

THE WILSON FUNCTION TRANSFORM

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ABSTRACT. Two unitary integral transforms with a very-well poised ${}_7F_6$ -function as a kernel are given. For both integral transforms the inverse is the same as the original transform after an involution on the parameters. The ${}_7F_6$ -function involved can be considered as a non-polynomial extension of the Wilson polynomial, and is therefore called a Wilson function. The two integral transforms are called a Wilson function transform of type I and type II. Furthermore, a few explicit transformations of hypergeometric functions are calculated, and it is shown that the Wilson function transform of type I maps a basis of orthogonal polynomials onto a similar basis of polynomials.

1. INTRODUCTION

The Jacobi polynomials are orthogonal polynomials that that are solutions to the hypergeometric second order differential equation. The Jacobi polynomials have an explicit expression as ${}_2F_1$ -hypergeometric series, see e.g. [1]. Wilson [23] gave a generalization of the Jacobi polynomials as orthogonal ${}_4F_3$ -polynomials. These polynomials are nowadays called the Wilson polynomials. The Wilson polynomials no longer satisfy a second order differential equation, but a second order difference equation. Another way to generalize the Jacobi polynomials is to consider non-polynomial solutions of the hypergeometric differential equation. Spectral analysis of the hypergeometric differential equation leads to a unitary integral transform, called the Jacobi function transform, see e.g. [12]. The kernel in this integral transform is given by a non-polynomial ${}_2F_1$ -functions called the Jacobi function. Koornwinder mentions in [11, §9] that there probably also exist functions, naturally called Wilson functions, that generalize the Wilson polynomials as well as the Jacobi functions.

In this paper we consider non-polynomial eigenfunctions to the Wilson second order difference operator. To study the associated Wilson polynomials, Ismail et al. [5], and Masson [14], showed that general solutions of the difference equation are given by very-well-poised ${}_7F_6$ -functions. The analytic part of a certain non-polynomial solution we call a Wilson function. We show that the Wilson functions are the kernel in two unitary integral transforms, which we call the Wilson function transform of type I and type II. The Wilson function satisfies the property, called the duality property, that after an involution on the parameters, the geometric and the spectral parameter can be interchanged. The parameters obtained from the involution are called the dual parameters. The inverse of the Wilson function transform of type I, respectively type II, is again a Wilson function transform of type I, respectively type II, with dual parameters. The transforms can be made completely self-dual by choosing a fixed point of the involution.

The Wilson function transform of type I is a generalized Fourier transform with respect to the same measure as the Wilson polynomials. This measure consists of an absolutely continuous part supported on $[0, \infty)$, and a finite number of discrete mass points. The Wilson function transform of type II is a generalized Fourier transform with respect to a one-parameter family of measures which consist of an absolutely continuous part supported on $[0, \infty)$, and an infinite number of discrete mass points. The extra parameter labels the different sets of discrete mass points.

The Wilson functions can be considered as a formal limit case of the Askey-Wilson functions for $q \rightarrow 1$. The Askey-Wilson functions are eigenfunction to the Askey-Wilson second order difference

operator for $0 < q < 1$. The Askey Wilson functions are the kernel in the Askey-Wilson function transform, which is found by Koelink and Stokman [9]. They show in [10] that the Askey-Wilson function has an interpretation as spherical function for the quantum $SU(1, 1)$ group. From this point of view the Askey-Wilson function is a q -analogue of the Jacobi function, since the Jacobi function has similar interpretation for the Lie group $SU(1, 1)$, see [12]. So the Wilson functions in this paper give a new limit case of the Askey-Wilson functions. We do not know if the Wilson function also has an interpretation as a spherical function.

The Wilson function can also be considered as a formal limit case of Ruijsenaars' R -function [17], [18], [19], which is an eigenfunction to the Askey-Wilson second order difference operator for $|q| = 1$. The R -function is given by a Barnes-type integral which is considered as a generalization of the Barnes integral representation for the ${}_2F_1$ -series. Using the Barnes-type integral representation for the ${}_7F_6$ -series [20, (4.7.1.3)], the R -function can also be considered as a generalization of the Wilson function.

In a future paper we will show that the Wilson functions, and also the Wilson polynomials, have an interpretation as Racah coefficients for tensor products of positive discrete series, negative discrete series, and principal unitary series representations of the Lie algebra $\mathfrak{su}(1, 1)$. Both Wilson function transforms in this paper have an interpretation in the context of Racah coefficients.

The organization of this paper is as follows. In section 2 we give some well-known properties of the Wilson polynomials. The Wilson polynomials are eigenfunctions of a second order difference operator Λ , and we show that the Wilson polynomials are also eigenfunction of the same difference operator with dual parameters.

In section 3 we consider a certain type of non-polynomial eigenfunction of a difference operator L which is closely related to Λ . These eigenfunctions, the Wilson functions, are also eigenfunctions of the difference operator L with dual parameters.

In section 4 we define a Hilbert space \mathcal{M} , and, using the asymptotic behaviour of the Wilson function, we show that a truncated inner product of two Wilson functions approximates a reproducing kernel. This leads to a unitary integral transform, which we call the Wilson function transform of type I.

In section 5 we define a different Hilbert space \mathcal{H} . Again using asymptotic behaviour of the Wilson function, we show that a truncated inner product of two Wilson functions approximates a reproducing kernel. This leads to the Wilson function transform of type II.

In section 6 the Wilson function transforms of a Jacobi function, and of a Wilson polynomial, are calculated explicitly, using an integral representation of the Wilson function. Also we show that the Wilson function transform of type I maps an orthogonal system of Wilson polynomials onto the same orthogonal system with dual parameters.

Notations. We use the standard notation for the hypergeometric series, i.e.

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n z^n}{(b_1)_n \dots (b_q)_n n!},$$

where $(a)_n$ denotes the Pochhammer symbol, defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+n-1), \quad n \in \mathbb{Z}_{\geq 0}.$$

A hypergeometric series is called very-well poised if $p = q + 1$, $a_1 + 1 = b_1 + a_2 = \dots = b_q + a_{q+1}$ and $b_1 = a_2/2$. For a very-well poised ${}_7F_6$ -series of argument 1 we use Bailey's W notation, see [2], i.e.

$$W(a; b, c, d, e, f) = {}_7F_6 \left(\begin{matrix} a, 1 + \frac{1}{2}a, b, c, d, e, f \\ \frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f \end{matrix} ; 1 \right).$$

If the series does not terminate, the condition for convergence is $\Re(2a + 2 - b - c - d - e - f) > 0$. Also, since the product $\Gamma(a + b)\Gamma(a - b)$ frequently occurs in this paper, we use for this product the shorthand notation $\Gamma(a \pm b)$.

2. WILSON POLYNOMIALS

In this section we recall some well-known properties of the Wilson polynomials. The Wilson polynomial satisfies a second order difference equation in its degree (the three-term recurrence relation), and also a second order difference equation in its argument. The goal of this section is to point out that the two difference equations are the same after a change of the parameters.

The Wilson polynomials $R_n(x)$, see [23], [1, §6.10], [8], are polynomials in x^2 of degree n . They can be defined by the initial values $R_{-1}(x) = 0$, $R_0(x) = 1$, and the recurrence relation

$$-(a^2 + x^2)R_n(x) = C_n[R_{n+1}(x) - R_n(x)] + D_n[R_{n-1}(x) - R_n(x)], \quad (2.1)$$

where

$$C_n = C_n(a, b, c, d) = \frac{(n + a + b + c + d - 1)(n + a + b)(n + a + c)(n + a + d)}{(2n + a + b + c + d - 1)(2n + a + b + c + d)},$$

$$D_n = D_n(a, b, c, d) = \frac{n(n + b + c - 1)(n + b + d - 1)(n + c + d - 1)}{(2n + a + b + c + d - 2)(2n + a + b + c + d - 1)}.$$

The explicit expression for the polynomials R_n is given by

$$R_n(x) = R_n(x; a, b, c, d) = {}_4F_3 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} ; 1 \right). \quad (2.2)$$

Let $a, b, c, d \in \mathbb{C}$ be such that non-real parameters appear in conjugate pairs with positive real part, and such that the pairwise sum of any two parameters has positive real part. We define the measure $d\mu(x) = d\mu(x; a, b, c, d)$ by

$$\int f(x)d\mu(x) = \frac{1}{2\pi} \int_0^\infty f(x)w(x)dx + i \sum_k f(x_k)w_k,$$

where

$$w(x) = w(x; a, b, c, d) = \frac{\Gamma(a \pm ix)\Gamma(b \pm ix)\Gamma(c \pm ix)\Gamma(d \pm ix)}{\Gamma(\pm 2ix)}.$$

The points x_k are of the form $x_k = i(e + k)$, where e is any of the parameters a, b, c, d with $e < 0$. The sum is over $k \in \mathbb{Z}_{\geq 0}$, such that $e + k < 0$. The weights w_k are the residue at $x = x_k$ of $w(x)$. In particular, if all parameters are positive, or occur in pairs of complex conjugates with positive real part, the measure $d\mu$ is absolutely continuous. The Wilson polynomials are orthogonal with respect to the measure $d\mu$, i.e.

$$\int R_m(x)R_n(x)d\mu(x) = \delta_{nm} \frac{a + b + c + d - 1}{2n + a + b + c + d - 1} \frac{n! \Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(b + c + n)\Gamma(b + d + n)\Gamma(c + d + n)}{(a + b)_n(a + c)_n(a + d)_n(a + b + c + d - 1)_n \Gamma(a + b + c + d)}. \quad (2.3)$$

The Wilson polynomials also satisfy a difference equation in x , given by

$$n(n + a + b + c + d - 1)R_n(x) = A(-x)[R_n(x + i) - R_n(x)] + A(x)[R_n(x - i) - R_n(x)], \quad (2.4)$$

where

$$A(x) = \frac{(a + ix)(b + ix)(c + ix)(d + ix)}{2ix(2ix + 1)}.$$

We define the difference operator Λ by

$$\Lambda = A(x)(T_{-i} - I) + A(-x)(T_i - I), \quad (2.5)$$

where I denotes the identity operator and T is the shift operator (i.e. $T_z f(x) = f(x+z)$). From the difference equation (2.4) it follows that the polynomials $R_n(x; a, b, c, d)$ are eigenfunctions of Λ for eigenvalue $n(n+a+b+c+d-1)$.

The recurrence relation can be written in a self-dual way. Given the parameters $a, b, c, d \in \mathbb{C}$, we define dual parameters by

$$\begin{aligned} \tilde{a} &= \frac{1}{2}(a+b+c+d-1), & \tilde{b} &= \frac{1}{2}(a+b-c-d+1), \\ \tilde{c} &= \frac{1}{2}(a-b+c-d+1), & \tilde{d} &= \frac{1}{2}(a-b-c+d+1). \end{aligned} \quad (2.6)$$

It is an easy verification that $(a, b, c, d) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ defines an involution on the parameters. For a function $f = f(a, b, c, d)$, we define \tilde{f} by $\tilde{f} = f(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$. We use the same notation for other objects, like measures, sets and operators. Denote the Wilson polynomial $R_n(x)$ by $P_\lambda(x)$, where $\lambda = i(n + \tilde{a})$, i.e.

$$P_\lambda(x) = {}_4F_3 \left(\begin{matrix} \tilde{a} + i\lambda, \tilde{a} - i\lambda, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} ; 1 \right),$$

then we see that $P_\lambda(x) = \tilde{P}_x(\lambda)$, since $a + e = \tilde{a} + \tilde{e}$, for $e = b, c, d$. The recurrence relation (2.1) and the difference equation (2.4) for the Wilson polynomials can now be written as

$$\begin{aligned} \tilde{\Lambda} \tilde{P}_x(\lambda) &= -(\lambda^2 + x^2) \tilde{P}_x(\lambda), \\ \Lambda P_\lambda(x) &= -(\tilde{a}^2 + \lambda^2) P_\lambda(x). \end{aligned}$$

So we see that, for $\lambda \in i(\tilde{a} + \mathbb{Z}_{\geq 0})$, the Wilson polynomial $P_\lambda(x)$ is an eigenfunction of both Λ and $\tilde{\Lambda}$.

3. WILSON FUNCTIONS

In the previous section we observed that, for $\lambda^2 = -(\tilde{a} + n)^2$, the Wilson polynomials are solutions to the eigenvalue equation

$$(\Lambda f)(x) = -(\lambda^2 + \tilde{a}^2) f(x), \quad (3.1)$$

For more general values of λ , solutions to (3.1) can be given in terms of very-well poised ${}_7F_6$ -series. This is shown by Ismail et al. [5] and by Masson [14], who investigate the associated Wilson polynomials. Let $\psi_\lambda(x)$ be the function defined by

$$\begin{aligned} \psi_\lambda(x) = \psi_\lambda(x; a, b, c, d) &= \frac{\Gamma(b+c)\Gamma(\tilde{a} + \tilde{b} + \tilde{c} + i\lambda)\Gamma(1 - \tilde{d} + i\lambda)\Gamma(1 - d \pm ix)}{\Gamma(\tilde{b} + c + i\lambda \pm ix)} \\ &\quad \times W(\tilde{a} + \tilde{b} + \tilde{c} - 1 + i\lambda; a + ix, a - ix, \tilde{a} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda). \end{aligned}$$

The dual parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are still defined by (2.6). By [5, Thm. 2], or [14, (2.5)], the function $\psi_\lambda(x)$ is a solution to the eigenvalue equation (3.1).

