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A BILATERAL GENERATING FUNCTION FOR THE ULTRASPHERICAL POLYNOMIALS

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The following differentiation formula for the Ultraspherical polynomials $P_n^\lambda(x)$ was given by Tricomi:

$$P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n(x^2-1)^{\lambda+1/2n}}{n!} D^n(x^2-1)^{-\lambda} . \tag{1.1}$$

The object of this paper is to point out that the formula of Tricomi leads us to the following bilateral generating function for the Ultraspherical polynomials:

THEOREM.
If $F(x, t) = \sum_{m=0}^\infty a_m t^m P_m^\lambda(x)$,
then

$$\rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right) = \sum_{r=0}^\infty t^r b_r(y) P_r^\lambda(x) , \tag{1.2}$$

where

$$b_r(y) = \sum_{m=0}^\infty \binom{r}{m} a_m y^m , \qquad \textit{and} \qquad \rho = (1-2xt+t^2)^{1/2} .$$

Starting from the formula (1.2), **one can derive a large number of bilateral generating functions for the Ultraspherical polynomials by attributing different values to** a_m.

2. Proof of the main formula (1.2). We first note from (1.1) that

$$\left(\frac{1}{\sqrt{x^2-1}}\right)^n P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n}{n!}(x^2-1)^\lambda D^n (x^2-1)^{-\lambda} . \tag{2.1}$$

Now let

$$F\left(\frac{x}{\sqrt{x^2-1}}, \quad \frac{t}{\sqrt{x^2-1}}\right) = \sum_{m=0}^\infty a_m \left(\frac{t}{\sqrt{x^2-1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right)$$

be a given generating function for $P_n^\lambda(x)$. Replacing t by ty and multiplying both sides by $(x^2-1)^{-\lambda}$ and then operating e^{-tD}, we get

$$\begin{aligned} &e^{-tD}(x^2-1)^{-\lambda} F\left(\frac{x}{\sqrt{x^2-1}}, \quad \frac{ty}{\sqrt{x^2-1}}\right) \\ &\quad = e^{-tD}(x^2-1)^{-\lambda} \sum_{m=0}^\infty a_m \left(\frac{ty}{\sqrt{x^2-1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) . \end{aligned} \tag{2.2}$$

Since we know that

$$e^{-tD}f(x) = f(x - t) , \tag{2.3}$$

the left member of (2.2) is equal to

$$\{(x - t)^2 - 1\}^{-\lambda} F\left(\frac{x - t}{\sqrt{(x - t)^2 - 1}} , \quad \frac{ty}{\sqrt{(x - t)^2 - 1}}\right) .$$

But the right member of (2.2) is equal to

$$\begin{aligned}
&\sum_{m=0}^{\infty} a_m (ty)^m e^{-tD} (x^2 - 1)^{-\lambda} \left(\frac{1}{\sqrt{x^2 - 1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right) \\
&\quad = \sum_{m=0}^{\infty} a_m (ty)^m e^{-tD} \left\{\frac{(-1)^m}{m\,!} D^m (x^2 - 1)^{-\lambda}\right\} \\
&\quad = \sum_{m=0}^{\infty} a_m (ty)^m \sum_{r=0}^{\infty} \frac{(-t)^r}{r\,!} D^r \left\{\frac{(-1)^m}{m\,!} D^m (x^2 - 1)^{-\lambda}\right\} \\
&\quad = \sum_{m=0}^{\infty} a_m y^m \sum_{r=0}^{\infty} \frac{(-t)^{r+m}}{r\,!\; m\,!} D^{r+m} (x^2 - 1)^{-\lambda} \\
&\quad = (x^2 - 1)^{-\lambda} \sum_{m=0}^{\infty} a_m y^m \sum_{r=0}^{\infty} \binom{r+m}{m} t^{r+m} \left(\frac{1}{\sqrt{x^2 - 1}}\right)^{r+m} P_{r+m}^\lambda \left(\frac{x}{\sqrt{x^2 - 1}}\right) \\
&\quad = (x^2 - 1)^{-\lambda} \sum_{r=0}^{\infty} \left(\frac{t}{\sqrt{x^2 - 1}}\right)^r P_r^\lambda \left(\frac{x}{\sqrt{x^2 - 1}}\right) \sum_{m=0}^{r} \binom{r}{m} a_m y^m .
\end{aligned}$$

It follows therefore that: If

$$F\left(\frac{x}{\sqrt{x^2 - 1}} , \quad \frac{t}{\sqrt{x^2 - 1}}\right) = \sum_{m=0}^{\infty} a_m \left(\frac{t}{\sqrt{x^2 - 1}}\right)^m P_m^\lambda \left(\frac{x}{\sqrt{x^2 - 1}}\right) ,$$

then

$$\begin{aligned}
&\left\{\frac{(x - t)^2 - 1}{x^2 - 1}\right\}^{-\lambda} F\left(\frac{x - t}{\sqrt{(x - t)^2 - 1}} , \quad \frac{ty}{\sqrt{(x - t)^2 - 1}}\right) \\
&\quad = \sum_{r=0}^{\infty} \left(\frac{t}{\sqrt{x^2 - 1}}\right)^r b_r(y) P_r^\lambda \left(\frac{x}{\sqrt{x^2 - 1}}\right) ,
\end{aligned} \tag{2.4}$$

where $b_r(y) = \sum_{m=0}^{r} \binom{r}{m} a_m y^m$. Now changing $x(x^2 - 1)^{-1/2}$ into x and then t into $t(x^2 - 1)^{-1/2}$, we obtain the theorem mentioned in the introduction.

