A CHARACTERIZATION OF ULTRASPHERICAL POLYNOMIALS

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ABSTRACT. We show that the only orthogonal polynomials with a generating function of the form $F\left(xz - \alpha z^2\right)$ are the ultraspherical, Hermite, and Chebyshev polynomials of the first kind. The generating function for the Chebyshev case is non-standard, although it is easily derived from the usual one.

1. THE QUESTION

Hermite polynomials $H_n(x)$ are one of the most important families of orthogonal polynomials in mathematics, appearing in probability theory, mathematical physics, differential equations, combinatorics, etc. One of the simplest ways to construct them is through their generating function,

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n = \exp\left(xz - z^2/2\right).$$

On the other hand, Chebyshev polynomials of the second kind $U_n(x)$ are another important family of orthogonal polynomials (appearing in numerical analysis, for example), with a generating function

$$\sum_{n=0}^{\infty} U_n(x) z^n = \frac{1}{1 - xz + z^2}.$$

Note that both of these functions have the form $F\left(xz - \alpha z^2\right)$, with $F(z) = e^z$, respectively, $F(z) = \frac{1}{1-z}$.

Question 1. What are all the orthogonal polynomials with generating functions of the form

$$F\left(xz - \alpha z^2\right)$$

for some number α and function (or, more precisely, formal power series) F?

A reader interested in further context and background for this question may want to start by reading Section 5.

2. THE METHOD

The following is the first fundamental theorem about orthogonal polynomials.

Theorem (Favard's theorem). *Let* $\{P_0(x), P_1(x), P_2(x), \ldots\}$ *be a* monic orthogonal polynomial family. *That is,*

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- *Each P_n is a polynomial of degree exactly n (polynomial family).*
- *Each*
$$P_n(x) = x^n + \text{ lower order terms}$$
(monic).
- *For some non-decreasing function M with $\lim_{x \to -\infty} M(x) = 0$, $\lim_{x \to \infty} M(x) = 1$, and infinitely many points of increase (equivalently, for some probability measure supported on infinitely many points) the Stieltjes integral*
$$\int_{-\infty}^{\infty} P_n(x) P_k(x) \, dM(x) = 0$$
for all $n \neq k$ (orthogonal).

Then there exist real numbers $\{\beta_0, \beta_1, \beta_2, \ldots\}$ and positive real numbers $\{\omega_1, \omega_2, \omega_3, \ldots\}$ such that the polynomials satisfy a three-term recursion relation
$$x P_n = P_{n+1} + \beta_n P_n + \omega_n P_{n-1}$$
(to make the formula work for $n = 0$, take $P_{-1} = 0$).

Actually, Favard's theorem also asserts that the converse to the statement above is true: if we have a monic polynomial family satisfying such a recursion, these polynomials are automatically orthogonal for some $M(x)$, and the case of M with only finitely many points of increase can also be included. See [Chi78, Chapter 1] or [Ism05, Chapter 2] for the proof. We only need the "easy" direction stated above, which we now prove.

Proof. Every polynomial P of degree k is a linear combination of $\{P_i : 0 \leq i \leq k\}$. Therefore P is orthogonal to all P_n with $k < n$. Since
$$x P_n(x) = x^{n+1} + \text{ lower order terms},$$
we can expand
$$x P_n(x) = P_{n+1}(x) + c_{n,n} P_n(x) + c_{n,n-1} P_{n-1}(x) + \ldots + c_{n,1} P_1(x) + c_{n,0} P_0(x)$$
for some coefficients $c_{n,n}, \ldots, c_{n,0}$. On the other hand, for $k < n - 1$ orthogonality implies
$$c_{n,k} \int_{-\infty}^{\infty} P_k(x) P_k(x) \, dM(x) = \int_{-\infty}^{\infty} \Big(x P_n(x) \Big) P_k(x) \, dM(x) = \int_{-\infty}^{\infty} P_n(x) \Big(x P_k(x) \Big) \, dM(x) = 0$$
since $\deg(x P_k) < n$. Since M has infinitely many points of increase, P_k cannot be zero at all of those points, and as a result
$$\int_{-\infty}^{\infty} P_k^2(x) \, dM(x) > 0.$$
Therefore $c_{n,k} = 0$ for $k < n - 1$, so in fact
$$x P_n(x) = P_{n+1}(x) + c_{n,n} P_n(x) + c_{n,n-1} P_{n-1}(x)$$