Instead of $\psi_\lambda(x)$, we study the closely related function

$$\begin{aligned} \phi_\lambda(x) = \phi_\lambda(x; a, b, c, d) &= \frac{\Gamma(\tilde{a} + \tilde{b} + \tilde{c} + i\lambda)}{\Gamma(a+b)\Gamma(a+c)\Gamma(1+a-d)\Gamma(1-\tilde{d}-i\lambda)\Gamma(\tilde{b}+c+i\lambda \pm ix)} \\ &\quad \times W(\tilde{a} + \tilde{b} + \tilde{c} - 1 + i\lambda; a + ix, a - ix, \tilde{a} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda). \end{aligned} \quad (3.2)$$

So $\phi_\lambda(x) = \psi_\lambda(x)/K(x, \lambda)$, where $K(x, \lambda)$ is the function given by

$$K(x, \lambda) = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(1+a-d)\Gamma(1-d \pm ix)\Gamma(1-\tilde{d} \pm i\lambda).$$

We call the function $\phi_\lambda(x)$ a Wilson function. The Wilson function $\phi_\lambda(x)$ has the advantage that it is symmetric in $a, b, c, 1 - d$, cf. Remark 4.5(iii), while $\psi_\lambda(x)$ is not. Also $\phi_\lambda(x)$ is an analytic function in $(x, \lambda) \in \mathbb{C}^2$.

The ${}_7F_6$ -series in the definition of $\phi_\lambda(x)$ converges absolutely for $\Re(1 - \tilde{d} - i\lambda) > 0$. Writing the ${}_7F_6$ -series as a sum of two balanced ${}_4F_3$ -series, we have an expression for $\phi_\lambda(x)$ which always converges:

$$\begin{aligned} \phi_\lambda(x) = & \frac{\Gamma(1 - a - d)}{\Gamma(a + b)\Gamma(a + c)\Gamma(1 - d \pm ix)\Gamma(1 - \tilde{d} \pm i\lambda)} {}_4F_3 \left(\begin{matrix} a + ix, a - ix, \tilde{a} + i\lambda, \tilde{a} - i\lambda \\ a + b, a + c, a + d \end{matrix} ; 1 \right) \\ & + \frac{\Gamma(a + d - 1)}{\Gamma(1 + b - d)\Gamma(1 + c - d)\Gamma(a \pm ix)\Gamma(\tilde{a} \pm i\lambda)} \\ & \times {}_4F_3 \left(\begin{matrix} 1 - d + ix, 1 - d - ix, 1 - \tilde{d} + i\lambda, 1 - \tilde{d} - i\lambda \\ 1 + b - d, 1 + c - d, 2 - a - d \end{matrix} ; 1 \right). \end{aligned} \quad (3.3)$$

This follows from [2, §4.4(4)] with parameters specified by

$$a \mapsto \tilde{a} + \tilde{b} + \tilde{c} - 1 + i\lambda, \quad c \mapsto \tilde{c} + i\lambda, \quad d \mapsto \tilde{b} + i\lambda, \quad e \mapsto \tilde{a} + i\lambda, \quad f \mapsto a - ix, \quad g \mapsto a + ix.$$

From (3.3) we see that $\phi_\lambda(x)$ is an analytic function in $(x, \lambda) \in \mathbb{C}^2$. Observe that for $\lambda = \pm i(\tilde{a} + n)$, the second term in (3.3) vanishes because of the factor $\Gamma(\tilde{a} \pm i\lambda)^{-1}$, and then we see that $\psi_\lambda(x) = K(x, \lambda)\phi_\lambda(x)$ reduces to a Wilson polynomial. So $\psi_\lambda(x)$ is the analytic continuation of the Wilson polynomial in its degree.

From (3.3) and (2.6) follows the duality property

$$\phi_\lambda(x) = \tilde{\phi}_x(\lambda). \quad (3.4)$$

This duality property is similar to the duality property for the Askey-Wilson functions in [9]. For the ${}_7F_6$ -series in the definition of the Wilson function, (3.4) is implied by Bailey's transformation [2, §7.5(1)].

Since $\phi_\lambda(x) = \psi_\lambda(x)/K(x, \lambda)$ and $\psi_\lambda(x)$ is a solution to eigenvalue equation (3.1), the Wilson function satisfies the equation

$$M_{K(x, \lambda)}^{-1} \circ \Lambda \circ M_{K(x, \lambda)} \phi_\lambda(x) = -(\lambda^2 + \tilde{a}^2)\phi_\lambda(x).$$

Here $M_{K(x, \lambda)}$ denotes multiplication by $K(x, \lambda)$. From this we obtain the following proposition.

Proposition 3.1. *The Wilson function $\phi_\lambda(x)$ is a solution to*

$$(Lf)(x) = (\lambda^2 + \tilde{a}^2)f(x),$$

where L is the difference operator defined by

$$\begin{aligned} L &= B(-x)T_i + [A(-x) + A(x)]I + B(x)T_{-i}, \\ A(x) &= \frac{(a + ix)(b + ix)(c + ix)(d + ix)}{2ix(2ix + 1)}, \\ B(x) &= \frac{(a + ix)(b + ix)(c + ix)(1 - d + ix)}{2ix(2ix + 1)}. \end{aligned}$$

In the next two sections we consider the action of the second order difference operator L on two different Hilbert spaces, which leads to two different integral transforms with the Wilson function as a kernel. The method we use comes down to approximating with the Fourier transform, using asymptotic expansions of the Wilson functions. This method is essentially the same method as used by Götze [6], and Braaksma and Meulenbeld [3], for the Jacobi function transform. A similar method is also used by Koelink and Stokman in [9] for the Askey-Wilson function transform, and in [7] for the continuous Hahn function transform.

4. THE WILSON FUNCTION TRANSFORM: TYPE I

4.1. **The Hilbert space \mathcal{M} .** Let $V \subset \mathbb{C}^4$ be the set of parameters (a, b, c, d) satisfying the following two conditions:

- The parameters $a, b, c, 1 - d$ are real except for pairs of complex conjugates with positive real part.
- The pairwise sum of $a, b, c, 1 - d$ is contained in $\mathbb{C} \setminus (-\infty, 0]$.

A direct verification shows that the assignment $(a, b, c, d) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, see (2.6), is an involution on V . Throughout this section we assume $(a, b, c, d) \in V$.

Let M be the weight given by

$$M(x) = M(x; a, b, c, d) = \frac{\Gamma(a \pm ix)\Gamma(b \pm ix)\Gamma(c \pm ix)\Gamma(1 - d \pm ix)}{\Gamma(\pm 2ix)}.$$

The weight M is positive for $x \in \mathbb{R}$. Observe that $M(\cdot; a, b, c, d) = w(\cdot; a, b, c, 1 - d)$, where w is the weight function for the Wilson polynomials. We assume that the function M has only simple poles. This imposes conditions on the parameters a, b, c, d that can be removed afterwards by continuity in the parameters. For $e \in \mathbb{C}$ define the set \mathcal{D}_e by

$$\mathcal{D}_e = \{i(e + n) \mid n \in \mathbb{Z}_{\geq 0}, e + n < 0\},$$

and let $\mathcal{D} = \mathcal{D}_a \cup \mathcal{D}_b \cup \mathcal{D}_c \cup \mathcal{D}_{1-d}$. We define the measure $dm(\cdot) = dm(\cdot; a, b, c, d)$ by

$$\int f(x)dm(x) = \frac{1}{2\pi} \int_0^\infty f(x)M(x)dx + i \sum_{x \in \mathcal{D}} f(x) \operatorname{Res}_{z=x} M(z).$$

If $x \in \mathcal{D}_a$, we have explicitly

$$\begin{aligned} i \operatorname{Res}_{z=i(a+n)} M(z) &= \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(1+a-d)\Gamma(b-a)\Gamma(c-a)\Gamma(1-d-a)}{\Gamma(-2a)} \\ &\quad \times \frac{(2a)_n(a+1)_n(a+b)_n(a+c)_n(1+a-d)_n}{n!(a)_n(1+a-b)_n(1+a-c)_n(a+d)_n}, \end{aligned}$$

and then we see that for $(a, b, c, d) \in V$, the measure dm is positive. Recall that $\phi_\lambda(x)$ is symmetric in $a, b, c, 1 - d$, and by (3.3) the Wilson function is obviously even in x and λ , therefore $\overline{\phi_\lambda(x)} = \phi_{\overline{\lambda}}(\overline{x})$. From this it follows that for $x \in \operatorname{supp} dm(\cdot)$ and $\lambda \in \operatorname{supp} d\tilde{m}(\cdot)$ the Wilson function $\phi_\lambda(x)$ is real valued.

We define the Hilbert space $\mathcal{M} = \mathcal{M}(a, b, c, d)$ to be the Hilbert space consisting of even functions that have finite norm with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ defined by

$$\langle f, g \rangle_{\mathcal{M}} = \int f(x)\overline{g(x)}dm(x).$$

4.2. **The Wronskian.** For $0 < N < \infty$, we define a pairing $\langle \cdot, \cdot \rangle_N$ by

$$\langle f, g \rangle_N = \frac{1}{2\pi} \int_0^N f(x)g(x)M(x)dx + i \sum_{x \in \mathcal{D}} f(x)g(x) \operatorname{Res}_{z=x} M(z).$$

If f and g are real valued functions in \mathcal{M} , the limit $N \rightarrow \infty$ gives the inner product $\langle f, g \rangle_{\mathcal{M}}$. For functions f, g that are analytic in \mathbb{C} , we define the Wronskian $[f, g]$ by

$$[f, g](z) = \frac{1}{2\pi} \int_z^{z+i} \{f(x)g(x-i) - f(x-i)g(x)\} B(x)M(x)dx.$$

Lemma 4.1. *Let f, g be analytic in \mathbb{C} and even, then $z \mapsto [f, g](z)$ is odd in z .*

Proof. Let I be the function given by

$$I(x) = \left(f(x)g(x-i) - f(x-i)g(x) \right) B(x)M(x),$$

then $[f, g](z) = \int_z^{z+i} I(x)dx$. Since $f(x), g(x)$ and $M(x)$ are even functions in x , and $B(-x)M(x) = B(x+i)M(x+i)$, we have $I(-x) = -I(x+i)$. Therefore

$$\int_z^{z+i} I(x)dx = - \int_{-z}^{-z-i} I(-x)dx = \int_{-z}^{-z-i} I(x+i)dx = - \int_{-z}^{-z+i} I(x)dx.$$

Hence $z \mapsto [f, g](z)$ is an odd function in z . \square

Proposition 4.2. *For $N \gg 0$ and for even analytic functions f and g we have*

$$\langle Lf, g \rangle_N - \langle f, Lg \rangle_N = [f, g](N).$$

Proof. Recall that we assume that the poles of $M(x)$ are simple. For even functions f and g we have

$$\langle f, g \rangle_N = \frac{1}{4\pi} \int_{\mathcal{C}_N} f(x)g(x)M(x)dx,$$

where \mathcal{C}_N is a contour in the complex plane defined as follows:

- \mathcal{C}_N starts at $x = -N$ and ends at $x = N$,
- \mathcal{C}_N is invariant under reflection in the origin,
- \mathcal{C}_N separates the sequence of poles $i(a+n)$, $n \in \mathbb{Z}_{\geq 0}$ from the sequence $-i(a+m)$, $m \in \mathbb{Z}_{\geq 0}$, and similarly for poles of $M(x)$ corresponding to $b, c, 1-d$.

Now we have

$$\begin{aligned} & \frac{1}{4\pi} \int_{\mathcal{C}_N} (Lf)(x)g(x)M(x)dx - \frac{1}{4\pi} \int_{\mathcal{C}_N} f(x)(Lg)(x)M(x)dx = \\ & \frac{1}{4\pi} \int_{\mathcal{C}_N} \left(f(x+i)g(x) - f(x)g(x+i) \right) B(-x)M(x)dx \\ & + \frac{1}{4\pi} \int_{\mathcal{C}_N} \left(f(x-i)g(x) - f(x)g(x-i) \right) B(x)M(x)dx. \end{aligned}$$

Using $B(-x+i)M(x-i) = B(x)M(x)$, we can write this as

$$\frac{1}{4\pi} \left(\int_{\mathcal{C}_N} - \int_{\mathcal{C}_{N+i}} \right) \left(f(x-i)g(x) - f(x)g(x-i) \right) B(x)M(x)dx.$$

Here $\mathcal{C}_N + i = \{x \in \mathbb{C} \mid x-i \in \mathcal{C}_N\}$. The integrand has its poles at $x = i(e+1+n)$, $x = -i(e+m)$, for $n, m \in \mathbb{Z}_{\geq 0}$ and $e = a, b, c, 1-d$. Now we make a closed contour by connecting \mathcal{C}_N and $\mathcal{C}_N + i$ at the end points in a straight line (there are no poles of M on these lines for N large enough), then the integrand $I(x)$ has no poles inside the closed contour. So, by Cauchy's Theorem,

$$\int_{\mathcal{C}_N} - \int_{\mathcal{C}_{N+i}} = \int_{-N}^{-N+i} - \int_N^{N+i},$$

and from this we obtain

$$\langle Lf, g \rangle_N - \langle f, Lg \rangle_N = \frac{1}{2}[f, g](N) - \frac{1}{2}[f, g](-N).$$

Now the proposition follows from Lemma 4.1. \square

Since the Wilson functions are eigenfunctions of L for eigenvalue $\tilde{a}^2 + \lambda^2$, we find from Proposition 4.2 the following.

