}\\ (w+p)(w+p+1)a_{21} & (w+n-p)(w+p+1)a_{22}+c\end{pmatrix}.$$

Now we introduce an useful basis for the differential operators of order at most two in the algebra $\mathcal{D}(W)$: the identity I and

$$D_1=\partial^2\begin{pmatrix}x^2 & x\\ -x & -1\end{pmatrix}+\partial\begin{pmatrix}(n+2)x & n-p+2\\ -p & 0\end{pmatrix}+\begin{pmatrix}p\,(n-p+1) & 0\\ 0 & 0\end{pmatrix},$$

$$D_2=\partial^2\begin{pmatrix}-1 & -x\\ x & x^2\end{pmatrix}+\partial\begin{pmatrix}0 & p-n\\ p+2 & (n+2)x\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & (p+1)(n-p)\end{pmatrix},$$

$$D_3=\partial^2\begin{pmatrix}-x & -1\\ x^2 & x\end{pmatrix}+\partial\begin{pmatrix}-p & 0\\ 2(p+1)x & p+2\end{pmatrix}+\begin{pmatrix}0 & 0\\ p(p+1) & 0\end{pmatrix},$$

$$D_4=\partial^2\begin{pmatrix}x & x^2\\ -1 & -x\end{pmatrix}+\partial\begin{pmatrix}n-p+2 & 2(n-p+1)x\\ 0 & p-n\end{pmatrix}+\begin{pmatrix}0 & (n-p)(n-p+1)\\ 0 & 0\end{pmatrix}.$$

The corresponding eigenvalues are

$$\Lambda_w(D_1)=\left(\begin{smallmatrix}(w+p)(w+n-p+1) & 0\\ 0 & 0\end{smallmatrix}\right),\qquad \Lambda_w(D_2)=\left(\begin{smallmatrix}0 & 0\\ 0 & (w+p+1)(w+n-p)\end{smallmatrix}\right),$$
$$\Lambda_w(D_3)=\left(\begin{smallmatrix}0 & 0\\ (w+p)(w+p+1) & 0\end{smallmatrix}\right),\qquad \Lambda_w(D_4)=\left(\begin{smallmatrix}0 & (w+n-p)(w+n-p+1)\\ 0 & 0\end{smallmatrix}\right).$$

Remark 6.8. The differential operator D appearing in Theorem 4.1 is

$$D=-D_1-D_2+p(n-p)I.$$

We observe here that, for example,

$$\Lambda_w(D_1)\Lambda_w(D_3)\neq\Lambda_w(D_3)\Lambda_w(D_1),\qquad \text{for all } w\geq 0.$$

From Proposition 6.2 we have an isomorphism from the algebra $\mathcal{D}(W)$ into the algebra $\mathrm{Mat}_2(\mathbb{C})^{\mathbb{N}_0}$. This isomorphism is clearly useful in any attempt to get the structure on our algebra. By using this we obtain that $D_1D_3\neq D_3D_1$. In particular we get the the following result.

Corollary 6.9. *The algebra $\mathcal{D}(W)$ is not commutative.*

By following the same argument, through the sequence of eigenvalues, we obtain the following relations among the differential operators D_1, D_2, D_3, D_4.

$$\begin{aligned}
&D_1D_2 = 0, \quad D_2D_1 = 0, \quad D_1D_3 = 0, \quad D_4D_1 = 0,\\
&D_2D_4 = 0, \quad D_3D_2 = 0, \qquad D_3^2 = 0, \qquad D_4^2 = 0,\\
&D_3D_1 = D_2D_3 - (n-2p)D_3, \qquad D_1D_4 = D_4D_2 - (n-2p)D_4,\\
&D_3D_4 = D_2^2 - (n-2p)D_2, \qquad D_4D_3 = D_1^2 + (n-2p)D_1.
\end{aligned}$$

With the help of symbolic computations, we prove that there are no operators of *order three* nor of *order five* in the algebra $\mathcal{D}(W)$ and we see that the vector space of differential operators in $\mathcal{D}(W)$ of *order four*, modulo differential operators of lower order, has dimension four. All of these operators are generated, in the algebra sense, by the four second order differential operators D_1, D_2, D_3 and D_4 given earlier. We interpret here that the $D^0 = \mathrm{I}$ is the identity.

Conjecture 6.10.

(1) *There are no operators of odd order in* $\mathcal{D}(W)$.
(2) *The second order differential operators in* $\mathcal{D}(W)$ *generate the algebra* $\mathcal{D}(W)$.

