



Transformations Preserving the Hankel Transform

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Abstract

We classify all polynomial transformations of integer sequences which preserve the Hankel transform, thus generalizing examples due to Layman and Spivey & Steil. We also show that such transformations form a group under composition.

1 Introduction

Given a sequence of integers $\{a_i\} = a_0, a_1, a_2, \dots$, the Hankel matrix of A is the infinite matrix whose (i, j) entry is a_{i+j} for $i \geq 0, j \geq 0$. The Hankel matrix of order n of A is the $n \times n$ matrix consisting of the first n rows and n columns of the Hankel matrix of A , and the Hankel sequence determined by A is the sequence of determinants of the Hankel matrices of order n . For example, the Hankel matrix of order 3 is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}.$$

Let $R = \mathbb{Z}[a_0, a_1, a_2, \dots]$. We will say that a sequence $\{b_i\} = \{b_0, b_1, b_2, \dots\}$ of elements in R is an *HTP* sequence (Hankel Transform Preserving) if the Hankel transform of $\{b_i\}$ is formally equal to that of $\{a_i\}$; that is, if the following identity holds in R for all n :

$$\begin{vmatrix} b_0 & b_1 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_{n+1} \\ \vdots & & \ddots & \vdots \\ b_n & b_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

An *HTP* sequence $\{b_i\}$ determines a ring homomorphism $b : R \rightarrow R$ by the equation $b(a_i) = b_i$. We will sometimes refer to the sequence $\{b_i\}$ as just b . It is easy to see that the composition of two ring homomorphisms associated to *HTP* sequences is the ring homomorphism associated to an *HTP* sequence, so that the set of all *HTP* sequences has a semigroup structure (with $b_i = a_i$ representing the identity). We will see (Theorem 3.6) that this is actually a group structure.

Example 1.1. Let $\sigma : R \rightarrow R$ be given by $\sigma(a_i) = (-1)^i a_i$. Then σ determines an *HTP* sequence. Indeed, the order n Hankel matrix of the sequence $\{(-1)^i a_i\}$ is given by conjugating the order n Hankel matrix A_n of $\{a_i\}$ by the diagonal matrix D_n with (i, i) -entry given by $(-1)^i$. The determinant of $D_n^{-1} A_n D_n$ is the same as that of A_n .

Example 1.2. For an integer k , the sequence defined by $b_n = \sum_{i=0}^n \binom{n}{i} k^{n-i} a_i$ is an *HTP* sequence. (Here, if $k = 0$, then we interpret 0^0 to be 1.) Spivey and Steil [5] called this the falling k -binomial transform, and they proved that this preserves the Hankel Transform. When $k = 1$, this gives the binomial transform, which Layman [3] originally proved preserves the Hankel transform.

Remark 1.3. In fact, Spivey and Steil allow k to be any real number. If k were not an integer, then b_n is not in $R = \mathbb{Z}[a_0, a_1, a_2, \dots]$, so in our language, $\{b_n\}$ would not be an *HTP* sequence. Since we are interested in integer sequences, we have restricted to $\mathbb{Z}[a_0, a_1, a_2, \dots]$.

Definition 1.4. An *HTP* sequence b preserves a_0 through a_n if $b_i = a_i$ for $0 \leq i \leq n$.

Suppose that $b(m)$ is a sequence of *HTP* sequences such that for each n , there is a number $M(n)$ such that $b(m)$ preserves a_0 through a_n for all $m > M(n)$. Then the *infinite composition*

$$\dots \circ b(m) \circ b(m-1) \circ \dots \circ b(0)$$

is itself a well-defined sequence. Indeed, the n th term in the sequence of this infinite composite is determined by $b(M(n)) \circ b(M(n)-1) \circ \dots \circ b(0)$. It is easy to see that this infinite composition is itself an *HTP* sequence.

In order to find all *HTP* sequences, we first describe a special set of sequences, parametrized by R and the positive integers. In fact, for each positive integer n , and each $c \in R$, we will define an *HTP* sequence $b(n, c)$ which preserves a_0 through a_{2n} . Example 1.2 above arises when $n = 0$ and $c \in \mathbb{Z}$. Our main theorem is

Theorem 1.5. If b is a given *HTP* sequence, then there is a sequence c_0, c_1, c_2, \dots in R and an $\epsilon \in \{0, 1\}$ such that

$$b = \dots \circ b(n, c_n) \circ \dots \circ b(1, c_1) \circ b(0, c_0) \circ \sigma^\epsilon.$$

Our goal in Section 2 is to define the sequences $b(n, c_n)$, which we do in Definition 2.5. In Section 3 we prove that the set of all *HTP* sequences forms a group (Theorem 3.6), and we prove our classification Theorem 1.5.