only. Denoting $\beta_n = c_{n,n}$ and $\omega_n = c_{n,n-1}$, we get the formula. It remains to note that

$$\begin{aligned} c_{n,n-1}\int_{-\infty}^{\infty} P_{n-1}(x)P_{n-1}(x)\,dM(x) &= \int_{-\infty}^{\infty} P_{n-1}(x)\Big(xP_n(x) - P_{n+1}(x) - c_{n,n}P_n(x)\Big)\,dM(x) \\ &= \int_{-\infty}^{\infty} P_{n-1}(x)\Big(xP_n(x)\Big)\,dM(x) \\ &= \int_{-\infty}^{\infty} \Big(xP_{n-1}(x)\Big)P_n(x)\,dM(x) \\ &= \int_{-\infty}^{\infty} P_n(x)P_n(x)\,dM(x). \end{aligned}$$

Since both $\int_{-\infty}^{\infty} P_{n-1}^2(x)\,dM(x)$ and $\int_{-\infty}^{\infty} P_n^2(x)\,dM(x)$ are positive, so is $\omega_n = c_{n,n-1}$. $\square$

3. Examples

Good references for polynomial families are Wikipedia and [KS98]. Beware that in the following examples, we use for monic polynomials the notation which usually appears for other normalizations, so our formulas may differ from the references by a re-scaling.

Example 1. The orthogonality relation for the Hermite polynomials is

$$\int_{-\infty}^{\infty} H_n(x)H_k(x)\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)\,dx = 0. \tag{1}$$

Thus they are orthogonal with respect to the normal (Gaussian) distribution. They satisfy a recursion

$$xH_n(x) = H_{n+1}(x) + nH_{n-1}(x).$$

Example 2. The orthogonality relation for the Charlier polynomials is

$$\sum_{i=0}^{\infty} C_n(i)C_k(i)\left(e^{-1}\frac{1}{i!}\right) = 0, \tag{2}$$

so that

$$M(x) = e^{-1}\sum_{i=0}^{[x]}\frac{1}{i!},$$

which is sometimes also written as

$$dM(x) = e^{-1}\sum_{i=0}^{\infty}\frac{1}{i!}\delta_i(x).$$

Thus the Charlier polynomials are orthogonal with respect to the Poisson distribution. They satisfy a recursion

$$xC_n(x) = C_{n+1}(x) + (n+1)\,C_n(x) + n\,C_{n-1}(x).$$

Example 3. The Legendre polynomials are the family of orthogonal polynomials a student is most likely to encounter in an undergraduate course. In a linear algebra course, one sees them in the applications of the Gram-Schmidt formula, since their orthogonality relation is simply

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = 0.$$

In a differential equations course, one sees them as a solution of the Legendre equation. This equation, in turn, arises after the separation of variables in the heat equation or Laplace's equation in three dimensions, or on a sphere. The recursion satisfied by the Legendre polynomials is

$$xP_n(x) = P_{n+1}(x) + \frac{n^2}{4n^2-1} P_{n-1}(x)$$

Example 4. The ultraspherical (also called Gegenbauer) polynomials $C_n^{(\lambda)}$ are orthogonal with respect to

$$M(x) = (1-x^2)^{\lambda - \frac{1}{2}}$$

(with integration restricted to $[-1, 1]$) for $\lambda > -\frac{1}{2}, \lambda \neq 0$. Note that the Legendre polynomials are a particular case corresponding to $\lambda = \frac{1}{2}$. Just as the Legendre polynomials are related to the sphere, the other ultraspherical families for half-integer λ are related to higher-dimensional spheres. They satisfy a recursion

$$xC_n^{(\lambda)}(x) = C_{n+1}^{(\lambda)}(x) + \frac{n(n+2\lambda-1)}{4(n+\lambda-1)(n+\lambda)} C_{n-1}^{(\lambda)}(x), \tag{3}$$

see Section 1.8.1 of [KS98]. Another important special case are the Chebyshev polynomials of the second kind U_n, which correspond to $\lambda = 1$ and are orthogonal with respect to $\sqrt{1-x^2}$.