For a differential operator of order two $D = \partial^2 F_2 + \partial F_1 + F_0 \in \mathcal{D}(W)$, the explicit expression of the adjoint operator D^* is

$$D^* = \partial^2 G_2 + \partial G_1 + G_0, \tag{30}$$

where the polynomials G_i, $i = 0, 1, 2$, are defined by

$$\begin{aligned}
G_0 &= \langle Q_0, Q_0\rangle \Lambda_0(D)^* \langle Q_0, Q_0\rangle^{-1},\\
G_1 &= \langle Q_1, Q_1\rangle \Lambda_1(D)^* \langle Q_1, Q_1\rangle^{-1} Q_1(x) - Q_1(x)G_0,\\
G_2 &= \langle Q_2, Q_2\rangle \Lambda_2(D)^* \langle Q_2, Q_2\rangle^{-1} Q_2(x) - \partial(Q_2)G_1(x) - Q_2(x)G_0,
\end{aligned}$$

see Theorem 4.3 in [15].

Also from Corollary 4.5 in [15], we obtain the expression for the corresponding eigenvalues for the adjoint operator D^*, in terms of the eigenvalues of the differential operator D and the norm of the polynomials Q_w,

$$\Lambda_w(D^*) = \langle Q_w, Q_w\rangle \Lambda_w(D)^* \langle Q_w, Q_w\rangle^{-1}, \quad \text{for all } w.$$

By using the expressions of $\langle Q_i, Q_i\rangle$, given at the end of Section 5, and making straightforward computations, we can verify that

$$D_1^* = D_1, \quad D_2^* = D_2, \quad \text{and} \quad D_3^* = \tfrac{p}{n-p} D_4.$$

Therefore

$$E_3 = (n-p)D_3 + pD_4 \qquad \text{and} \qquad E_4 = i\big((n-p)D_3 - pD_4\big)$$

are also symmetric operators, because for any $D \in \mathcal{D}(W)$ the operators $D + D^*$ and $i(D - D^*)$ are symmetric operators.

Explicitly,

$$\begin{aligned}
E_3 &= (n-p)D_3 + pD_4 \\
&= \partial^2 \begin{pmatrix} -x(n-2p) & x^2p-n+p \\ x^2(n-p)-p & x(n-2p) \end{pmatrix} + \partial \begin{pmatrix} 2p & 2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
-iE_4 &= (n-p)D_3 - pD_4 \\
&= \partial^2 \begin{pmatrix} -nx & -x^2p-n+p \\ x^2(n-p)+p & nx \end{pmatrix} + \partial \begin{pmatrix} -2p(n-p+1) & -2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p)(p+1) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & -p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix}.
\end{aligned}$$

The corresponding eigenvalues are

$$\begin{aligned}
\Lambda_w(E_3) &= \begin{pmatrix} 0 & p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix}, \\
\Lambda_w(-iE_4) &= \begin{pmatrix} 0 & -p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix}.
\end{aligned}$$

Remark 6.11. In [17] the authors study matrix valued orthogonal polynomials related to spherical functions on the group $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{SU}(2))$. Particularly, in Subsection 8.3 an example of a weight matrix of size 3×3, that reduces to smaller size, appears. The irreducible 2×2 block is

$$W_1 = (1-x)^{1/2}(1+x)^{1/2}\begin{pmatrix} 4x^2+3 & 3\sqrt{2}x \\ 3\sqrt{2}x & x^2+2 \end{pmatrix}, \quad x\in[-1,1].$$

It is a particular case of the examples considered in the present paper. In fact let $n=3$ and $p=1$ in the weight W, given in (11)

$$W_{1,3} = (1-x^2)^{1/2}\begin{pmatrix} x^2+2 & -3x \\ -3x & 2x^2+1 \end{pmatrix}.$$

Therefore, with $L = \left(\begin{smallmatrix} 0 & \sqrt{2} \\ -1 & 0 \end{smallmatrix}\right)$ we get $W_1 = LW_{1,3}L^*$.

Let us denote $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$ be the differential operators D_1,D_2 and D_3 appearing in Theorem 8.1 in [17]. Then we have the following relations with our operators D_1, D_2,D_3 and D_4 for $n=3$ and $p=1$

$$\widetilde{D}_1 = L(D_1+D_2-3)L^{-1}, \quad \widetilde{D}_2 = LD_2L^{-1}, \quad \widetilde{D}_3 = -\sqrt{2}\,L(2D_3+D_4)L^{-1}.$$

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