J. Austral. Math. Soc. (*Series A*) **68** (2000), 349–356

APÉRY SEQUENCES AND LEGENDRE TRANSFORMS

YUAN JIN and H. DICKINSON

(Received 29 June 1998; revised 22 September 1999)

Communicated by W. Chen

Abstract

A lower bound for the minimal length of the polynomial recurrence of a binomial sum is obtained.

2000 *Mathematics subject classification*: primary 11B37.

A sequence a_n satisfies a *polynomial recurrence of length r and degree m* if there exist r polynomials $P_0, P_1, \dots, P_{r-1}$, with degree at most m such that

$$P_0(n)a_n + P_1(n)a_{n-1} + \cdots + P_{r-1}(n)a_{n-r} = 0 \tag{1}$$

for $n \geq r$. For a sequence a_n the recurrence (1) is called *minimal* if it has minimal length and minimal degree.

It is well known (see [1, 8]) that the Apéry sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the three term polynomial recurrence

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)\, a_{n-1} + (n-1)^3 a_{n-2} = 0$$

for $n \geq 2$, where as usual $\binom{p}{q}$ denotes a binomial coefficient. Since the characteristic polynomial $x^2 - 34x + 1$ has roots $(1 \pm \sqrt{2})^4$, it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \left(1 + \sqrt{2}\right)^4$$

is irrational and that a_n cannot satisfy a two term recurrence. Apéry used these facts in his celebrated proof of the irrationality of $\zeta(3)$ (see [8]) and stimulated much interest in recursive sequences.

Wilf and Zeilberger [9] and others have shown that certain hypergeometric sums, including the binomial sum

$$(2)\quad a(n) = \sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \binom{n+2k}{k}^{r_2} \cdots \binom{n+tk}{k}^{r_t} = \sum_{k=0}^{n} \prod_{i=0}^{t} \binom{n+ik}{k}^{r_i},$$

where $r_0, r_1, r_2, \ldots, r_t$ are nonnegative integers, satisfy polynomial recurrences, without however any bounds on their lengths and degrees. It is not easy to find recurrences even for $a_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r$ and

$$(3)\qquad a_{r,s}(n) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s,$$

and at present no nontrivial lower bounds for the minimal lengths of the recurrences for $a_{r,s}(n)$ exist.

The sums $a_r(n)$ (see above) for $n \geq 0$ have been studied by many people. Apart from the trivial recurrences

$$a_1(n+1) - 2a_1(n) = 0, \quad \text{and} \quad (n+1)a_2(n+1) - (4n+2)a_2(n) = 0$$

with $n \geq 0$, Franel [2, 3] was the first to obtain recurrences for $a_3(n)$ and $a_4(n)$, namely

$$P_0(n)a_r(n+1) + P_1(n)a_r(n) + P_2(n)a_r(n-1) = 0$$

for $n \geq 1$, where, for $r = 3$

$$P_0(n) = (n+1)^2, \quad P_1(n) = -(7n^2 + 7n + 2), \quad P_2(n) = -8n^2$$

and for $r = 4$

$$P_0(n) = (n+1)^3, \quad P_1(n) = -2(2n+1)(3n^2 + 3n + 1),$$

and

$$P_2(n) = -4n(4n+1)(4n-1).$$

For $r = 5$ and 6, Perlstadt [4] found recurrences of length 4 while Schmidt and Yuan [6] showed that the recurrences stated for $r = 3, 4, 5$ and 6 are minimal and that the minimal lengths for $r > 6$ are at least 3. In this paper a nontrivial lower bound for the minimal length of the sequence (2) is obtained. We prove the following result.

THEOREM 1. *Let $r_0, r_1 \geq 1$ and $m, r_2, \ldots, r_t$ be nonnegative integers. Then there exist no nontrivial integer polynomials*

$$P_0(n) = c_0 + c_1 n + \cdots + c_m n^m, \quad P_1(n) = d_0 + d_1 n + \cdots + d_m n^m$$

such that

$$P_0(n)\,a(n+1) + P_1(n)\,a(n) = 0 \tag{4}$$

for $n \geq 0$.