For $\lambda = 0$, the coefficient ω_1 as written in formula (3) is undefined. The polynomials orthogonal with respect to $\frac{1}{\sqrt{1-x^2}}$ are the Chebyshev polynomials of the first kind T_n, for which the recursion (3) holds for $n \geq 2$ (with $\lambda = 0$), but

$$xT_1(x) = T_2(x) + \frac{1}{2} T_0(x).$$

Remark 1. Suppose $\{P_n\}$ form a monic orthogonal polynomial family, with orthogonality given by a function M and the recursion

$$xP_n = P_{n+1} + \beta_n P_n + \omega_n P_{n-1}. \tag{4}$$

Then for $r \neq 0$, the polynomials

$$Q_n(x) = r^n P_n(x/r)$$

are also monic, orthogonal with respect to $N(x) = M(x/r)$, and satisfy

$$xQ_n = Q_{n+1} + r\beta_n Q_n + r^2 \omega_n Q_{n-1}.$$

Indeed, from equation (4),

$$r^{n+1}(x/r)P_n(x/r) = r^{n+1}P_{n+1}(x/r) + \beta_n r^{n+1} P_n(x/r) + \omega_n r^{n+1} P_{n-1}(x/r),$$

which implies the recursion for $\{Q_n\}$.

4. The result

Theorem 1. *Let $\alpha > 0$ and $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be a formal power series with $c_0 = 1$, $c_1 = c \neq 0$. Define the polynomials $\{P_n : n \geq 0\}$ via*

$$F\left(xz - \alpha z^2\right) = \sum_{n=0}^{\infty} c_n P_n(x) z^n \tag{5}$$

(if $c_n = 0$, P_n is undefined). These polynomials form an orthogonal polynomial family (which is automatically monic) if and only if

- $\{P_n\}$ *are re-scaled ultraspherical polynomials,*

$$P_n(x) = C_n^{(\lambda)}\left(\sqrt{\frac{b}{4\alpha}}\,x\right),$$

for $\lambda > -\frac{1}{2}$, $\lambda \neq 0$, *and* $b > 0$. *In this case*

$$F(z) = 1 + \frac{c}{\lambda b}\left(\frac{1}{(1-bz)^{\lambda}} - 1\right)$$

for $c \neq 0$. *The choice* $c = \lambda b$ *gives simply* $F(z) = \frac{1}{(1-bz)^{\lambda}}$.
- $\{P_n\}$ *are re-scaled Chebyshev polynomials of the first kind,*

$$P_n(x) = T_n\left(\sqrt{\frac{b}{4\alpha}}\,x\right),$$

for $b > 0$. *In this case*

$$F(z) = 1 + \frac{c}{b}\ln\left(\frac{1}{1-bz}\right)$$

for $c \neq 0$. *The choice* $c = b$ *gives simply* $F(z) = 1 + \ln\left(\frac{1}{1-bz}\right)$.
- $\{P_n\}$ *are re-scaled Hermite polynomials,*

$$P_n(x) = H_n\left(\sqrt{\frac{a}{2\alpha}}\,x\right)$$

for $a > 0$. *In this case*

$$F(z) = 1 + \frac{c}{a}\left(e^{az} - 1\right)$$

for $c \neq 0$. *The choice* $c = a$ *gives simply* $F(z) = e^{az}$.