Proposition 4.3. For $\lambda \neq \bar{\lambda}$,

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_N = \frac{[\phi_\lambda, \phi_{\lambda'}](N)}{\lambda^2 - \lambda'^2}.$$

Next we want to let $N \rightarrow \infty$ in Proposition 4.3, so we need the asymptotic behaviour of the Wilson function and of $B(x + iy)M(x + iy)$ for $x \rightarrow \infty$ and $0 \leq y \leq 1$. We have

$$\begin{aligned} M(x + iy) &= 16\pi^3 x^{2a+2b+2c-2d-1} e^{-2\pi x - 2i\pi y} \left(1 + \mathcal{O}\left(\frac{1}{x}\right) \right), \\ B(x + iy) &= -\frac{x^2}{4} \left(1 + \mathcal{O}\left(\frac{1}{x}\right) \right). \end{aligned} \quad (4.1)$$

The asymptotic behaviour of the weight function M can be obtained from [16, §4.5]

$$\frac{\Gamma(a + z)}{\Gamma(b + z)} = z^{a-b} \left(1 + \frac{1}{2z}(a-b)(a+b-1) + \mathcal{O}\left(\frac{1}{z^2}\right) \right), \quad |z| \rightarrow \infty, \quad |\arg(z)| < \pi, \quad (4.2)$$

and from applying Euler's reflection formula for the Γ -function

$$\Gamma(a \pm ix) = \frac{\pi \Gamma(a + ix)}{\Gamma(1 - a + ix) \sin \pi(a - ix)}.$$

To determine the asymptotic behaviour of $\phi_\lambda(x)$, we expand $\phi_\lambda(x)$ in functions with nice asymptotic behaviour.

Proposition 4.4.

$$\phi_\lambda(x) = \tilde{c}(\lambda)\Phi_\lambda(x) + \tilde{c}(-\lambda)\Phi_{-\lambda}(x),$$

where

$$\begin{aligned} \Phi_\lambda(x) &= \frac{1}{\Gamma(\tilde{b} + c + i\lambda \pm ix)} {}_4F_3 \left(\begin{matrix} \tilde{a} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda, 1 - \tilde{d} + i\lambda \\ \tilde{b} + c + i\lambda + ix, \tilde{b} + c + i\lambda - ix, 1 + 2i\lambda \end{matrix} ; 1 \right), \\ \tilde{c}(\lambda) &= \frac{\Gamma(-2i\lambda)}{\Gamma(\tilde{a} - i\lambda)\Gamma(\tilde{b} - i\lambda)\Gamma(\tilde{c} - i\lambda)\Gamma(1 - \tilde{d} - i\lambda)}. \end{aligned}$$

Proof. This follows from transforming the ${}_7F_6$ -function in the definition (3.2) of $\phi_x(\lambda)$ by [2, §4.4(4)] with parameters specified by

$$a \mapsto \tilde{a} + \tilde{b} + \tilde{c} - 1 + i\lambda, \quad c \mapsto a + ix, \quad d \mapsto a - ix, \quad e \mapsto \tilde{a} + i\lambda, \quad f \mapsto \tilde{b} + i\lambda, \quad g \mapsto \tilde{c} + i\lambda. \quad \square$$

Remark 4.5. (i) Note that the dual weight function \tilde{M} can be expressed in terms of the \tilde{c} -function as

$$\tilde{M}(\lambda) = \frac{1}{\tilde{c}(\lambda)\tilde{c}(-\lambda)}.$$

(ii) The functions Φ_λ and $\Phi_{-\lambda}$ are in general not solutions to the eigenvalue equation in Proposition 3.1. Let us define

$$\begin{aligned} \Psi_\lambda(x) &= \\ &= \frac{\Gamma(1 - a - ix)\Gamma(1 - b - ix)\Gamma(1 - c - ix)\Gamma(1 - d - ix)\Gamma(2 - d - ix + 2i\lambda)}{\Gamma(1 + \tilde{a} - d + i\lambda - ix)\Gamma(1 + \tilde{d} - d + i\lambda - ix)\Gamma(2 - \tilde{b} - d + i\lambda - ix)\Gamma(2 - \tilde{c} - d + i\lambda - ix)\Gamma(d - ix)} \\ &\times \frac{\sin \pi(1 + \tilde{a} - d + ix + i\lambda)}{\sin \pi(d - ix)} W(1 - d - ix + 2i\lambda; 1 - d - ix, 1 - \tilde{a} + i\lambda, 1 - \tilde{d} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda). \end{aligned}$$

From [14, (2.12)] it follows that Ψ_λ and $\Psi_{-\lambda}$ are solutions to the eigenvalue equation (3.1), so $\Psi_\lambda(x)/K(x, \lambda)$ is an eigenfunction of L for eigenvalue $\lambda^2 + \tilde{a}^2$. The Wilson function $\phi_\lambda(x)$ can be expanded in terms of $\Psi_\lambda(x)$ and $\Psi_{-\lambda}(x)$ as follows:

$$\Gamma(1 - d \pm ix)\phi_\lambda(x) = \tilde{c}(\lambda)\Psi_\lambda(x) + \tilde{c}(-\lambda)\Psi_{-\lambda}(x),$$

where the \tilde{c} -function is the same as in Proposition 4.4. This expansion follows from [20, (4.3.7.8)], with parameters given by

$$\begin{aligned} a &\mapsto 1 - d + 2i\lambda - ix, & b &\mapsto 1 - d - ix, & c &\mapsto 1 - \tilde{a} + i\lambda, \\ d &\mapsto 1 - \tilde{d} + i\lambda, & e &\mapsto \tilde{b} + i\lambda, & f &\mapsto \tilde{c} + i\lambda, \end{aligned}$$

and Euler's reflection formula for the Γ -function. From this expansion it is possible to determine the asymptotic behaviour of $\phi_\lambda(x)$ for $x \rightarrow \infty$. We will use the simpler functions $\Phi_{\pm\lambda}$ from Proposition 4.4 to compute the asymptotic behaviour of the Wilson function.

(iii) Writing $c = \frac{1}{2}(\tilde{a} - \tilde{b} + \tilde{c} - \tilde{d} + 1)$ in Proposition 4.4 we see that the Wilson function $\phi_\lambda(x; a, b, c, d)$ is symmetric in $\tilde{a}, \tilde{b}, \tilde{c}, 1 - \tilde{d}$, and by the duality property (3.4) then also in $a, b, c, 1 - d$.

To determine the asymptotic behaviour of $\Phi_\lambda(x)$ we use Euler's reflection formula to rewrite the Γ -functions in front of the ${}_4F_3$ -function and we use (4.2). Then we find for Φ_λ , for $y \in \mathbb{R}$ and $x \rightarrow \infty$,

$$\Phi_\lambda(x + iy) = \frac{1}{2i\pi} x^{d-a-b-c-2i\lambda} e^{\pi x + i\pi y} \left(1 + \frac{y}{ix} (a + b + c - d + 2i\lambda) + \mathcal{O}\left(\frac{1}{x^2}\right) \right). \quad (4.3)$$

From Proposition 4.4 it follows that the asymptotic behaviour of the Wronskian $[\phi_\lambda, \phi_{\lambda'}](N)$ can be computed from the four Wronskians $[\Phi_{\pm\lambda}, \Phi_{\pm\lambda'}](N)$. So we must compute $[\Phi_\lambda, \Phi_{\lambda'}](N)$ for $N \rightarrow \infty$.

Lemma 4.6. *For $N \rightarrow \infty$,*

$$[\Phi_\lambda, \Phi_{\lambda'}](N) = i(\lambda - \lambda') N^{-2i(\lambda + \lambda')} \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right).$$

Proof. Let $G(x)$ be the function given by

$$G(x) = \Phi_\lambda(x)\Phi_{\lambda'}(x - i) - \Phi_\lambda(x - i)\Phi_{\lambda'}(x).$$

From the asymptotic behaviour of Φ_λ given above we find, for $0 \leq y \leq 1$ and $x \rightarrow \infty$,

$$G(x + iy) = -\frac{(\lambda - \lambda')}{2\pi^2} x^{2d-2a-2b-2c-1-2i(\lambda + \lambda')} e^{2\pi x + 2i\pi y} \left(1 + \mathcal{O}\left(\frac{1}{x}\right) \right).$$

Using the asymptotic behaviour of $B(x + iy)M(x + iy)$ for $x \rightarrow \infty$ gives

$$G(x + iy)B(x + iy)M(x + iy) = 2\pi(\lambda - \lambda') x^{-2i(\lambda + \lambda')} \left(1 + \mathcal{O}\left(\frac{1}{x}\right) \right).$$

Note that the main term in the asymptotic expansion is independent of y . Next we write

$$[\Phi_\lambda, \Phi_{\lambda'}](N) = \frac{1}{2\pi} \int_N^{N+i} G(x)B(x)M(x)dx = -\frac{1}{2\pi i} \int_0^1 G(N + iy)B(N + iy)M(N + iy)dy,$$

and we apply dominated convergence to obtain the result. \square

4.3. Continuous spectrum. In this subsection we assume $\lambda, \lambda' \in \mathbb{R}$, and since ϕ_λ is even in λ , we may assume $\lambda, \lambda' \geq 0$.

Proposition 4.7. *Let f be a continuous function, satisfying*

$$f(\lambda) = \begin{cases} \mathcal{O}(\lambda^{\tilde{a}+\tilde{b}+\tilde{c}-\tilde{d}-\frac{1}{2}-\varepsilon} e^{-\pi\lambda}), & \lambda \rightarrow \infty, \quad \varepsilon > 0, \\ \mathcal{O}(\lambda^\delta), & \lambda \downarrow 0, \quad \delta > 0. \end{cases}$$

Then

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda = \frac{f(\lambda')}{\tilde{M}(\lambda')}.$$

Proof. From the c -function expansion in Proposition 4.4 we find

$$[\phi_\lambda, \phi_{\lambda'}](N) = \sum_{\varepsilon, \xi \in \{-1, 1\}} \tilde{c}(\varepsilon\lambda) \tilde{c}(\xi\lambda') [\Phi_{\varepsilon\lambda}, \Phi_{\xi\lambda'}](N),$$

and then we obtain from Lemma 4.6 and Proposition 4.3

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_N = i \sum_{\varepsilon, \xi \in \{-1, 1\}} \tilde{c}(\varepsilon\lambda) \tilde{c}(\xi\lambda') \frac{N^{-2i(\varepsilon\lambda + \xi\lambda')}}{\varepsilon\lambda + \xi\lambda'} \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right).$$

We multiply both sides with an arbitrary function $f(\lambda)$, and we integrate over λ from 0 to ∞ . The function f must satisfy certain conditions that we determine later on. Letting $N \rightarrow \infty$ then gives

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda = \\ \lim_{N \rightarrow \infty} i \int_0^\infty f(\lambda) \left\{ \psi_1(\lambda) \cos(2[\lambda + \lambda'] \ln(N)) + \psi_2(\lambda) \sin(2[\lambda + \lambda'] \ln(N)) \right. \\ \left. + \psi_3(\lambda) \cos(2[\lambda - \lambda'] \ln(N)) + \psi_4(\lambda) \frac{\sin(2[\lambda - \lambda'] \ln(N))}{\lambda - \lambda'} \right\} d\lambda \end{aligned}$$

where

$$\begin{aligned} \psi_1(\lambda) &= \frac{1}{\lambda + \lambda'} \left(\tilde{c}(\lambda) \tilde{c}(\lambda') - \tilde{c}(-\lambda) \tilde{c}(-\lambda') \right), \\ \psi_2(\lambda) &= \frac{-i}{\lambda + \lambda'} \left(\tilde{c}(\lambda) \tilde{c}(\lambda') + \tilde{c}(-\lambda) \tilde{c}(-\lambda') \right), \\ \psi_3(\lambda) &= \frac{1}{\lambda - \lambda'} \left(\tilde{c}(\lambda) \tilde{c}(-\lambda') - \tilde{c}(-\lambda) \tilde{c}(\lambda') \right), \\ \psi_4(\lambda) &= -i \left(\tilde{c}(\lambda) \tilde{c}(-\lambda') + \tilde{c}(-\lambda) \tilde{c}(\lambda') \right). \end{aligned}$$

Observe that $\tilde{c}(\lambda') \tilde{c}(-\lambda') - \tilde{c}(-\lambda') \tilde{c}(\lambda') = 0$, so ψ_3 has a removable singularity at $\lambda = \lambda'$. From the Riemann-Lebesgue lemma we find that the terms with ψ_i , $i = 1, 2, 3$, vanish, provided that $f\psi_i \in L^1(0, \infty)$. We recognize the term with ψ_4 as a Dirichlet integral. Using the well-known property (see e.g. [22, §9.7]) for Dirichlet integrals

$$\lim_{t \rightarrow \infty} \frac{1}{\pi} \int_0^\infty g(x) \frac{\sin[t(x-y)]}{x-y} dx = g(y), \quad (4.4)$$

for a continuous function $g \in L^1(0, \infty)$, we obtain

$$\lim_{N \rightarrow \infty} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda = \pi \psi_4(\lambda') f(\lambda') = 2\pi \tilde{c}(\lambda') \tilde{c}(-\lambda') f(\lambda') = 2\pi \frac{f(\lambda')}{\tilde{M}(\lambda')}.$$

The asymptotic behaviour of ψ_i , for $i = 1, 2, 3, 4$, can be obtained in the same way as the asymptotic behaviour of \tilde{M} , see (4.1). Then we find, for $i = 1, 2, 3, 4$,

$$|\psi_i(\lambda)| = \begin{cases} \mathcal{O}(\lambda^{-\frac{1}{2}-\tilde{a}-\tilde{b}-\tilde{c}+\tilde{d}}e^{\pi\lambda}), & \lambda \rightarrow \infty, \\ \mathcal{O}(\lambda^{-1}), & \lambda \downarrow 0. \end{cases}$$

So, if f satisfies the conditions given in the proposition, then $f\psi_i \in L^1(0, \infty)$. \square

Let f be a continuous function. We define a linear operator \mathcal{F} by

$$(\mathcal{F}f)(\lambda) = \int f(x)\phi_\lambda(x)dm(x).$$

We call \mathcal{F} the Wilson function transform of type I. We denote the continuous part of the above integral by $\mathcal{F}_c f$, i.e.

$$(\mathcal{F}_c f)(\lambda) = \frac{1}{2\pi} \int_0^\infty f(x)\phi_\lambda(x)M(x)dx.$$

From the asymptotic behaviour of ϕ_λ and $M(x)$ we find that both $\mathcal{F}f$ and $\mathcal{F}_c f$ are well defined if f satisfies the conditions

$$f(x) = \begin{cases} \mathcal{O}(x^{d-a-b-c-\varepsilon}e^{\pi x}), & x \rightarrow \infty, \quad \varepsilon > 0, \\ \mathcal{O}(x^{\delta-1}), & x \downarrow 0, \quad \delta > 0. \end{cases}$$

Observe that if f satisfies the condition given above, then $f \in \mathcal{M}$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be the space consisting of continuous functions satisfying the above conditions. Then \mathcal{M}_0 is a dense subspace of \mathcal{M} (it contains for instance the Wilson polynomials $R_n(x; a, b, c, 1-d)$, which form an orthogonal basis for \mathcal{M}).