Every sequence (c_k) has an associated Legendre transform $L(n)$ defined by

$$L(n) = \sum_{k=0}^{n} c_k \binom{n}{k}\binom{n+k}{k}.$$

For $r \in \mathbb{Z}$, $r \geq 2$ numerical evidence indicates that each of the sequences $a_{r,r}(n)$ defined as in (3) is the Legendre transform of an integer sequence $\left(c_k^{(r)}\right)$. Schmidt [5] and Strehl [7] proved independently that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

that is, $c_k^{(2)} = \sum_{j=0}^{k} \binom{k}{j}^3$.

The next theorem, proved later, shows that this is the only case of this form.

THEOREM 2. *Let* $r, s \geq 1$ *be integers. There exists an integer* $l \geq 1$ *such that the sequence* $a_{r,s}(n)$ *(defined as in* (3)*) for* $n \geq 0$ *is the Legendre transform of the integer sequence*

$$c_k = \sum_{j=0}^{k} \binom{k}{j}^l,$$

if and only if $s = 2$, $r = 2$ *and* $l = 3$.

Before the theorems are proved the congruence properties of $a(n)$, defined as in (2) are determined.

LEMMA 1. *For any prime* $p > t$, $r_0, r_1 \in \mathbb{N}$, *and* $r_2, \ldots, r_t$ *nonnegative integers, the following hold*:

(i) $a(p-1) \equiv 1 \pmod{p}$;
(ii) $a(jp) \equiv a(j) \pmod{p}$;
(iii) $a(p+1) \equiv a(1)^2 \pmod{p}$;
(iv) $a(2p-1) \equiv a(1) \pmod{p}$.

PROOF. Firstly case (i) is considered. If $1 \leq i \leq p-1$, then $p \mid (p-1+i)!$ but $p \nmid i!(p-1)!$ implying that

$$p \mid \binom{p-1+i}{i}.$$

Using this we have that

$$a(p-1) = \sum_{k=0}^{p-1} \prod_{i=0}^{t} \binom{p-1+ik}{k}^{r_i} \equiv \prod_{i=0}^{t} \binom{p-1}{0}^{r_i} \equiv 1 \pmod{p}.$$

To prove (ii), write

$$\begin{aligned} a(jp) &= \sum_{k=0}^{jp} \prod_{i=0}^{t} \binom{jp+ik}{k}^{r_i} \\ &= \sum_{k=0}^{j} \prod_{i=0}^{t} \binom{jp+ikp}{kp}^{r_i} + \sum_{\substack{0\le k<j \\ 1\le l<p}} \prod_{i=0}^{t} \binom{jp+i(kp+l)}{kp+l}^{r_i}. \end{aligned} \tag{5}$$

For $0 \le k < j$, $1 \le l < p$ we have

$$p^{k+1} \mid (jp)(jp-1)\cdots(jp-(kp+l)+1)$$

but $p^{k+1} \nmid (kp+l)!$, so

$$\binom{jp}{kp+l} \equiv 0 \pmod{p}. \tag{6}$$

It is readily verified, using the fact that $\prod_{l=1}^{p-1} l \equiv -1 \pmod{p}$, that

$$\prod_{\substack{0\le m<k \\ 0<l<p}} \frac{jp-(mp+l)}{kp-(mp+l)} \equiv 1 \pmod{p}.$$

Hence, as

$$\binom{jp}{kp} = \prod_{0\le m<k} \frac{(j-m)p}{(k-m)p} \prod_{\substack{0\le m<k \\ 0<l<p}} \frac{jp-(mp+l)}{kp-(mp+l)} = \binom{j}{k} \prod_{\substack{0\le m<k \\ 0<l<p}} \frac{jp-(mp+l)}{kp-(mp+l)}$$

we have

$$\binom{jp}{kp} \equiv \binom{j}{k} \pmod{p}. \tag{7}$$

Using (5), (6) and (7) we obtain

$$a(jp) \equiv \sum_{k=0}^{j} \prod_{i=0}^{t} \binom{j+ik}{k}^{r_i} = a(j) \pmod{p}.$$