Remark 2. If equation (5) holds, so that

$$1 + \sum_{n=1}^{\infty} c_n \left(xz - \alpha z^2\right)^n = F\left(xz - \alpha z^2\right) = 1 + \sum_{n=1}^{\infty} c_n P_n(x) z^n,$$

then clearly also

$$1 + \sum_{n=1}^{\infty} C c_n \left(xz - \alpha z^2\right)^n = F_C\left(xz - \alpha z^2\right) = 1 + \sum_{n=1}^{\infty} C c_n P_n(x) z^n$$

for any $C \neq 0$ and $F_C(z) = 1 + C(F(z) - 1)$. This is the source of the free parameter c in the theorem.

Proof of the Theorem. Using the binomial formula, expand

$$\begin{aligned} F\left(xz - \alpha z^2\right) &= \sum_{n=0}^{\infty} c_n \left(xz - \alpha z^2\right)^n \\ &= \sum_{n=0}^{\infty} c_n x^n z^n - \sum_{n=1}^{\infty} c_n n (xz)^{n-1} \alpha z^2 + \dots \\ &= 1 + \sum_{n=1}^{\infty} \Big(c_n x^n - \alpha (n-1) c_{n-1} x^{n-2} \Big) z^n + \dots \end{aligned}$$

If for some $n \geq 2$, $c_n = 0$ while $c_{n-1} \neq 0$, then comparing the coefficients of z^n in the preceding equation and expansion (5), we see that the coefficient is non-zero on the left and zero on the right. So all $c_n \neq 0$, and we may denote $d_n = \frac{c_n}{c_{n-1}}$. Then the same coefficient comparison gives $P_0(x) = 1$, and for $n \geq 1$

$$P_n(x) = x^n - \alpha (n-1) d_n^{-1} x^{n-2} + \dots$$

Using this equation for both n and $n+1$, we then get

$$\begin{aligned} x P_n &= x^{n+1} - \alpha (n-1) d_n^{-1} x^{n-1} + \dots \\ &= P_{n+1} - \alpha (n-1) d_n^{-1} x^{n-1} + \alpha n d_{n+1}^{-1} x^{n-1} + \dots \end{aligned}$$

If we want the polynomials to be orthogonal, by Favard's theorem they have to satisfy a three-term recursion relation

$$x P_n = P_{n+1} + \beta_n P_n + \omega_n P_{n-1}$$

(note that $\{P_n\}$ are clearly monic). We see that $\beta_n = 0$, and

$$\omega_n = \alpha \Big(n d_{n+1}^{-1} - (n-1) d_n^{-1} \Big)$$

for $n \geq 1$.

Now expanding further in the binomial formula,

$$\begin{aligned} \sum_{n=0}^{\infty} c_n \left(xz - \alpha z^2\right)^n &= \sum_{n=0}^{\infty} c_n x^n z^n - \sum_{n=1}^{\infty} c_n n (xz)^{n-1} \alpha z^2 + \sum_{n=2}^{\infty} c_n \frac{n(n-1)}{2} (xz)^{n-2} \left(\alpha z^2\right)^2 + \dots \\ &= 1 + c_1 x z + \left(c_2 x^2 - \alpha c_1\right) z^2 + \left(c_3 x^3 - 2\alpha c_2 x\right) z^3 \\ &\quad + \sum_{n=4}^{\infty} \left(c_n x^n - \alpha (n-1) c_{n-1} x^{n-2} + \alpha^2 \frac{(n-2)(n-3)}{2} c_{n-2} x^{n-4} \right) z^n + \dots \end{aligned}$$

Thus

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \alpha d_2^{-1}, \quad P_3(x) = x^3 - 2\alpha d_3^{-1} x$$

and for $n \geq 4$,

$$P_n(x) = x^n - \alpha (n-1) d_n^{-1} x^{n-2} + \alpha^2 \frac{(n-2)(n-3)}{2} d_n^{-1} d_{n-1}^{-1} x^{n-4} + \dots$$