Proposition 4.8. *Let $g \in \tilde{\mathcal{M}}_0$ and $\lambda \geq 0$, then $(\mathcal{F}(\tilde{\mathcal{F}}_c g))(\lambda) = g(\lambda)$.*

Proof. Define $f(\lambda) = \tilde{M}(\lambda)g(\lambda)$, then f satisfies the conditions given in Proposition 4.7. Then we have

$$\begin{aligned} (\mathcal{F}(\tilde{\mathcal{F}}_c g))(\lambda') &= \int \phi_{\lambda'}(x) \left(\frac{1}{2\pi} \int_0^\infty g(\lambda)\phi_\lambda(x)\tilde{M}(\lambda)d\lambda \right) dm(x) \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^\infty f(\lambda) \left(\frac{1}{2\pi} \int_0^N \phi_\lambda(x)\phi_{\lambda'}(x)M(x)dx + i \sum_{x \in \mathcal{D}} \phi_\lambda(x)\phi_{\lambda'}(x) \operatorname{Res}_{z=x} M(z) \right) d\lambda \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda \\ &= \frac{f(\lambda')}{\tilde{M}(\lambda')} = g(\lambda'). \end{aligned}$$

Note that the first integral converges absolutely for $g \in \tilde{\mathcal{M}}_0$, so in that case interchanging the order of integration is allowed. \square

In the next subsection we show that the dual Wilson function transform $\tilde{\mathcal{F}}$ is the inverse of the Wilson function transform \mathcal{F} . To do this, we must consider the discrete spectrum of the difference operator L .

4.4. Discrete spectrum. From the asymptotic behaviour of $\Phi_\lambda(x)$ and $M(x)$, see (4.3) and (4.1), we obtain

$$|\Phi_\lambda(x)|^2 M(x) = \mathcal{O}(x^{4\Im(\lambda)-1}), \quad x \rightarrow \infty.$$

So for $\Im(\lambda) < 0$ we have $\Phi_\lambda \in \mathcal{M}$. In this subsection we assume that $\lambda \in \tilde{\mathcal{D}}$ and that the set $\tilde{\mathcal{D}}$ is not empty, so $\Im(\lambda) < 0$. In this case $\tilde{c}(-\lambda) = 0$, and therefore $\phi_\lambda(x) = \tilde{c}(\lambda)\Phi_\lambda(x) \in \mathcal{M}$. First we show that ϕ_λ is orthogonal to $\phi_{\lambda'}$ if $\lambda' \neq \lambda$.

Proposition 4.9. *For $\lambda \in \tilde{\mathcal{D}}$, $\lambda' \in \text{supp}(d\tilde{m})$, $\lambda \neq \lambda'$, we have*

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_{\mathcal{M}} = 0.$$

Proof. From Propositions 4.3 and 4.4 we obtain

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_N = \frac{\tilde{c}(\lambda)\tilde{c}(\lambda')[\Phi_\lambda, \Phi_{\lambda'}](N) + \tilde{c}(\lambda)\tilde{c}(-\lambda')[\Phi_\lambda, \Phi_{-\lambda'}](N)}{\lambda^2 - \lambda'^2}.$$

Then Lemma 4.6 gives for large N

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_N = \frac{i(\lambda - \lambda')\tilde{c}(\lambda)\tilde{c}(\lambda')N^{-2i(\lambda+\lambda')} + i(\lambda + \lambda')\tilde{c}(\lambda)\tilde{c}(-\lambda')N^{-2i(\lambda-\lambda')}}{(\lambda + \lambda')(\lambda - \lambda')} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \quad (4.5)$$

Recall that for $\lambda \in \tilde{\mathcal{D}}$ we have $\lambda \in i\mathbb{R}_{<0}$. Then it is clear that for $\lambda' \in \mathbb{R}$ the right hand side tends to zero for $N \rightarrow \infty$. In case $\lambda' \in \tilde{\mathcal{D}}$ the second term vanishes, and we have $\Im(\lambda + \lambda') < 0$. So in this case the right hand side also tends to zero for $N \rightarrow \infty$. \square

It remains to calculate the squared norm of ϕ_λ in case $\lambda \in \tilde{\mathcal{D}}$.

Proposition 4.10. *For $\lambda \in \tilde{\mathcal{D}}$*

$$\langle \phi_\lambda, \phi_\lambda \rangle_{\mathcal{M}} = \left(i \operatorname{Res}_{\lambda'=\lambda} \tilde{M}(\lambda') \right)^{-1}.$$

Proof. We use expression (4.5), where we let $\lambda' \rightarrow \lambda$. Then for large N

$$\lim_{\lambda' \rightarrow \lambda} \langle \phi_\lambda, \phi_{\lambda'} \rangle_N = \left(\frac{i}{2\lambda} \tilde{c}(\lambda)\tilde{c}(\lambda)N^{4i\lambda} + i\tilde{c}(\lambda) \left(\operatorname{Res}_{\lambda'=\lambda} \frac{1}{\tilde{c}(-\lambda')} \right)^{-1} \right) \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right).$$

Letting $N \rightarrow \infty$ gives the result. \square

Remark 4.11. If $\lambda, \lambda' \in \mathcal{D}_a$ then Propositions 4.9 and 4.10 give orthogonality relations for a finite number of Wilson polynomials with respect a measure that has only finitely many moments. Explicitly, if we assume $a, b, c, 1 - d > 0$, we have for $n, m < \frac{1}{2}(1 - a - b - c - d)$

$$\begin{aligned} & \frac{1}{2\pi} \int_0^\infty R_n(x)R_m(x) \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(1-d+ix)\Gamma(2ix)} \right|^2 dx = \\ & \delta_{nm} \frac{a+b+c+d-1}{a+b+c+d+2n-1} \frac{n!(b+c)_n(b+d)_n(c+d)_n}{(a+b)_n(a+c)_n(a+d)_n(a+b+c+d-1)_n} \\ & \times \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(1-a-b-c-d)}{\Gamma(1-a-d)\Gamma(1-b-d)\Gamma(1-c-d)}, \end{aligned}$$

where R_n is the Wilson polynomial defined by (2.2). Neretin [15] found this orthogonality relation using an explicit evaluation of the Barnes-type integral

$$\frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(1-d+ix)\Gamma(2ix)} \right|^2 dx.$$

Note that the above orthogonality relations remain valid for $n + m < 1 - a - b - c - d$.

4.5. **The Wilson function transform: type I.** Combining Propositions 4.9 and 4.10 with Proposition 4.8, gives the following theorem.

Theorem 4.12. *The Wilson function transform of type I, $\mathcal{F} : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$, defined by*

$$(\mathcal{F}f)(\lambda) = \int f(x)\phi_\lambda(x)dm(x),$$

is unitary, and its inverse is given by $\mathcal{F}^{-1} = \tilde{\mathcal{F}}$.

Proof. First we show that $\mathcal{F} \circ \tilde{\mathcal{F}}$ is the identity operator on $\tilde{\mathcal{M}}_0$. For the continuous part of $\tilde{\mathcal{F}}$ this is Proposition 4.8, therefore we just write down the proof for the discrete part of $\tilde{\mathcal{F}}$. Let $g \in \tilde{\mathcal{M}}_0$. Recall that \tilde{D} is a finite set, then we obtain from Propositions 4.9 and 4.10,

$$\begin{aligned} \int \phi_{\lambda'}(x) \left(i \sum_{\lambda \in \tilde{D}} g(\lambda) \phi_\lambda(x) \operatorname{Res}_{\mu=\lambda} \tilde{M}(\mu) \right) dm(x) &= i \sum_{\lambda \in \tilde{D}} g(\lambda) \operatorname{Res}_{\mu=\lambda} \tilde{M}(\mu) \left(\int \phi_\lambda(x) \phi_{\lambda'}(x) dm(x) \right) \\ &= \begin{cases} 0, & \lambda' \in [0, \infty), \\ g(\lambda'), & \lambda' \in \tilde{D}. \end{cases} \end{aligned}$$

Combining this with Proposition 4.8 we obtain the desired result. By duality we obtain $(\tilde{\mathcal{F}}(\mathcal{F}f))(x) = f(x)$ for $f \in \mathcal{M}_0$.

Next we show that $\tilde{\mathcal{F}}$ is an isometry on $\tilde{\mathcal{M}}_0$. For simplicity we assume that the measures dm and $d\tilde{m}$ are absolutely continuous. From Proposition 4.7 we obtain

$$\begin{aligned} \langle \tilde{\mathcal{F}}f, \tilde{\mathcal{F}}g \rangle_{\mathcal{M}} &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^N (\tilde{\mathcal{F}}f)(x) \overline{(\tilde{\mathcal{F}}g)(x)} M(x) dx \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty f(\lambda) \overline{g(\lambda')} \langle \phi_\lambda, \phi_{\lambda'} \rangle_N \tilde{M}(\lambda) \tilde{M}(\lambda') d\lambda d\lambda' \\ &= \frac{1}{2\pi} \int_0^\infty f(\lambda') \overline{g(\lambda')} \tilde{M}(\lambda') d\lambda' = \langle f, g \rangle_{\tilde{\mathcal{M}}}. \end{aligned}$$

In case dm and $d\tilde{m}$ have discrete mass points, the proof runs along the same lines.

So the operator $\mathcal{F} : \mathcal{M}_0 \rightarrow \tilde{\mathcal{M}}_0$ is unitary. Since \mathcal{M}_0 is dense in \mathcal{M} , \mathcal{F} extends uniquely to a unitary operator on \mathcal{M} . \square

Remark 4.13. An interesting special case is the case $a = b + c + d - 1$. Then $(a, b, c, d) = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, so the Wilson function transform of type I is then completely self-dual.

5. THE WILSON FUNCTION TRANSFORM: TYPE II

In this section we give another unitary integral transform which has the Wilson function as a kernel, by considering the action of the difference operator L on (a dense subspace of) a different Hilbert space than in section 4. The method we use is the same method as in section 4, therefore we omit some details.

5.1. **The Hilbert space \mathcal{H} .** Let $V^+ \subset \mathbb{C}^5$ be the set of parameters a, b, c, d, t satisfying the following conditions:

$$\bar{a} = 1 - d, \quad \bar{b} = c, \quad t \in \mathbb{R}.$$

The dual parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are still defined by (2.6), and we define the dual parameter \tilde{t} by

$$\tilde{t} = 1 - \tilde{b} - c - t.$$

It is easily verified that the assignment $(a, b, c, d, t) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})$ is an involution on V^+ . Throughout this section we assume $(a, b, c, d, t) \in V^+$.

Let H be the weight given by

$$H(x) = H(x; a, b, c, d; t) = \frac{\Gamma(a \pm ix)\Gamma(b \pm ix)\Gamma(c \pm ix)\Gamma(1 - d \pm ix)}{\sin \pi(t \pm ix)\Gamma(\pm 2ix)}.$$

Here we use the notation $\sin(a \pm b) = \sin(a + b)\sin(a - b)$. The weight H is positive for $x \in \mathbb{R}$. Observe that $H(x; a, b, c, d; t) = M(x; a, b, c, d)/\sin \pi(t \pm ix)$, where M is the weight function defined in section 4. We assume that H has only simple poles. This imposes conditions on the parameters that can be removed afterwards by continuity. Let \mathcal{D}^+ be the infinite discrete set defined by

$$\mathcal{D}^+ = \{i(t - n) \mid n \in \mathbb{Z}, t - n < 0\}.$$

We define the measure $dh(\cdot) = dh(\cdot; a, b, c, d; t)$ by

$$\int f(x)dh(x) = \frac{C}{2\pi} \int_0^\infty f(x)H(x)dx + iC \sum_{x \in \mathcal{D}^+} f(x) \operatorname{Res}_{z=x} H(z).$$

Here C is the normalizing constant

$$C = \sqrt{\sin \pi(a + t) \sin \pi(b + t) \sin \pi(c + t) \sin \pi(1 - d + t)}.$$

Note that $\tilde{C} = C$.

For $x \in \mathcal{D}^+$ we have explicitly,

$$\begin{aligned} i \operatorname{Res}_{z=i(t-k)} H(z) &= \Gamma(a + t)\Gamma(a - t)\Gamma(b + t)\Gamma(b - t)\Gamma(c + t)\Gamma(c - t)\Gamma(1 - d + t)\Gamma(1 - d - t) \\ &\times \frac{k - t}{2t^2\pi^2} \frac{(a - t)_k(b - t)_k(c - t)_k(1 - d - t)_k}{(1 - a - t)_k(1 - b - t)_k(1 - c - t)_k(d - t)_k} \end{aligned}$$

and then we see that for $(a, b, c, d, t) \in V^+$, the measure dh is positive. For $x \in \operatorname{supp} dh$ and $\lambda \in \operatorname{supp} d\tilde{h}$ the Wilson function $\phi_\lambda(x)$ is real valued.