To prove case (iii), the formulae

$$\binom{n+1}{j+1} = \binom{n}{j+1} + \binom{n}{j} \tag{8}$$

and

$$\binom{jp+l}{m} = \sum_{k=0}^{m} \binom{l}{k}\binom{jp}{m-k} \tag{9}$$

are used. The latter equation follows from the identity $(1+z)^{jp+l} = (1+z)^l(1+z)^{jp}$. By (6), (7), (8), (9) and the fact that $\binom{p}{j} \equiv 0 \pmod p$, for $1 \le j \le p-1$ it is readily verified that

$$\begin{aligned} a(p+1) &= \sum_{k=0}^{p+1} \prod_{i=0}^{t} \binom{p+1+ik}{k}^{r_i} \\ &\equiv \sum_{k=0}^{p+1} \left[\binom{p}{k} + \binom{p}{k-1}\right]^{r_0} \prod_{i=1}^{t} \binom{p+1+ik}{k}^{r_i} \\ &\equiv \binom{p}{0}^{r_0} \prod_{i=1}^{t} \binom{p+1}{0}^{r_i} + \binom{p}{0}^{r_0} \prod_{i=1}^{t} \binom{p+1+i}{1}^{r_i} \\ &\quad + \binom{p}{p}^{r_0} \prod_{i=1}^{t} \binom{p+1+ip}{p}^{r_i} + \binom{p}{p}^{r_0} \prod_{i=1}^{t} \binom{p+1+i(p+1)}{p+1}^{r_i} \\ &\equiv 1 + 2^{r_1+1}3^{r_2} \cdots (1+l)^{r_t} + 2^{2r_1}3^{2r_2} \cdots (l+1)^{2r_t} \\ &= (1 + 2^{r_1}3^{r_2} \cdots (l+1)^{r_t})^2 = a(1)^2 . \end{aligned}$$

Finally, case (iv) is considered. For $1 \le k \le p-1$,

$$p \mid \binom{p+(p-1)+k}{k} \quad \text{and} \quad p \mid \binom{p+(p-1)+(p+k)}{p+k} \tag{10}$$

since $p^2 \mid [2p+(k-1)]!$ and $p^3 \mid [3p+(k-1)]!$ but $p^2 \nmid k!(2p-1)!$ and $p^3 \nmid (p+k)!(2p-1)!$.

By definition

$$a(2p-1) = \sum_{k=0}^{p+(p-1)} \prod_{i=0}^{t} \binom{p+(p-1)+ik}{k}^{r_i}$$

and by (10) the terms $k = 1, \ldots, p-1$ and $k = p+1, \ldots, 2p-1$ are congruent to zero, leaving the terms $k = 0$ and $k = p$. From (6), (7) and (9) it follows that

$$a(2p-1) \equiv 1 + \binom{1}{1}^{r_0}\binom{2}{1}^{r_1} \cdots \binom{l+1}{1}^{r_t} = a(1) \pmod p$$

proving the lemma. □

PROOF (of Theorem 1). We prove the theorem using induction on m. Assume first that there is a recurrence relation with $m = 1$, then for any prime p

$$\begin{cases} (c_0 + c_1)\, a(2) + (d_0 + d_1)\, a(1) = 0 \\ (c_0 + c_1(p-1))\, a(p) + (d_0 + d_1(p-1))\, a(p-1) = 0 \\ (c_0 + c_1 p)\, a(p+1) + (d_0 + d_1 p)\, a(p) = 0 \\ (c_0 + c_1(2p-1))\, a(2p) + (d_0 + d_1(2p-1))\, a(2p-1) = 0. \end{cases}$$

Therefore, using Lemma 1, for any prime $p > t$ we have

$$\left.\begin{aligned} (c_0 + c_1)a(2) + (d_0 + d_1)a(1) &\equiv 0 \\ (c_0 - c_1)a(1) + (d_0 - d_1) &\equiv 0 \\ c_0 a(1)^2 + d_0 a(1) &\equiv 0 \\ (c_0 - c_1)a(2) + (d_0 - d_1)a(1) &\equiv 0 \end{aligned}\right\} \pmod p$$

It is readily verified that

$$a(2) \neq a(1)^2 \tag{11}$$

and with some manipulation it follows from (11) that $c_0 = c_1 = d_0 = d_1 = 0$, which proves the claim for $m = 1$.