Therefore for $n \geq 4$,

$$xP_n - P_{n+1} - \omega_n P_{n-1} = \Bigg(\alpha^2 \frac{(n-2)(n-3)}{2} d_n^{-1} d_{n-1}^{-1} - \alpha^2 \frac{(n-1)(n-2)}{2} d_{n+1}^{-1} d_n^{-1} - \alpha\Big((n-1)d_n^{-1} - n d_{n+1}^{-1}\Big)\alpha(n-2)d_{n-1}^{-1}\Bigg)x^{n-3} + \dots,$$

a formula which also holds for $n = 3$. For this to be zero we need

$$\frac{(n-3)}{2} d_n^{-1} d_{n-1}^{-1} - \frac{(n-1)}{2} d_{n+1}^{-1} d_n^{-1} - \Big((n-1)d_n^{-1} - n d_{n+1}^{-1}\Big)d_{n-1}^{-1} = 0,$$

or

$$\frac{(n-3)}{2} d_{n+1} - \frac{(n-1)}{2} d_{n-1} - \Big((n-1)d_{n+1} - n d_n\Big) = 0.$$

Thus for $n \geq 3$,

$$(n+1)d_{n+1} = 2n d_n - (n-1)d_{n-1}.$$

The general solution of this recursion is

$$n d_n = a + b(n-1)$$

for $n \geq 2$. Since all $d_n \neq 0$, a, b cannot both be zero. Therefore

$$c_n = \frac{a + b(n-1)}{n} c_{n-1} = \frac{\prod_{i=1}^{n-1}(a+ib)}{n!} c_1 = \frac{\prod_{i=1}^{n-1}(a+ib)}{n!} c$$

for $n \geq 2$ and

$$F(z) = 1 + cz + c\sum_{n=2}^{\infty} \frac{\prod_{i=1}^{n-1}(a+ib)}{n!} z^n.$$

If $a \neq 0$, $b \neq 0$, then

$$(6) \qquad \begin{aligned} F(z) &= 1 + c\sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-1}(-a/b - i)}{an!}(-bz)^n \\ &= 1 + \frac{c}{a}\left((1-bz)^{-a/b} - 1\right) = 1 + \frac{c}{a}\left(\frac{1}{(1-bz)^{a/b}} - 1\right). \end{aligned}$$

If $a = 0$, $b \neq 0$, then

$$(7) \qquad F(z) = 1 + c\sum_{n=1}^{\infty} \frac{b^{n-1}}{n} z^n = 1 - \frac{c}{b}\ln(1-bz) = 1 + \frac{c}{b}\ln\left(\frac{1}{1-bz}\right),$$

which can also be obtained from the preceding formula by using L'Hôpital's rule. Finally, if $a \neq 0$, $b = 0$, then

$$(8) \qquad F(z) = 1 + c\sum_{n=1}^{\infty} \frac{a^{n-1}}{n!} z^n = 1 + \frac{c}{a}\left(e^{az} - 1\right).$$

Moreover,

$$\omega_n = \alpha n \frac{(n-1)b + 2a}{((n-1)b + a)(nb + a)}.$$

Since for orthogonality, we need $\omega_n \geq 0$, clearly $b \geq 0$. If $b = 0$, then

$$\omega_n = \frac{2\alpha}{a} n > 0$$

as long as $a > 0$. The polynomials with this recursion re-scaled Hermite polynomials. We recall [KS98, Section 1.13] that the generating function for standard (monic) Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n = \exp\left(xz - z^2/2\right),$$

which is of the form (5) with $F(z) = e^z$, and the generating function (8) is obtained from it by a re-scaling and a shift from Remark 2.

If $b > 0$, $a \neq 0$, we denote $\lambda = a/b$ and get

$$\omega_n = \frac{\alpha}{b} \frac{n(n + 2\lambda - 1)}{(n + \lambda - 1)(n + \lambda)}.$$

Since

$$\omega_1 = \frac{\alpha}{b} \frac{2}{1 + \lambda},$$

we have $\lambda > -1$. Since

$$\omega_2 = \frac{\alpha}{b} \frac{2(1 + 2\lambda)}{(1 + \lambda)(2 + \lambda)},$$

we have moreover $\lambda > -\frac{1}{2}$. It is now easy to see that this condition suffices for the positivity of all ω_n; indeed, the corresponding polynomials are re-scaled ultraspherical polynomials. We recall [KS98, Section 1.8.1] that the generating function for standard (monic) ultraspherical polynomials is

$$\sum_{n=0}^{\infty} \frac{2^n \prod_{i=0}^{n-1} (\lambda - i)}{n!} C_n^{(\lambda)}(x) z^n = \frac{1}{(1 - 2xz + z^2)^{\lambda}},$$

which is of the form (5) with $F(z) = \frac{1}{(1-2z)^{\lambda}}$, and the generating function (6) is obtained from it by a re-scaling and a shift from Remark 2.