We define the Hilbert space $\mathcal{H} = \mathcal{H}(a, b, c, d; t)$ to be the Hilbert space consisting of even functions that have finite norm with respect to inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\langle f, g \rangle_{\mathcal{H}} = \int f(x)\overline{g(x)}dh(x).$$

5.2. The Wronskian. We denote $H_k = iC \operatorname{Res}_{z=i(t-k)} H(z)$, and we define k_0 to be the smallest integer such that $t - k_0 < 0$. For $N \geq 0$ and $K \geq k_0$ we define a pairing $\langle \cdot, \cdot \rangle_{N,K}$ by

$$\langle f, g \rangle_{N,K} = \frac{C}{2\pi} \int_0^N f(x)g(x)M(x)dx + \sum_{k=k_0}^K f(i(t-k))g(i(t-k))H_k.$$

If f and g are real valued functions in \mathcal{H} , the limits $N, K \rightarrow \infty$ give the inner product $\langle f, g \rangle_{\mathcal{H}}$. We denote $\lim_{N \rightarrow \infty} \langle f, g \rangle_{N,K} = \langle f, g \rangle_K$, assuming that the limit exists.

For analytic functions f and g we define the Wronskian $[f, g]$ by

$$[f, g](k) = \left\{ f(i(t-k))g(i(t-k-1)) - f(i(t-k-1))g(i(t-k)) \right\} B(i(t-k))H_k.$$

Proposition 5.1. *For even analytic functions f and g we have*

$$\langle Lf, g \rangle_{N,K} - \langle f, Lg \rangle_{N,K} = \frac{C}{2\pi} \int_N^{N+i} \{f(x)g(x-i) - f(x-i)g(x)\} B(x)H(x)dx - [f, g](K).$$

Proof. We follow the proof of Proposition 4.2. For even functions f and g we have

$$\langle f, g \rangle_{N,K} = \frac{1}{4\pi} \int_{\mathcal{C}_{N,K}} f(x)g(x)H(x)dx,$$

where $\mathcal{C}_{N,K}$ is a contour in the complex plane defined as follows:

- $\mathcal{C}_{N,K}$ starts at $x = -N$ and ends at $x = N$,
- $\mathcal{C}_{N,K}$ is invariant under reflection in the origin,
- $\mathcal{C}_{N,K}$ separates the sequence of poles $i(a+n)$, $n \in \mathbb{Z}_{\geq 0}$, from the sequence $-i(a+m)$, $m \in \mathbb{Z}_{\geq 0}$, and similarly for poles of $H(x)$ corresponding to $b, c, 1-d$,
- $\mathcal{C}_{N,K}$ separates the sequence $i(t-n)$, $n \in \mathbb{Z}$, $n \leq K$, from the sequence $i(t-n)$, $n \in \mathbb{Z}$, $n \geq K+1$,
- $\mathcal{C}_{N,K}$ separates the sequence of poles $i(t-n)$, $n \leq K$, from the sequence $-i(t-m)$, $m \leq K$,

Using $B(-x+i)H(x-i) = B(x)H(x)$, we have

$$\langle Lf, g \rangle_{N,K} - \langle f, Lg \rangle_{N,K} = \frac{C}{4\pi} \left(\int_{\mathcal{C}_{N,K}} - \int_{\mathcal{C}_{N,K}+i} \right) \left(f(x-i)g(x) - f(x)g(x-i) \right) B(x)H(x)dx.$$

Now we make a counterclockwise oriented closed contour by connecting $\mathcal{C}_{N,K}$ and $\mathcal{C}_{N,K}+i$ at the end points, then the integrand $I(x)$ has two poles inside the closed contour; at $x = i(t-K)$ and $x = -i(t-K-1)$. So,

$$\frac{C}{4\pi} \left(\int_{\mathcal{C}_N} - \int_{\mathcal{C}_N+i} \right) I(x)dx = \frac{C}{4\pi} \left(\int_{-N}^{-N+i} - \int_N^{N+i} \right) I(x)dx + \frac{iC}{2} \operatorname{Res}_{x=i(t-K)} I(x) + \frac{iC}{2} \operatorname{Res}_{x=-i(t-K-1)} I(x).$$

In the same way as in the proof Lemma 4.1 we can show that the first integral on the right hand side is equal to the second integral with opposite sign. For the residues we have

$$\frac{iC}{2} \operatorname{Res}_{x=i(t-K)} I(x) = -\frac{1}{2}[f, g](K),$$

and, since $I(x) = -I(-x+i)$,

$$\begin{aligned} \frac{iC}{2} \operatorname{Res}_{x=-i(t-K-1)} I(x) &= - \lim_{x \rightarrow -i(t-K-1)} (x+i(t-K-1))I(-x+i) \\ &= \lim_{y \rightarrow i(t-K)} (y-i(t-K))I(y) = \operatorname{Res}_{y=i(t-K)} I(y) = -\frac{1}{2}[f, g](K). \end{aligned}$$

This gives the desired result. \square

From $H(x) = M(x)/\sin \pi(t \pm ix)$ and (4.1) we find, for $y \in \mathbb{R}$,

$$H(x+iy) = \mathcal{O}(x^{2a+2b+2c-2d-1}e^{-4\pi x}), \quad x \rightarrow \infty. \quad (5.1)$$

This gives the following for the Wilson functions.

Proposition 5.2. For $\lambda \neq \lambda'$

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_K = \frac{[\phi_\lambda, \phi_{\lambda'}](K)}{\lambda'^2 - \lambda^2}.$$

Proof. Since the Wilson function is an eigenfunction of L for eigenvalue $\tilde{a}^2 + \lambda^2$, the result follows from Proposition 5.2, because the integral on the right hand side in Proposition 5.2 is equal to zero. Indeed, using (5.1) and the asymptotic behaviour of ϕ_λ , which follows from Proposition 4.4 and (4.3), and applying dominated convergence, we obtain

$$\lim_{N \rightarrow \infty} \int_N^{N+i} \{ \phi_\lambda(x)\phi_{\lambda'}(x-i) - \phi_\lambda(x-i)\phi_{\lambda'}(x) \} B(x)H(x)dx = 0.$$

\square

To find the asymptotic behaviour of the Wronskian $[\phi_\lambda, \phi_{\lambda'}](K)$ for $K \rightarrow \infty$, we need the asymptotic behaviour of H_K , $B(i(t-K))$ and $\phi_\lambda(i(t-K))$. First we expand ϕ_λ in functions with nice asymptotic behaviour, as in Proposition 4.4.

Proposition 5.3. For $x = i(t - k) \in \mathcal{D}^+$

$$\phi_\lambda(x) = \tilde{d}(\lambda)\Theta_\lambda(k) + \tilde{d}(-\lambda)\Theta_{-\lambda}(k),$$

where

$$\Theta_\lambda(k) = \frac{(-1)^k}{\pi} \frac{\Gamma(1 - \tilde{b} - c - i\lambda - t + k)}{\Gamma(\tilde{b} + c + i\lambda - t + k)} {}_4F_3 \left(\begin{matrix} \tilde{a} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda, 1 - \tilde{d} + i\lambda \\ \tilde{b} + c + i\lambda - t + k, \tilde{b} + c + i\lambda + t - k, 1 + 2i\lambda \end{matrix} ; 1 \right),$$

$$\tilde{d}(\lambda) = \frac{\Gamma(-2i\lambda) \sin \pi(\tilde{t} - i\lambda)}{\Gamma(\tilde{a} - i\lambda)\Gamma(\tilde{b} - i\lambda)\Gamma(\tilde{c} - i\lambda)\Gamma(1 - \tilde{d} - i\lambda)}.$$

Proof. This follows from Proposition 4.4, and Euler reflection formula;

$$\frac{1}{\Gamma(\tilde{b} + c + i\lambda \pm ix)} = (-1)^k \frac{\Gamma(1 - \tilde{b} - c - i\lambda + ix) \sin \pi(\tilde{b} + c + t + i\lambda)}{\pi \Gamma(\tilde{b} + c + i\lambda + ix)},$$

for $x = i(t - k)$, $k \in \mathbb{Z}$. □

Remark 5.4. Observe that

$$\frac{1}{\tilde{d}(\lambda)\tilde{d}(\lambda)} = \tilde{H}(\lambda).$$

For $k \rightarrow \infty$ we find from the explicit expression for Θ_λ and (4.2), for $y \in \mathbb{Z}$,

$$\Theta_\lambda(k + y) = \frac{(-1)^k}{\pi} k^{d-a-b-c-2i\lambda} \left(1 + \frac{1}{2k} (d - a - b - c - 2i\lambda)(1 - 2t + 2y) + \mathcal{O}\left(\frac{1}{k^2}\right) \right). \quad (5.2)$$

Furthermore, for $k \rightarrow \infty$,

$$H_k = \frac{2\pi^2}{C} k^{2a+2b+2c-2d-1} \left(1 + \left(\frac{1}{k}\right) \right),$$

$$B(i(t - k)) = \frac{k^2}{4} \left(1 + \left(\frac{1}{k}\right) \right).$$

We can find the Wronskian $[\phi_\lambda, \phi_{\lambda'}](K)$, for $K \rightarrow \infty$, from the four Wronskians $[\Theta_{\pm\lambda}, \Theta_{\pm\lambda'}](K)$.

Lemma 5.5. For $K \rightarrow \infty$,

$$[\Theta_\lambda, \Theta_{\lambda'}](K) = \frac{i}{C} (\lambda' - \lambda) K^{-2i(\lambda+\lambda')} \left(1 + \left(\frac{1}{K}\right) \right).$$

Proof. We have

$$[\Theta_\lambda, \Theta_{\lambda'}](K) = \left\{ \Theta_\lambda(K)\Theta_{\lambda'}(K+1) - \Theta_\lambda(K+1)\Theta_{\lambda'}(K) \right\} B(i(t - K)) H_K.$$

The lemma follows from this expression using the asymptotic behaviour of $\Theta_\lambda(K)$, $B(i(t - K))$ and H_K . □

5.3. Continuous spectrum. In this subsection we assume $\lambda, \lambda' \geq 0$.

Proposition 5.6. Let f be an even continuous function, satisfying

$$f(\lambda) = \begin{cases} \mathcal{O}(\lambda^{\tilde{a}+\tilde{b}+\tilde{c}-\tilde{d}-\frac{1}{2}-\varepsilon} e^{-2\pi\lambda}), & \lambda \rightarrow \infty, \quad \varepsilon > 0, \\ \mathcal{O}(\lambda^\delta), & \lambda \downarrow 0, \quad \delta > 0. \end{cases}$$

Then

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda = \frac{f(\lambda')}{C \tilde{H}(\lambda')}.$$

Proof. From Proposition 5.3 we find,

$$[\phi_\lambda, \phi_{\lambda'}](K) = \sum_{\epsilon, \xi \in \{-1, 1\}} \tilde{d}(\epsilon\lambda) \tilde{d}(\xi\lambda') [\Phi_{\epsilon\lambda}, \Phi_{\xi\lambda'}](K),$$

and then we obtain from Lemma 5.5 and Proposition 5.2

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_K = \frac{i}{C} \sum_{\epsilon, \xi \in \{-1, 1\}} \tilde{d}(\epsilon\lambda) \tilde{d}(\xi\lambda') \frac{K^{-2i(\epsilon\lambda + \xi\lambda')}}{\epsilon\lambda + \xi\lambda'} \left(1 + \mathcal{O}\left(\frac{1}{K}\right) \right).$$

We multiply both sides with an arbitrary function $f(\lambda)$, and we integrate over λ from 0 to ∞ . The function f must satisfy certain conditions that we determine later on. Letting $K \rightarrow \infty$ then gives

$$\begin{aligned} \lim_{K \rightarrow \infty} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_K d\lambda = \\ \lim_{K \rightarrow \infty} \frac{i}{C} \int_0^\infty f(\lambda) \left\{ \psi_1(\lambda) \cos(2[\lambda + \lambda'] \ln(K)) + \psi_2(\lambda) \sin(2[\lambda + \lambda'] \ln(K)) \right. \\ \left. + \psi_3(\lambda) \cos(2[\lambda - \lambda'] \ln(K)) + \psi_4(\lambda) \frac{\sin(2[\lambda - \lambda'] \ln(K))}{\lambda - \lambda'} \right\} d\lambda \end{aligned}$$

where

$$\begin{aligned} \psi_1(\lambda) &= \frac{1}{\lambda + \lambda'} \left(\tilde{d}(\lambda) \tilde{d}(\lambda') - \tilde{d}(-\lambda) \tilde{d}(-\lambda') \right), \\ \psi_2(\lambda) &= \frac{-i}{\lambda + \lambda'} \left(\tilde{d}(\lambda) \tilde{d}(\lambda') + \tilde{d}(-\lambda) \tilde{d}(-\lambda') \right), \\ \psi_3(\lambda) &= \frac{1}{\lambda - \lambda'} \left(\tilde{d}(\lambda) \tilde{d}(-\lambda') - \tilde{d}(-\lambda) \tilde{d}(\lambda') \right), \\ \psi_4(\lambda) &= -i \left(\tilde{d}(\lambda) \tilde{d}(-\lambda') + \tilde{d}(-\lambda) \tilde{d}(\lambda') \right). \end{aligned}$$