2 A collection of HTP sequences

Definition 2.1. Let $T : R[x] \rightarrow R$ be the R -linear homomorphism defined by $T(x^k) = a_k$. For integers $i \geq j \geq 0$, define $T_{ij} : R[x] \rightarrow R$ to be the R -linear homomorphism given by

$$T_{ij}(x^k) = (-1)^{i+j} \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_i \\ a_1 & a_2 & a_3 & \cdots & a_{i+1} \\ \vdots & & & \ddots & \vdots \\ a_{j-1} & a_j & a_{j+1} & \cdots & a_{j-1+i} \\ a_{j+1} & a_{j+2} & a_{j+3} & \cdots & a_{j+1+i} \\ \vdots & & & \ddots & \vdots \\ a_i & a_{i+1} & a_{i+2} & \cdots & a_{2i} \\ a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+i} \end{vmatrix}.$$

Remark 2.2. If $0 \leq k \leq i$ and $k \neq j$, then $T_{ij}(x^k) = 0$, since two rows in the matrix defining $T_{ij}(x^k)$ are equal. Also, keeping track of signs, one sees that $T_{ij}(x^j) = T_{ii}(x^i)$. Finally, we note that $T_{00} = T$.

Definition 2.3. For each $c \in R$ and each integer $i \geq 1$, we define a sequence of polynomials $f_{m,i,c}(x) \in R[x]$ recursively by

$$f_{0,i,c} = 1, \quad f_{m+1,i,c} = f_{m,i,c} \cdot (x + cT_{i-1,i-1}(x^{i-1})) - c \left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{m,i,c}) \cdot x^j \right).$$

Also, we define $f_{m,0,c} \in R[x]$ by $f_{m,0,c} = (x+c)^m$, or (equivalently) recursively by

$$f_{0,0,c} = 1, \quad f_{m+1,0,c} = f_{m,0,c} \cdot (x+c).$$

We will show in Lemma 2.9 that for each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree m polynomial in x with leading coefficient 1.

Definition 2.4. Fixing a choice of i and c , define $U_{k,m}$ for each k, m by

$$f_{m,i,c}(x) = \sum U_{k,m} x^k.$$

Let U be the infinite matrix whose (k, m) entry is given by $U_{k,m}$.

Thus U is upper triangular, with diagonal entries all 1. Let A be the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We will show in Lemma 2.14 that the product $U^t A U$ is a Hankel matrix.

Definition 2.5. Let $b(i, c)$ denote the sequence whose Hankel matrix is the matrix U^tAU , where U is defined in terms of i and c as above.

We show in Corollary 2.15 that $b(i, c)$ is an HTP sequence preserving a_0 through a_{2i} .

Example 2.6. If $i = 1$ and $c = 1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,1,1} = f_{m,1,1} \cdot (x + a_0) - T(f_{m,1,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,1,1} = (x + a_0) - T(1) = x + a_0 - a_0 = x,$$

$$f_{2,1,1} = x(x + a_0) - T(x) = x^2 + a_0x - a_1,$$

$$f_{3,1,1} = (x^2 + a_0x - a_1)(x + a_0) - T(x^2 + a_0x - a_1) = x^3 + 2a_0x^2 + (a_0^2 - a_1)x - a_0a_1 - a_2.$$

Therefore, the upper left 4×4 submatrix of U is

$$\begin{pmatrix} 1 & 0 & -a_1 & -a_0a_1 - a_2 \\ 0 & 1 & a_0 & a_0^2 - a_1 \\ 0 & 0 & 1 & 2a_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left 4×4 submatrix of U^tAU to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 - a_1^2 + a_0a_2 \\ a_1 & a_2 & a_3 - a_1^2 + a_0a_2 & x \\ a_2 & a_3 - a_1^2 + a_0a_2 & x & y \\ a_3 - a_1^2 + a_0a_2 & x & y & z \end{pmatrix}$$

where

$$x = a_4 - a_0a_1^2 + 2a_0a_3 - 2a_1a_2 + a_0^2a_2$$

$$y = a_5 - 4a_0a_1a_2 + 3a_0a_4 - 2a_1a_3 - a_0^2a_1^2 + 3a_0a_3 - a_2^2 + a_0^3a_2 + a_1^3$$

$$z = a_6 + a_0^4a_2 + 3a_1^2a_2 - a_0^3a_1^2 + 4a_0^3a_3 - 2a_2a_3 - 2a_1a_4 - 6a_0^2a_1a_2$$

$$+ 6a_0^2a_4 + 2a_0a_1^3 - 3a_0a_2^2 - 6a_0a_1a_3 + 4a_0a_5$$

Thus, the first seven terms of $b(1, 1)$ are $a_0, a_1, a_2, a_3 - a_1^2 + a_0a_2, x, y, z$.