Finally, if $b > 0$, $\lambda = a = 0$, then

$$\omega_n = \frac{\alpha}{b}$$

for $n \geq 2$, but $\omega_1 = 2\frac{\alpha}{b}$. These are precisely recursion coefficients for the re-scaled Chebyshev polynomials of the fist kind. The standard generating function [KS98, Section 1.8.2] for (monic) Chebyshev polynomials of the first kind is

$$\sum_{n=0}^{\infty} 2^n T_n(x) z^n = \frac{1 - xz}{1 - 2xz + z^2},$$

so it is not of the form (5). However,

$$\sum_{n=1}^{\infty} 2^n T_n(x) z^{n-1} = \frac{1}{z} \left(\frac{1 - xz}{1 - 2xz + z^2} - 1 \right) = \frac{x - z}{1 - 2xz + z^2}.$$

Term-by-term integration with respect to z gives

$$C + \sum_{n=1}^{\infty} \frac{2^n}{n} T_n(x) z^n = -\frac{1}{2} \ln\left(1 - 2xz + z^2\right) = \frac{1}{2} \ln\left(\frac{1}{1 - 2\left(xz - z^2/2\right)} \right).$$

with $C = 0$, which is of the form (5) with $F(z) = \frac{1}{2} \ln\left(\frac{1}{1-2z}\right)$. The generating function (7) is obtained from it by a re-scaling and a shift from Remark 2. $\square$

5. THE HISTORY

The question of characterizing various classes of orthogonal polynomials has a long and distinguished history, see [AS90] for an excellent survey up to 1990. The study of general polynomial families goes back to Paul Appell in 1880 [App80], who looked at polynomials with generating functions of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) z^n = A(z) \exp(xz)$$

for some function $A(z)$. These are now called Appell polynomials. Later, they were generalized to Sheffer families with generating functions

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) z^n = A(z) \exp(xU(z)) \tag{9}$$

for some functions $A(z), U(z)$. The prototypical "orthogonal polynomials characterization result" is Meixner's 1934 description of all *orthogonal* polynomials with the Sheffer-type generating functions [Mei34]. On the other hand, among Appell polynomials, only Hermite polynomials are orthogonal. See Figure 1.

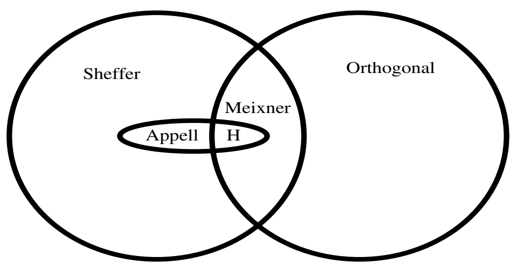

FIGURE 1. The relationship between the class of orthogonal polynomials and Sheffer, Appell and Meixner classes. "H" stands for Hermite.

Example 5. One way to state Meixner's result is that orthogonal polynomials with generating functions (9) satisfy a three-term recursion

$$xP_n(x) = P_{n+1}(x) + (na + \beta_0)\, P_n(x) + n((n-1)b + 1)\, P_{n-1}(x),$$

for some a, b, β_0 (up to re-scaling). The function M_{a,b,β_0} for which these polynomials are orthogonal can be written down explicitly, but for different values of the parameters these functions look quite different. For example, for $a = b = \beta_0 = 0$, we get the Hermite polynomials, with a continuous orthogonality relation (1). On the other hand, for $a = \beta_0 = 1$, $b = 0$, we get the Charlier polynomials, with a discrete orthogonality relation (2). Other polynomials in the Meixner class carry the names of Laguerre, Krawtchouk, Meixner, and Pollaczek, and are orthogonal with respect to gamma, binomial, negative binomial, and Meixner distributions.