ψ_3 has a removable singularity at $\lambda = \lambda'$. From the Riemann-Lebesgue lemma we find that the terms with ψ_i , $i = 1, 2, 3$, vanish, provided that $f\psi_i \in L^1(0, \infty)$. Using (4.4) for the term with ψ_4 , we obtain

$$\lim_{N \rightarrow \infty} \int_0^\infty f(\lambda) \langle \phi_\lambda, \phi_{\lambda'} \rangle_N d\lambda = \frac{\pi}{C} \psi_4(\lambda') f(\lambda') = \frac{2\pi}{C} \tilde{d}(\lambda') \tilde{d}(-\lambda') f(\lambda') = \frac{2\pi}{C} \frac{f(\lambda')}{\tilde{H}(\lambda')}.$$

Using, for $i = 1, 2, 3, 4$,

$$|\psi_i(\lambda)| = \begin{cases} \mathcal{O}(\lambda^{-\frac{1}{2} - \bar{a} - \bar{b} - \bar{c} + \bar{d}} e^{2\pi\lambda}), & \lambda \rightarrow \infty, \\ \mathcal{O}(\lambda^{-1}), & \lambda \downarrow 0, \end{cases}$$

we see that if f satisfies the conditions given in the proposition, then $f\psi_i \in L^1(0, \infty)$. \square

Let \mathcal{H}_0 be the dense subspace of \mathcal{H} defined by

$$\mathcal{H}_0 = \{f \in \mathcal{H} \mid f(i(t - k)) = 0 \text{ for } k \gg 0\}.$$

For $f \in \mathcal{H}_0$ we define a linear operator \mathcal{G} by

$$(\mathcal{G}f)(\lambda) = \int f(x) \phi_\lambda(x) dh(x).$$

We call \mathcal{G} the Wilson function transform of type II. We denote the continuous part of the above integral by $\mathcal{G}_c f$, i.e.

$$(\mathcal{G}_c f)(\lambda) = \frac{C}{2\pi} \int_0^\infty f(x) \phi_\lambda(x) H(x) dx.$$

From the asymptotic behaviour of ϕ_λ and $H(x)$ we find that both $\mathcal{G}f$ and $\mathcal{G}_c f$ are well defined if $f \in \mathcal{H}_0$.

Proposition 5.7. *Let g be a continuous function satisfying*

$$g(\lambda) = \begin{cases} \mathcal{O}(\lambda^{\tilde{d}-\tilde{a}-\tilde{b}-\tilde{c}+\frac{1}{2}-\varepsilon}e^{2\pi\lambda}), & \lambda \rightarrow \infty, \quad \varepsilon > 0, \\ \mathcal{O}(\lambda^{\delta-1}), & \lambda \downarrow 0, \quad \delta > 0, \end{cases}$$

then $(\mathcal{F}(\tilde{\mathcal{F}}_c g))(\lambda) = g(\lambda)$.

Proof. The proof runs along the same lines as the proof of Proposition 4.8. \square

Note that if $g \in \tilde{\mathcal{H}}_0$, then $g = o(\lambda^{\tilde{d}-\tilde{a}-\tilde{b}-\tilde{c}}e^{2\pi\lambda})$ for $\lambda \rightarrow \infty$. So if $g \in \mathcal{H}_0$, then g satisfies the conditions of Proposition 5.7.

5.4. Discrete spectrum. In this subsection we assume $\lambda \in \tilde{\mathcal{D}}^+$. In this case $\tilde{d}(-\lambda) = 0$ and $\Im\lambda < 0$, and then it follows from Proposition 5.3 and the asymptotic behaviour (5.2) of Θ_λ , that $\phi_\lambda \in \mathcal{H}$. The following two propositions are proved in a similar way as Propositions 4.9 and 4.10, therefore we omit the proofs here.

Proposition 5.8. *For $\lambda \in \tilde{\mathcal{D}}^+$, $\lambda' \in \text{supp}(d\tilde{h})$, $\lambda \neq \lambda'$, we have*

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle_{\mathcal{H}} = 0.$$

Proposition 5.9. *For $\lambda \in \tilde{\mathcal{D}}^+$,*

$$\langle \phi_\lambda, \phi_\lambda \rangle_{\mathcal{H}} = \left(iC \text{Res}_{\lambda'=\lambda} \tilde{H}(\lambda') \right)^{-1}.$$

5.5. The Wilson function transform: type II. In the same way as in the proof of Theorem 4.12 we can combine Propositions 5.8 and 5.9 with Proposition 5.7, to find that the operator $\mathcal{G} : \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0$ is a unitary operator with inverse $\tilde{\mathcal{G}}$. Since \mathcal{H}_0 is dense in \mathcal{H} , \mathcal{G} extends uniquely to a unitary operator on \mathcal{H} .

Theorem 5.10. *The Wilson function transform of type II, $\mathcal{G} : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$, defined by*

$$(\mathcal{G}f)(\lambda) = \int f(x)\phi_\lambda(x)dh(x),$$

is unitary, and its inverse is given by $\mathcal{G}^{-1} = \tilde{\mathcal{G}}$.

Remark 5.11. For $a = b + c + d - 1$, $t = \frac{1}{2}(1 - b - c - s)$, $s \in \mathbb{R}$, the Wilson function transform of type II is completely self-dual.

6. EXPLICIT TRANSFORMATIONS

In this section we calculate explicitly the Wilson function transforms of certain functions. First we give an integral representation for the Wilson function related to Jacobi functions. This leads to two explicit transformations in Theorems 6.2 and 6.5. Then, in Theorem 6.7, we show that the Wilson function transform of type I maps an orthogonal basis of polynomials in the Hilbert space \mathcal{M} to itself, with dual parameters.

6.1. Transformations related to Jacobi functions. The Jacobi functions, see [12], are defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1 \left(\begin{matrix} \frac{1}{2}(\alpha + \beta + 1 - i\lambda), \frac{1}{2}(\alpha + \beta + 1 + i\lambda) \\ \alpha + 1 \end{matrix} ; -x \right). \quad (6.1)$$

For $x \geq 1$ we use here the unique one valued analytic continuation of the ${}_2F_1$ -function. The Jacobi functions are the kernel in an integral transform pair, called the Jacobi transform, given by

$$\begin{cases} (\mathcal{J}f)(\lambda) = \int_0^\infty f(z)\varphi_\lambda^{(\alpha,\beta)}(x)\Delta_{\alpha,\beta}(x)dx \\ f(x) = \frac{1}{2\pi} \int (\mathcal{J}f)(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)d\nu(\lambda) \end{cases} \quad (6.2)$$

where $\alpha > -1$, $\beta \in \mathbb{R} \cup i\mathbb{R}$,

$$\Delta_{\alpha,\beta}(x) = x^\alpha(1+x)^\beta,$$

and $d\nu(\lambda)$ is the measure given by

$$\begin{aligned} \frac{1}{2\pi} \int g(\lambda)d\nu(\lambda) &= \frac{1}{2\pi} \int_0^\infty g(\lambda)|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda - i \sum_{\lambda \in \mathcal{E}} g(\lambda) \operatorname{Res}_{\mu=\lambda} (c_{\alpha,\beta}(\mu)c_{\alpha,\beta}(-\mu))^{-1}, \\ c_{\alpha,\beta}(\lambda) &= \frac{2^{-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}, \\ \mathcal{E} &= \left\{ i(|\beta| - \alpha - 1 - 2j) \mid j \in \mathbb{Z}_{\geq 0}, |\beta| - \alpha - 1 - 2j > 0 \right\}. \end{aligned}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the measure $(1-x)^\alpha(1+x)^\beta dx$. They have an explicit expression as ${}_2F_1$ -series, see [1], [8]. In this paper we need Jacobi polynomials of argument $(1-x)/(1+x)$. Using Pfaff's transformation, we find that the explicit expression for these polynomials is

$$P_n^{(\alpha,\beta)}\left(\frac{1-x}{1+x}\right) = \frac{(\alpha+1)_n}{n!} (1+x)^{-n} {}_2F_1\left(\begin{matrix} -n, -\beta-n \\ 2\alpha \end{matrix}; -x\right). \quad (6.3)$$

The orthogonality relation for these polynomials reads, for $\Re(\alpha), \Re(\beta) > -1$,

$$\begin{aligned} \int_0^\infty P_n^{(\alpha,\beta)}\left(\frac{1-x}{1+x}\right) P_m^{(\alpha,\beta)}\left(\frac{1-x}{1+x}\right) x^\alpha(1+x)^{-\alpha-\beta-2} dx &= \\ \frac{1}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \delta_{nm}. \end{aligned} \quad (6.4)$$

The following proposition gives a representation of a very-well poised ${}_7F_6$ -series as an integral over a product of two ${}_2F_1$ -series.

Proposition 6.1. For $\beta, \mu \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\gamma + \delta \pm \rho) > 0$,

$$\begin{aligned} &\int_0^\infty {}_2F_1\left(\begin{matrix} \alpha+\mu+\gamma, \alpha+\mu-\gamma \\ 2\alpha \end{matrix}; -u\right) {}_2F_1\left(\begin{matrix} \alpha+\beta+\rho, \alpha+\beta-\rho \\ 2\alpha \end{matrix}; -u\right) u^{2\alpha-1}(1+u)^{\beta-\delta+\mu} du \\ &= \frac{\Gamma(2\alpha)\Gamma(2\alpha+\delta+\rho+\gamma)\Gamma(\delta+\gamma\pm\rho)\Gamma(\delta-\gamma+\rho)}{\Gamma(\alpha+\delta+\rho\pm\mu)\Gamma(\alpha+\delta+\gamma\pm\beta)} \\ &\quad \times W(2\alpha+\delta+\gamma-1+\rho; \alpha+\gamma+\mu, \alpha+\gamma-\mu, \alpha+\beta+\rho, \alpha-\beta+\rho, \delta+\gamma+\rho). \end{aligned} \quad (6.5)$$

Proof. We start with the following formula, for $\Re(\alpha) > 0$, $\Re(\gamma + \delta \pm \rho) > 0$,

$$\begin{aligned} &\int_0^1 y^{2\alpha-1}(1-y)^{\gamma+\delta-\alpha-\beta-1} {}_2F_1\left(\begin{matrix} \alpha+\beta+\rho, \alpha+\beta-\rho \\ 2\alpha \end{matrix}; \frac{y}{1-y}\right) dy \\ &= \frac{\Gamma(2\alpha)\Gamma(\gamma+\delta+\rho)\Gamma(\gamma+\delta-\rho)}{\Gamma(\alpha-\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}. \end{aligned} \quad (6.6)$$

To prove this identity we transform the ${}_2F_1$ -series by Pfaff's transformation [1, Thm.2.2.5], then we obtain a ${}_2F_1$ -series that converges uniformly on $[0, 1]$ for $\Re(\rho) < 0$. We interchange summation and integration, then the result follows from using the Beta-integral and Gauss's summation formula [1, Thm.2.2.2]. The condition on ρ can be removed using the symmetry in ρ and $-\rho$, and continuity in ρ .

Next we write the integral in the theorem as

$$I = \int_0^1 y^{2\alpha-1} (1-y)^{\delta-\mu-\beta-2\alpha-1} \times {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; \frac{y}{y-1} \right) {}_2F_1 \left(\begin{matrix} \alpha + \beta + \rho, \alpha + \beta - \rho \\ 2\alpha \end{matrix} ; \frac{y}{y-1} \right) dy,$$

where we substituted $u \mapsto y/(1-y)$. By [4, §2.10(3)] the first ${}_2F_1$ -function can be expanded in terms of ${}_2F_1$ -functions of argument $1-y$

$$\begin{aligned} & {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; \frac{y}{y-1} \right) = \\ & \frac{\Gamma(2\alpha)\Gamma(-2\gamma)}{\Gamma(\alpha + \mu - \gamma)\Gamma(\alpha - \mu - \gamma)} (1-y)^{\alpha+\mu+\gamma} {}_2F_1 \left(\begin{matrix} \alpha + \gamma + \mu, \alpha + \gamma - \mu \\ 1 + 2\gamma \end{matrix} ; 1-y \right) \\ & + \frac{\Gamma(2\alpha)\Gamma(2\gamma)}{\Gamma(\alpha + \mu + \gamma)\Gamma(\alpha - \mu + \gamma)} (1-y)^{\alpha+\mu-\gamma} {}_2F_1 \left(\begin{matrix} \alpha - \gamma + \mu, \alpha - \gamma - \mu \\ 1 - 2\gamma \end{matrix} ; 1-y \right). \end{aligned} \quad (6.7)$$

Observe that the second term is equal to the first term with γ replace by $-\gamma$. The integral I splits according to this as $I = I_\gamma + I_{-\gamma}$. We use formula (6.6) to evaluate I_γ . Interchanging summation and integration, which is allowed for $\Re(\alpha) < \frac{1}{2}$ since then the ${}_2F_1$ -series converges uniformly on $[0, 1]$, and using (6.6) leads to

$$\begin{aligned} I_\gamma &= \frac{\Gamma(2\alpha)^2 \Gamma(-2\gamma) \Gamma(\delta + \gamma + \rho) \Gamma(\delta + \gamma - \rho)}{\Gamma(\alpha - \gamma + \mu) \Gamma(\alpha - \gamma - \mu) \Gamma(\alpha + \beta + \gamma + \delta) \Gamma(\alpha - \beta + \gamma + \delta)} \\ & \times {}_4F_3 \left(\begin{matrix} \alpha + \gamma + \mu, \alpha + \gamma - \mu, \delta + \gamma + \rho, \delta + \gamma - \rho \\ 2\gamma + 1, \alpha + \beta + \gamma + \delta, \alpha - \beta + \gamma + \delta \end{matrix} ; 1 \right). \end{aligned} \quad (6.8)$$

Then $I = I_\gamma + I_{-\gamma}$ is the sum of two balanced ${}_4F_3$ -functions, which by [2, §4.4(4)] can be written as the very-well poised ${}_7F_6$ -series given in the proposition.