3. Some applications of the theorem.

(A) First we consider the generating function of Truesdell:

$$e^{xt}\, {}_0F_1\left(-; \lambda + \frac{1}{2} ; \frac{t^2(x^2 - 1)}{4}\right) = \sum_{m=0}^{\infty} \frac{t^m}{(2\lambda)_m} P_m^\lambda(x) . \tag{3.1}$$

Thus if we take $a_m = 1/(2\lambda)_m$ in our theorem, we obtain

$$\rho^{-2} \exp\left\{\frac{yt(x - t)}{\rho^2}\right\} {}_0F_1\left(-; \lambda + \frac{1}{2} ; \frac{y^2 t^2 (x^2 - 1)}{4\rho^4}\right) = \sum_{r=0}^{\infty} t^r b_r(y) P_r^\lambda(x) .$$

But we notice that

$$b_r(y) = {}_1F_1(-r; 2\lambda; -y) = \frac{r!}{(2\lambda)_r} L_r^{(2\lambda-1)}(-y) .$$

Hence we derive the following generating function of Weisner [3].

$$\rho^{-2\lambda} \exp\left\{\frac{-yt(x-t)}{\rho^2}\right\} {}_0F_1\left(-; \lambda + \frac{1}{2}; \frac{y^2t^2(x^2-1)}{4\rho^4}\right) = \sum_{r=0}^{\infty} \frac{r!\, L_r^{(2\lambda-1)}(y)}{(2\lambda)_r} t^r P_r^{\lambda}(x) . \tag{3.2}$$

Thus we remark that the bilateral generating function of Weisner is a particular case of our theorem. Moreover we have obtained the theorem by a method different from that used by Weisner or from that used by Rainville [2].

(B) If we consider the formula of Brafman:

$$(1 - xt)^{-\gamma} {}_2F_1\left(\frac{1}{2}\gamma, \frac{1}{2}\gamma + \frac{1}{2}; \lambda + \frac{1}{2}; \frac{t^2(x^2-1)}{(1-xt)^2}\right) = \sum_{m=0}^{\infty} \frac{(\gamma)_m t^m}{(2\lambda)_m} P_m^{\lambda}(x) , \tag{3.3}$$

then we put $a_m = (\gamma)_m/(2\lambda)_m$ in our theorem and we obtain

$$\rho^{2(\gamma-\lambda)}\{\rho^2 + yt(x-t)\}^{-\gamma} {}_2F_1\left(\frac{1}{2}\gamma, \frac{1}{2}\gamma + \frac{1}{2}; \lambda + \frac{1}{2}; \frac{y^2t^2(x^2-1)}{(\rho^2 + yt(x-t))^2}\right) = \sum_{r=0}^{\infty} {}_2F_1(-r, \gamma; 2\lambda; y) t^r P_r^{\lambda}(x) . \tag{3.4}$$

(C) Next we consider the following generating function of Bateman:

$${}_0F_1\left(-; \lambda + \frac{1}{2}; \frac{t(x-1)}{2}\right) {}_0F_1\left(-; \lambda + \frac{1}{2}; \frac{t(x+1)}{2}\right) = \sum_{m=0}^{\infty} \frac{t^m}{(2\lambda)_m\left(\lambda + \frac{1}{2}\right)_m} P_m^{\lambda}(x) . \tag{3.5}$$

Here we set $a_m = 1/\{(2\lambda)_m(\lambda + 1/2)_m\}$ in our theorem and we derive

$$\rho^{-2\lambda} {}_0F_1\left(-; \lambda + \frac{1}{2}; \frac{yt(t-x+\rho)}{2\rho^2}\right) {}_0F_1\left(-; \lambda + \frac{1}{2}; \frac{yt(t-x-\rho)}{2\rho^2}\right) = \sum_{r=1}^{\infty} {}_1F_2\left(-r; 2\lambda, \lambda + \frac{1}{2}; y\right) t^r P_r^{\lambda}(x) . \tag{3.6}$$

(D) Lastly if we consider the following generating function of Brafman:

$$
\begin{aligned}
&{}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1-t-\rho}{2}\right) x \\
(3.7) \qquad &{}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1+t-\rho}{2}\right) \\
&= \sum_{m=0}^{\infty} \frac{(\gamma)_m (2\lambda - \gamma)_m}{(2\lambda)_m \left(\lambda + \frac{1}{2}\right)_m} t^m P_m^{\lambda}(x) \; ;
\end{aligned}
$$

we put

$$
a_m = \frac{(\gamma)_m (2\lambda - \gamma)_m}{(2\lambda)_m \left(\lambda + \frac{1}{2}\right)_m}
$$

in our theorem and thus we obtain

$$
\begin{aligned}
&\rho^{-2\lambda}\, {}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{\rho + yt - \omega}{2\rho}\right) x \\
(3.8) \qquad &{}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{\rho - yt - \omega}{2\rho}\right) \\
&= \sum_{r=0}^{\infty} {}_3F_2\left(-r, \gamma, 2\lambda - \gamma; 2\lambda, \lambda + \frac{1}{2}; y\right) t^r P_r^{\lambda}(x) \; ;
\end{aligned}
$$

where

$$
\omega = [1 - 2xt(1-y) + t^2(1-y)^2]^{1/2} \, .
$$

References

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