Besides nice generating functions, the Meixner class has many other characterizations and applications, see [DKSC08] for an excellent (but advanced) survey. Perhaps for this reason, many generalizations of this class have been attempted. The most popular of these are probably the q-deformed families. One approach (there are several) extends the Sheffer class by looking at the generating functions of the form

$$A(z)\prod_{k=0}^{\infty}\frac{1}{1-(1-q)U\left(q^{k}z\right)z}$$

(after appropriate normalization, one gets the Sheffer form for $q \to 1$). In this case the analog of the Meixner class are the Al-Salam and Chihara polynomials [ASC87]. For the study of two different types of q-Appell polynomials, see [AS67, AS95].

A different generalization of the Sheffer class are generating functions of the general Boas-Buck [BB64] type:

$$\sum_{n=0}^{\infty} c_n P_n(x) z^n = A(z)F(xU(z))$$

for $F(z) = \sum_{n=0}^{\infty} c_n z^n$ with $c_0 = 1$. The usual case corresponds to $F(z) = e^z$. In the Boas-Buck setting, the problem of describing all orthogonal polynomials is wide open. The Appell-type class (with $U(z) = z$) in this case consists of the Brenke polynomials, and at least in that case all the orthogonal polynomials are known [Chi68].

Now note that in the Sheffer/Meixner case in equation (9), corresponding to $F(z) = e^z$, the generating function has an alternative form

$$\sum_{n=0}^{\infty}\frac{1}{n!}P_n(x)z^n = A(z)\exp(xU(z)) = \exp(xU(z)+\log A(z)).$$

So another interesting class to look at are (all or just orthogonal) polynomials with generating functions

$$\sum_{n=0}^{\infty} c_n P_n(x) z^n = F(xu(z)-R(z)),$$

which again gives the Sheffer/Meixner families for $F(z) = e^z$.

The case $F(z) = \frac{1}{1-z}$ appears in Free Probability [NS06], see Section 3 of [Ans03] for the author's description of the "free Meixner class", which is in a precise bijection with the Meixner class (except for the binomial case [BB06]). Here again, one can write the generating function in two ways:

$$A(z)\frac{1}{1-xU(z)} = \frac{1}{1-\left(x\frac{U(z)}{A(z)}-\frac{1-A(z)}{A(z)}\right)}.$$

More generally, Boas and Buck proved the following result.

Theorem. [BB56] *The only functions F with $F(0) = 1$ such that*

$$A(z)F(xU(z)) = F(xu(z)-R(z)) \tag{10}$$

are $F(z) = e^z$ and $F(z) = \frac{1}{(1-z)^\lambda}$ for some λ.

So as an alternative to the Boas-Buck formulation, we are interested in orthogonal polynomials with generating functions of the form $F(xu(z) - R(z))$, or at least in the Appell-type subclass $F(xz - R(z))$. For general R, even this seems to be a hard question. However, the orthogonal Appell polynomials are only the Hermite polynomials, with the exponential generating function

$$\exp\left(xz - z^2/2\right).$$

On the other hand, the orthogonal free Appell polynomials are only the Chebyshev polynomials of the second kind, with the ordinary generating function

$$\frac{1}{1-(xz-z^2)}.$$

Moreover, $R(z) = \alpha z^2$ appears naturally in combinatorial proofs of the usual, free, and other central-limit-type theorems (see for example Lecture 8 of [NS06]). Thus it is reasonable to consider $F(xz - \alpha z^2)$ first, which leads to Question 1. Conversely, the answer to that question indicates that interesting generating functions (and also, potentially, interesting non-commutative probability theories) arise precisely for F covered by the Boas-Buck theorem above, plus in the exceptional case $F(z) = 1 + \log \frac{1}{1-z}$ not covered by that theorem. On the other hand, see [ASV86, Dem09] for some negative results.

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