Note that the expression in (6.8) is an analytic function in α for $\Re(\alpha) > 0$. The integrand of the integral I is analytic in α for $\Re(\alpha) > 0$, and continuous in y . So differentiation with respect to α and integration with respect to y can be interchanged, see e.g. [22, §4.2]. We see that the integral I is an analytic function in α for $\Re(\alpha) > 0$, and therefore the condition $\Re(\alpha) < \frac{1}{2}$ can be removed by analytic continuation. \square

Both ${}_2F_1$ -series in Proposition 6.1 can be considered as Jacobi functions. Using the substitutions

$$\alpha \mapsto \frac{1}{2}(a+1-d), \quad \beta \mapsto \frac{1}{2}(c-b), \quad \gamma \mapsto \frac{1}{2}(a+d-1), \quad \delta \mapsto \frac{1}{2}(b+c), \quad \mu \mapsto ix, \quad \rho \mapsto i\lambda, \quad (6.9)$$

and the definition (3.2) of the Wilson functions, the right hand side of (6.5) can be written as

$$\Gamma(1+a-d)^2 \Gamma(\tilde{a} \pm i\lambda) \Gamma(1 - \tilde{d} \pm i\lambda) \phi_\lambda(x; a, b, c, d).$$

So Proposition 6.1 gives a representation of the Wilson function as the Jacobi function transform of a Jacobi function. From the inverse Jacobi transform we obtain the following.

Theorem 6.2. Let $\varphi(x) = \varphi_{2x}^{(b+c-1, c-b+1)}(u)$, $u \geq 0$, and $\psi(x) = \sin \pi(t \pm ix)\varphi(x)$, then

$$(\mathcal{F}\varphi)(\lambda) = (1+u)^{\frac{1}{2}-\tilde{c}+i\lambda} {}_2F_1 \left(\begin{matrix} \tilde{a} + i\lambda, 1 - \tilde{d} + i\lambda \\ 1 + \tilde{a} - \tilde{d} \end{matrix} ; -u \right), \quad (a, b, c, d) \in V,$$

$$(\mathcal{G}\psi)(\lambda) = C(1+u)^{\frac{1}{2}-\tilde{c}+i\lambda} {}_2F_1 \left(\begin{matrix} \tilde{a} + i\lambda, 1 - \tilde{d} + i\lambda \\ 1 + \tilde{a} - \tilde{d} \end{matrix} ; -u \right), \quad (a, b, c, d, t) \in V^+.$$

Observe that for $u = 0$ the first statement gives $(\mathcal{F}1)(\lambda) = 1$.

Proof. In Proposition 6.1 we replace ρ by $i\lambda \in i\mathbb{R}$. Then using the definition of the Jacobi functions (6.1) we can write Proposition 6.1 as

$$\int_0^\infty (1+u)^{\mu-\delta-\beta} {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; -u \right) \varphi_{2\lambda}^{(2\alpha-1, 2\beta)}(u) \Delta_{2\alpha-1, 2\beta}(u) du = F(\lambda),$$

where $F(\lambda)$ denotes the right hand side of (6.5). Note that from (6.7) we find for $u \rightarrow \infty$

$$(1+u)^{\mu-\delta-\beta} {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; -u \right) \sim C_1 u^{-\alpha-\beta-\delta-\gamma} + C_2 u^{-\alpha-\beta-\delta+\gamma},$$

where C_1 and C_2 are independent of u . So for $\Re(\delta \pm \gamma) > 0$ we see that this function is an element of $L^2([0, \infty), \Delta_{2\alpha-1, 2\beta}(u) du)$. Taking the inverse Jacobi transform, assuming for simplicity that the discrete set \mathcal{E} is empty, gives

$$(1+u)^{\mu-\delta-\beta} {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; -u \right) = \frac{1}{2\pi} \int_0^\infty F(\lambda) \varphi_{2\lambda}^{(2\alpha-1, 2\beta)}(u) \left| \frac{\Gamma(\alpha + \beta + i\lambda)\Gamma(\alpha - \beta + i\lambda)}{\Gamma(2\alpha)\Gamma(2i\lambda)} \right|^2 d\lambda.$$

We write out $F(\lambda)$ and $\varphi_{2\lambda}^{(2\alpha-1, 2\beta)}(t)$ as hypergeometric functions and use the substitutions given in (6.9), without the last substitution, then we obtain

$$(1+u)^{\frac{1}{2}-c+ix} {}_2F_1 \left(\begin{matrix} a + ix, 1 - d + ix \\ 1 + a - d \end{matrix} ; -u \right) = \frac{1}{2\pi} \int_0^\infty {}_2F_1 \left(\begin{matrix} \tilde{c} + i\lambda, \tilde{c} - i\lambda \\ \tilde{b} + \tilde{c} \end{matrix} ; -u \right) \phi_\lambda(x) \tilde{M}(\lambda) d\lambda.$$

This is the first statement in the theorem with dual parameters, and x and λ interchanged. Writing the above integral as a contour integral (with the contour as in the proof of Proposition 4.2 with $N \rightarrow \infty$), and using analytic continuation in the parameters a, b, c, d , the identity can be extended to the case $(a, b, c, d) \in V$. Deforming the contour again to the real line might add a finite number of discrete mass points.

The second statement follows from the first. We use $H(x) = M(x)/\sin \pi(t \pm ix)$ and we use that the function $\sin \pi(t \pm ix)\varphi(x)$ vanishes on the infinite discrete set \mathcal{D}^+ . \square

We substitute $\mu = -\alpha - \gamma - n$, $n \in \mathbb{Z}_{\geq 0}$, in Proposition 6.1, then the first ${}_2F_1$ -series on the left hand side of (6.5) terminates and can be written as a Jacobi polynomial by (6.3). Writing the right hand side as $I_\gamma + I_{-\gamma}$, with I_γ as in (6.8), we see that $I_{-\gamma} = 0$ because of the factor $\Gamma(\alpha + \gamma + \mu)^{-1}$. Now we obtain, for $\Re(\alpha) > 0$ and $\Re(\gamma + \delta \pm \rho) > 0$,

$$\begin{aligned} \int_0^\infty u^{2\alpha-1} (1+u)^{\beta-\delta-\alpha-\gamma} P_n^{(2\alpha-1, 2\gamma)} \left(\frac{1-u}{1+u} \right) {}_2F_1 \left(\begin{matrix} \alpha + \beta + \rho, \alpha + \beta - \rho \\ 2\alpha \end{matrix} ; -u \right) du \\ = \frac{(-1)^n (2\gamma + 1)_n \Gamma(2\alpha) \Gamma(\gamma + \delta + \rho) \Gamma(\gamma + \delta - \rho)}{n! \Gamma(\alpha + \beta + \gamma + \delta) \Gamma(\alpha - \beta + \gamma + \delta)} \\ \times {}_4F_3 \left(\begin{matrix} -n, 2\alpha + 2\gamma + n, \delta + \gamma + \rho, \delta + \gamma - \rho \\ 2\gamma + 1, \alpha + \beta + \gamma + \delta, \alpha - \beta + \gamma + \delta \end{matrix} ; 1 \right). \end{aligned} \quad (6.10)$$

This is Koornwinder's formula [13, (3.3)] stating that Jacobi polynomials are mapped onto Wilson polynomials by the Jacobi function transform. We use this formula to expand the integral in Proposition 6.1 in terms of terminating ${}_4F_3$ -functions. This leads to an expansion formula for the Wilson function.

Proposition 6.3. *Let $f, g \in \mathbb{C}$ satisfy $\Re(f) > 0$, $\Re(f + g) > 0$, $\Re(\tilde{a} + g) > 0$, $\Re(1 - \tilde{d} + g) > 0$. Then*

$$\begin{aligned} \phi_\lambda(x; a, b, c, d) &= \sum_{n=0}^{\infty} C_n(x, \lambda) {}_4F_3 \left(\begin{matrix} -n, a + f + g - d + n, f + i\lambda, f - i\lambda \\ f + g, \tilde{b} + f, \tilde{c} + f \end{matrix} ; 1 \right) \\ &\quad \times {}_4F_3 \left(\begin{matrix} -n, a - d + f + g + n, g + \tilde{a}, g + 1 - \tilde{d} \\ f + g, g + \tilde{b} + c + ix, g + \tilde{b} + c - ix \end{matrix} ; 1 \right), \end{aligned}$$

where

$$\begin{aligned} C_n(x, \lambda) &= \frac{\Gamma(a - d + f + g + 1)\Gamma(g + \tilde{a})\Gamma(g + 1 - \tilde{d})\Gamma(f \pm i\lambda)}{\Gamma(1 + a - d)\Gamma(f + g)\Gamma(f + \tilde{b})\Gamma(f + \tilde{c})\Gamma(g + \tilde{b} + c \pm ix)\Gamma(\tilde{a} \pm i\lambda)\Gamma(1 - \tilde{d} \pm i\lambda)} \\ &\quad \times \frac{a + f + g - d + 2n}{a + f + g - d} \frac{(a + f + g - d)_n (f + g)_n}{n! (a + 1 - d)_n}. \end{aligned}$$

Proof. Let I be the integral in Proposition 6.1. We expand the first ${}_2F_1$ -function in I in terms of Jacobi polynomials of argument $(1 - u)/(1 + u)$:

$$(1 + u)^{\alpha + \mu + \eta - \sigma + 1} {}_2F_1 \left(\begin{matrix} \alpha + \mu + \gamma, \alpha + \mu - \gamma \\ 2\alpha \end{matrix} ; -u \right) = \sum_{n=0}^{\infty} c_n P_n^{(2\alpha-1, 2\eta)} \left(\frac{1 - u}{1 + u} \right).$$

Note that from (6.7) we obtain that the left hand behaves for large t as $C_1 u^{\eta - \sigma - \gamma + 1} + C_2 u^{\eta - \sigma + \gamma + 1}$, where C_1 and C_2 are independent of t . So for $\Re(\eta + \sigma \pm \gamma) > 0$ the function on the left hand side is an element of $L^2((0, \infty), u^{2\alpha-1}(1+u)^{-2\alpha-2\eta-1} du)$. We see that the expansion on the right hand side converges uniformly for u in compact intervals. The coefficients c_n are found using (6.10) and the orthogonality relation for the Jacobi polynomials (6.4), and this gives for $\Re(\alpha) > 0$, $\Re(\eta) > -\frac{1}{2}$, and $\Re(\sigma + \eta \pm \gamma) > 0$,

$$\begin{aligned} c_n &= \frac{\Gamma(2\alpha + 2\eta + 1)\Gamma(\eta + \sigma \pm \gamma)}{\Gamma(2\eta + 1)\Gamma(\alpha + \eta + \sigma \pm \mu)} \frac{(2\alpha + 2\eta + 2n)}{(2\alpha + 2\eta)} \frac{(-1)^n (2\alpha + 2\eta)_n}{(2\alpha)_n} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -n, 2\alpha + 2\eta + n, \sigma + \eta + \gamma, \sigma + \eta - \gamma \\ 2\eta + 1, \alpha + \mu + \eta + \sigma, \alpha - \mu + \eta + \sigma \end{matrix} ; 1 \right) \end{aligned}$$

From this expansion we find

$$I = \sum_{n=0}^{\infty} c_n \int_0^{\infty} u^{2\alpha-1} (1+u)^{\beta-\alpha-\delta-\eta+\sigma-1} P_n^{(2\alpha-1, 2\eta)} \left(\frac{1-u}{1+u} \right) {}_2F_1 \left(\begin{matrix} \alpha + \beta + \rho, \alpha + \beta - \rho \\ 2\alpha \end{matrix} ; -u \right) du$$

Using (6.10) again gives the following expansion for I , for $\Re(\alpha) > 0$, $\Re(\eta) > -\frac{1}{2}$, $\Re(\sigma + \eta \pm \gamma) > 0$, $\Re(\delta - \sigma + \eta + 1 \pm \rho) > 0$,

$$\begin{aligned} I &= \frac{\Gamma(2\alpha)\Gamma(2\alpha + 2\eta + 1)\Gamma(\eta + \sigma \pm \gamma)\Gamma(\eta + \delta - \sigma + 1 \pm \rho)}{\Gamma(2\eta + 1)\Gamma(\alpha + \eta + \sigma \pm \mu)\Gamma(\alpha + \eta + \delta - \sigma + 1 \pm \beta)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{2\alpha + 2\eta + 2n}{2\alpha + 2\eta} \frac{(2\alpha + 2\eta)_n (2\eta + 1)_n}{n! (2\alpha)_n} {}_4F_3 \left(\begin{matrix} -n, 2\alpha + 2\eta + n, \sigma + \eta + \gamma, \sigma + \eta - \gamma \\ 2\eta + 1, \alpha + \mu + \eta + \sigma, \alpha - \mu + \eta + \sigma \end{matrix} ; 1 \right) \\ &\quad \times {}_4F_3 \left(\begin{matrix} -n, 2\alpha + 2\eta + n, \delta - \sigma + 1 + \eta + \rho, \delta - \sigma + 1 + \eta - \rho \\ 2\eta + 1, \alpha + \beta + \eta + \delta - \sigma + 1, \alpha - \beta + \eta + \delta - \sigma + 1 \end{matrix} ; 1 \right). \end{aligned}$$

The proposition then follows from Proposition 6.1, the substitutions $\eta - \sigma + 1 + \delta \mapsto f$, $\eta + \sigma - \delta \mapsto g$ and (6.9). \square

Corollary 6.4. For $\Re(f) > 0$

$$\phi_\lambda(x; a, b, c, d) = \sum_{n=0}^{\infty} C_n(x, \lambda) {}_3F_2 \left(\begin{matrix} -n, f + i\lambda, f - i\lambda \\ \tilde{b} + f, \tilde{c} + f \end{matrix}; 1 \right) {}_3F_2 \left(\begin{matrix} -n, f - \tilde{a}, f + \tilde{d} - 1 \\ f + \tilde{c} - c - ix, f + \tilde{c} - c + ix \end{matrix}; 1 \right),$$

where

$$C_n(x, \lambda) = \frac{(f + \tilde{c} - c + ix)_n (f + \tilde{c} - c - ix)_n \Gamma(f \pm i\lambda)}{n! \Gamma(1 + a - d + n) \Gamma(f + \tilde{b}) \Gamma(f + \tilde{c}) \Gamma(\tilde{a} \pm i\lambda) \Gamma(1 - \tilde{d} \pm i\lambda)}.$$

Proof. We use Whipple's transformation [1, Thm.3.3.3] to write the second ${}_4F_3$ -function in Proposition 6.3 as

$$\frac{(f + \tilde{c} - c + ix)_n (f + \tilde{c} - c - ix)_n}{(g + \tilde{b} + c + ix)_n (g + \tilde{b} + c + ix)_n} {}_4F_3 \left(\begin{matrix} -n, a - d + f + g + n, f - \tilde{a}, f + \tilde{d} - 1 \\ f + g, f + \tilde{c} - c - ix, f + \tilde{c} - c + ix \end{matrix}; 1 \right).$$

Then the corollary follows from letting $g \rightarrow \infty$, using (4.2) for the Γ -functions. \square

We consider one of the ${}_4F_3$ -functions in Proposition 6.3 as a Wilson polynomial, then we obtain the following Wilson function transform of a Wilson polynomial.