Now suppose that the claim is true for $\deg(P_0) \le m - 1$ and $\deg(P_1) \le m - 1$ and assume that there exists a recurrence with $\deg(P_0) = m$ and $\deg(P_1) = m$. Therefore, (4) holds for all $n \ge 0$, and in particular, for $n = p - 1$ and $n = 2p - 1$, where $p > t$ is any prime. Then by Lemma 1

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(1) + d_0 - d_1 + \cdots + (-1)^m d_m \equiv 0 \pmod p$$

and

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(2) + (d_0 - d_1 + \cdots + (-1)^m d_m)\, a(1) \equiv 0 \pmod p.$$

Hence, as this holds for all $p > t$

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(1) + d_0 - d_1 + \cdots + (-1)^m d_m = 0$$

and

$$(c_0 - c_1 + \cdots + (-1)^m c_m)\, a(2) + (d_0 - d_1 + \cdots + (-1)^m d_m)\, a(1) = 0.$$

Using (11) it is not difficult to show that

$$c_0 - c_1 + \cdots + (-1)^m c_m = d_0 - d_1 + \cdots + (-1)^m d_m = 0,$$

that is, -1 is a root of P_0 and P_1. Whence there exist integer polynomials $\tilde{P}_0$ and $\tilde{P}_1$ with degree $m-1$ such that

$$P_0 = (n+1)\tilde{P}_0(n) \quad \text{and} \quad P_1(n) = (n+1)\tilde{P}_1(n),$$

and

$$\tilde{P}_0(n)\,a(n+1) + \tilde{P}_1(n)\,a(n) = 0$$

for $n \geq 0$. By the induction hypothesis $\tilde{P}_0 = \tilde{P}_1 = 0$, which implies that $P_0 = P_1 = 0$ and completes the proof of the theorem. □

PROOF (of Theorem 2). Assume that the sequence

$$a_{r,s}(n) = \sum_{j=0}^{n} \binom{n}{j}^r \binom{n+j}{j}^s$$

is the Legendre transform of the integral sequence

$$c_j = \sum_{k=0}^{j} \binom{j}{k}^l.$$

Then

$$a_{r,s}(n) = \sum_{j=0}^{n} c_j \binom{n}{j}\binom{n+j}{j}.$$

Therefore, for any prime $p > 2$

$$a_{r,s}(p) = \sum_{j=0}^{p} c_j \binom{p}{j}\binom{p+j}{j}$$

and hence by (7) and since $p \mid \binom{p}{j}$ for $1 \leq j \leq p-1$,

$$a_{r,s}(p) \equiv 1 + 2^s \equiv c_0 + 2c_p \pmod{p}.$$

As

$$c_0 = 1 \quad \text{and} \quad c_p = \sum_{k=0}^{p} \binom{p}{k}^l \equiv 2 \pmod{p},$$

it follows that $1 + 2^s \equiv 1 + 4 \pmod{p}$ and therefore, as p is an arbitrary prime, that $1 + 2^s = 5$ implying that $s = 2$.

Since $c_0 = 1$, $c_1 = 2$, $c_2 = 2 + 2^l$, $s = 2$,

$$a_{r,s}(2) = \sum_{k=0}^{2} \binom{2}{k}^r \binom{2+k}{k}^s = 1 + 2^r 3^s + 6^s$$

and

$$a_{r,s}(2) = \sum_{k=0}^{2} c_k \binom{2}{k} \binom{2+k}{k} = c_0 + 6c_1 + 6c_2$$

we get $2^{l+1} = 4 + 3.2^r$ and it is easy to show that this equation has only one solution, namely $r = 2$ and $l = 3$ which completes the proof. □

Acknowledgements

We are grateful to the Royal Society for providing a three month visiting fellowship for Yuan Jin to visit York and to the EPSRC who supports the first author. We would also like to thank Maurice Dodson for his advice and support.

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Department of Mathematics
Northwest University
710069 Xi'an
Shaanxi Province
P. R. China
e-mail: yuanj@nwu.edu.cn

Department of Mathematics
University of York
Heslington
York
YO10 5DD
England
e-mail: hd3@york.ac.uk