Theorem 6.5. Let $f, g \in \mathbb{R}$ such that $g + a, g + b, g + c, g + 1 - d$ has positive real part. Furthermore, let φ_n and ψ_n be the function defined by

$$\varphi_n(x) = \Gamma(g \pm ix) R_n(x; f, b, c, g), \quad \psi_n(x) = \sin \pi(t \pm ix) \varphi_n(x).$$

Then

$$\begin{aligned} (\mathcal{F}\varphi_n)(\lambda) &= C^{-1}(\mathcal{G}\psi_n)(\lambda) = \frac{\Gamma(g + a)\Gamma(g + b)\Gamma(g + c)\Gamma(g + 1 - d)}{\Gamma(g + \tilde{b} + c \pm i\lambda)} \frac{(g + b)_n (g + c)_n}{(f + b)_n (f + c)_n} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -n, b + c + f + g + n - 1, g + 1 - d, g + a \\ f + g, g + \tilde{b} + c + i\lambda, g + \tilde{b} + c - i\lambda \end{matrix}; 1 \right). \end{aligned}$$

Here we assume for \mathcal{F} that $(a, b, c, d) \in V$, and for \mathcal{G} that $(a, b, c, d, t) \in V^+$.

Proof. We recognize the first ${}_4F_3$ -function in Proposition 6.3 as the Wilson polynomial $R_n(\lambda) = R_n(\lambda; f, \tilde{b}, \tilde{c}, g)$. We multiply by $R_n(\lambda)$ and integrate against the orthogonality measure $d\mu(\lambda; f, \tilde{b}, \tilde{c}, g)$. Using the orthogonality relation (2.3) then gives

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty R_n(\lambda) \phi_\lambda(x; a, b, c, d) \frac{\Gamma(\tilde{a} \pm i\lambda) \Gamma(\tilde{b} \pm i\lambda) \Gamma(\tilde{c} \pm i\lambda) \Gamma(1 - \tilde{d} \pm i\lambda) \Gamma(g \pm i\lambda)}{\Gamma(\pm 2i\lambda)} d\lambda &= \\ \frac{\Gamma(g + \tilde{a}) \Gamma(g + \tilde{b}) \Gamma(g + \tilde{c}) \Gamma(g + 1 - \tilde{d})}{\Gamma(g + \tilde{b} + c \pm ix)} \frac{(g + \tilde{b})_n (g + \tilde{c})_n}{(f + \tilde{b})_n (f + \tilde{c})_n} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -n, a - d + f + g + n, g + 1 - \tilde{d}, g + \tilde{a} \\ f + g, g + \tilde{b} + c + ix, g + \tilde{b} + c - ix \end{matrix}; 1 \right). \end{aligned}$$

The theorem follows from this by replacing the parameters a, b, c, d by their dual parameters, and interchanging λ and x .

Note that the integral can again be written as a contour integral. Then by analytic continuation the result remains true when $g \in \mathbb{R}$ and the pairwise sum of a, b, c, d, g has positive real part. \square

From Corollary 6.4 a similar theorem can be derived for the Wilson function transform of a continuous dual Hahn polynomial, which is a limit case of the Wilson polynomial, see [1], [8]. We leave this to the reader.

6.2. The Wilson function transform of type I of an orthogonal system. Koornwinder's formula (6.10) shows that the Jacobi function transform maps a (bi)orthogonal system of polynomials (the Jacobi polynomials) in $L^2((0, \infty), \Delta_{\alpha, \beta}(t) dt)$, onto an orthogonal system of polynomials (the Wilson polynomials) in $L^2((0, \infty), d\nu)$. We show that there exists a similar formula for the Wilson function transform of type I.

The Wilson polynomials $R_n(x) = R_n(x; a, b, c, 1 - d)$ form an orthogonal basis for the Hilbert space \mathcal{M} . We show that the Wilson function transform of type I maps a Wilson polynomial R_n onto itself, with dual parameters. The proof is based on the following proposition.

Proposition 6.6. *Let L be the difference operator as in Proposition 3.1, then*

$$(LR_n)(x) = C_n R_{n+1}(x) + (\tilde{C}_n + \tilde{D}_n) R_n(x) + D_n R_{n-1}(x),$$

where

$$C_n = \frac{(n+a+b+c-d)(n+a+b)(n+a+c)(n+a+1-d)}{(2n+a+b+c-d)(2n+a+b+c-d+1)},$$

$$D_n = \frac{n(n+b+c-1)(n+b-d)(n+c-d)}{(2n+a+b+c-d)(2n+a+b+c-d-1)}.$$

Proof. Applying the difference operator L on R_n gives

$$\begin{aligned} (LR_n)(x) &= B(-x)R_n(x+i) + [A(-x) + A(x)]R_n(x) + B(x)R_n(x-i) \\ &= [n(n+a+b+c-d) + A(x) + A(-x) + B(x) + B(-x)]R_n(x). \end{aligned}$$

Here we use the difference equation (2.4) for the Wilson polynomials $R_n(x; a, b, c, 1 - d)$. From the explicit expressions for $A(x)$ and $B(x)$ we obtain

$$A(x) + B(x) = \frac{(a+ix)(b+ix)(c+ix)}{2ix},$$

and this gives

$$\begin{aligned} A(x) + A(-x) + B(x) + B(-x) &= \frac{(a+ix)(b+ix)(c+ix) - (a-ix)(b-ix)(c-ix)}{2ix} \\ &= ab + ac + bc - x^2. \end{aligned}$$

Then, using the three-term recurrence relation (2.1) for the Wilson polynomials, we find

$$\begin{aligned} (LR_n)(x) &= [n(n+a+b+c-d) + ab + ac + bc + a^2]R_n(x) - (a^2 + x^2)R_n(x) \\ &= C_n R_{n+1}(x) + D_n R_{n-1}(x) \\ &\quad + [n(n+a+b+c-d) + ab + ac + bc + a^2 - C_n - D_n]R_n(x). \end{aligned}$$

Note that we use here C_n and D_n as in (2.1), but with d replaced by $1 - d$. A long, but straightforward calculation shows that

$$C_n + D_n + \tilde{C}_n + \tilde{D}_n = n(n+a+b+c-d) + a^2 + ab + ac + bc,$$

and from this the proposition follows. \square

Theorem 6.7. *For $(a, b, c, d) \in V$, the Wilson function transform of type I of the Wilson polynomial $R_n(x) = R_n(x; a, b, c, 1 - d)$ is given by*

$$(\mathcal{F}R_n)(\lambda) = (-1)^n \frac{(b+c)_n}{(1+a-d)_n} \tilde{R}_n(\lambda).$$

Proof. We use $(\mathcal{F}R_n)(\lambda) = \lim_{N \rightarrow \infty} \langle \phi_\lambda, R_n \rangle_N$, where $\langle \cdot, \cdot \rangle_N$ is the pairing defined in section 4. From Proposition 4.2 we find,

$$\lim_{N \rightarrow \infty} \langle L\phi_\lambda, R_n \rangle_N - \langle \phi_\lambda, LR_n \rangle_N = \lim_{N \rightarrow \infty} [\phi_\lambda, R_n](N).$$

Using (4.1), (4.3), $R_n(x) = \mathcal{O}(x^{2n})$, and applying dominated convergence, we obtain for the Wronskian

$$\lim_{N \rightarrow \infty} [\phi_\lambda, R_n](N) = 0.$$

The Wilson function ϕ_λ is an eigenfunction of L for eigenvalue $\tilde{a}^2 + \lambda^2$, so we have

$$(\tilde{a}^2 + \lambda^2)(\mathcal{F}R_n)(\lambda) = \lim_{N \rightarrow \infty} \langle L\phi_\lambda, R_n \rangle_N = \lim_{N \rightarrow \infty} \langle \phi_\lambda, LR_n \rangle_N = (\mathcal{F}(LR_n))(\lambda).$$

Then from Proposition 6.6 and linearity of \mathcal{F} , it follows that the function $\mathcal{F}R_n$ satisfies the recurrence relation

$$(\tilde{a}^2 + \lambda^2) y_n(\lambda) = C_n y_{n+1}(\lambda) + (\tilde{C}_n + \tilde{D}_n) y_n(\lambda) + D_n y_{n-1}(\lambda). \quad (6.11)$$

From $R_{-1}(x) = 0$, we obtain $(\mathcal{F}R_{-1})(\lambda) = 0$. By Theorem 6.2 we have $(\mathcal{F}R_0)(\lambda) = (\mathcal{F}1)(\lambda) = 1$, so we find from the recurrence relation that $(\mathcal{F}R_n)(\lambda) = P_n(\lambda)$. Here $P_n(\lambda)$ is a polynomial in λ^2 of degree n satisfying the three-term recurrence relation (6.11), with initial values $P_{-1}(\lambda) = 0$ and $P_0(\lambda) = 1$. From the recurrence relation for $\tilde{R}_n(\lambda) = R_n(\lambda; \tilde{a}, \tilde{b}, \tilde{c}, 1 - \tilde{d})$ it follows that $P_n(\lambda) = (-1)^n \frac{(b+c)_n}{(1+a-d)_n} \tilde{R}_n(\lambda)$. \square

Remark 6.8. Theorem 6.7 gives in fact a new proof for the unitarity of \mathcal{F} , since an orthogonal basis in \mathcal{M} is mapped to an orthogonal basis in $\tilde{\mathcal{M}}$ with the same norm. Note that Theorem 4.12 is not used to proof Theorem 6.7.

Theorem 6.7 can be considered as the $q = 1$ analogue of [21, Prop.4.1], which is used by Stokman to obtain an expansion of the Askey-Wilson function in Askey-Wilson polynomials, see [21, Thm.4.2]. So one might expect that from Theorem 6.7 a similar expansion can be proved for the Wilson functions. However, if we formally expand

$$\phi_\lambda(x; a, b, c, d) = \sum_{n=0}^{\infty} d_n(x, \lambda) R_n(x; a, b, c, 1-d) R_n(\lambda; \tilde{a}, \tilde{b}, \tilde{c}, 1-\tilde{d}),$$

and we calculate $d_n(x, \lambda)$ using Theorem 6.7 and the orthogonality relation for the Wilson polynomials, we find an expansion that in general does not converge, but it does converge if $x \in \mathcal{D}$, or $\lambda \in \tilde{\mathcal{D}}$. The convergence of the expansion formula for the Askey-Wilson functions is due to a Gaussian factor $q^{m(m+1)/2}$, $0 < q < 1$. In the limit $q \rightarrow 1$ this factor disappears.

From Theorem 6.7 we can also find the Wilson function transform of type II of a Wilson polynomial.

Theorem 6.9. *For $(a, b, c, d, t) \in V^+$, the Wilson function transform of type II of the Wilson polynomial $R_n(x) = R_n(x; a, b, c, 1-d)$ is given by*

$$(\mathcal{G}R_n)(\lambda) = (-1)^n \frac{(b+c)_n}{(1+a-d)_n} \frac{\sin \pi(\tilde{t} \pm i\lambda)}{\sqrt{\sin \pi(a+t) \sin \pi(b+t) \sin \pi(c+t) \sin \pi(1-d+t)}} \tilde{R}_n(\lambda).$$

Proof. From Theorem 6.7 we obtain

$$(\mathcal{G} \sin \pi(t \pm ix) R_n)(\lambda) = (-1)^n \frac{C(b+c)_n}{(1+a-d)_n} \tilde{R}_n(\lambda).$$

Taking the inverse of this gives

$$(\tilde{\mathcal{G}}\tilde{R}_n)(x) = (-1)^n \frac{(1+a-d)_n}{C(b+c)_n} \sin \pi(t \pm ix) R_n(x).$$

Replacing all parameters by their dual and interchanging x and λ then gives the statement in the theorem. \square

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