Example 2.7. If $i = 2$ and $c = 1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,2,1} = f_{m,2,1} \cdot (x + T_{1,1}(x)) - T_{1,1}(f_{m,2,1})x - T_{1,0}(f_{m,2,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,2,1} = (x + a_0a_2 - a_1^2) - 0x - (a_0a_2 - a_1^2) = x,$$

$$f_{2,2,1} = x(x + a_0a_2 - a_1^2) - (a_0a_2 - a_1^2)x + 0 = x^2,$$

$$\begin{aligned} f_{3,2,1} &= x^2(x + a_0a_2 - a_1^2) - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2 \\ &= x^3 + (a_0a_2 - a_1^2)x^2 - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2. \end{aligned}$$

Therefore, the upper left 4×4 submatrix of U is

$$\begin{pmatrix} 1 & 0 & 0 & a_1a_3 - a_2^2 \\ 0 & 1 & 0 & a_1a_2 - a_0a_3 \\ 0 & 0 & 1 & a_0a_2 - a_1^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left 4×4 submatrix of U^tAU to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & x \\ a_3 & a_4 & x & y \end{pmatrix}$$

where

$$\begin{aligned} x &= a_5 - a_2^3 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 + a_0a_2a_4, \\ y &= a_6 - 2a_1^3a_2a_3 - 2a_2^2a_3 + a_1^4a_4 - a_0^2a_2a_3^2 + a_0^2a_2^2a_4 + 2a_0a_1a_2^2a_3 + 2a_1a_2^3 + 2a_1a_2a_4 \\ &\quad + a_1^2a_2^3 + a_0a_1^2a_3^2 - 2a_0a_1^2a_2a_4 - 2a_1^2a_5 - a_0a_2^4 - 2a_0a_3a_4 + 2a_0a_2a_5. \end{aligned}$$

Thus the first seven terms of $b(2, 1)$ are $a_0, a_1, a_2, a_3, a_4, x, y$.

Lemma 2.8. *Suppose f and g are two polynomials in $R[x]$, and $i \geq 0$ is any integer. Then*

$$\sum_{j=0}^{i-1} T(f \cdot x^j) \cdot T_{i-1,j}(g) = \sum_{j=0}^{i-1} T(g \cdot x^j) \cdot T_{i-1,j}(f).$$

Proof. Since both sides of the equation are R -linear in both f and g , it suffices to consider the case when $f = x^m$ and $g = x^n$, so we need to show that

$$\sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n) = \sum_{j=0}^{i-1} T(x^{n+j}) \cdot T_{i-1,j}(x^m).$$

Consider the matrix below:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{i-1} & a_m \\ a_1 & a_2 & a_3 & \cdots & a_i & a_{m+1} \\ a_2 & a_3 & a_4 & \cdots & a_{i+1} & a_{m+2} \\ \vdots & & & \ddots & & \vdots \\ a_{i-1} & a_i & a_{i+1} & \cdots & a_{2i-2} & a_{m+i-1} \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+i-1} & 0 \end{pmatrix}$$

If we compute the determinant by expanding along the rightmost column, we get

$$\sum_{j=0}^{i-1} (-1)^{i+j} a_{m+j} \cdot T_{i-1,j}(x^n) = - \sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n).$$

If we compute the determinant by expanding along the bottom row, we get

$$\sum_{j=0}^{i-1} (-1)^{i+j} a_{n+j} \cdot T_{i-1,j}(x^m) = - \sum_{j=0}^{i-1} T(x^{n+j}) \cdot T_{i-1,j}(x^m).$$

□

Lemma 2.9. For each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree m polynomial in x with leading coefficient 1.

Proof. When $i = 0$ or $m = 0$, the claim is clear. We assume $i \geq 1$. Assume by induction on m that $f_{m,i,c}(x)$ has degree m and leading coefficient 1. It then suffices to show by the recursive definition of $f_{m+1,i,c}(x)$ that $T_{i-1,j}(f_{m,i,c}) = 0$ whenever $i > j > m$. This follows at once from Remark 2.2 and the inductive hypothesis. □

Lemma 2.10. $f_{m,i,c} = x^m$ whenever $m \leq i$.

Proof. Using the recursive definition, we see that it is enough to show that

$$x^m T_{i-1,i-1}(x^{i-1}) = \sum_{j=0}^{i-1} T_{i-1,j}(x^m) x^j$$

whenever $m \leq i - 1$. This follows from Remark 2.2, since all terms in the right hand sum are 0, except for the term when $j = m$, and this is $T_{i-1,m}(x^m) x^m = T_{i-1,i-1}(x^{i-1}) x^m$. □

Lemma 2.11. For $m, n \geq 0$,

$$T(f_{m+1,i,c} \cdot f_{n,i,c}) = T(f_{m,i,c} \cdot f_{n+1,i,c}).$$

Proof. First,

$$f_{m+1,0,c} \cdot f_{n,0,c} = (x+c)^{m+1}(x+c)^n = (x+c)^m(x+c)^{n+1} = f_{m,0,c} \cdot f_{n+1,0,c}.$$

We now assume $i \geq 1$. We then have

$$\begin{aligned} & f_{m+1,i,c} \cdot f_{n,i,c} - f_{m,i,c} \cdot f_{n+1,i,c} = \\ & \left(f_{m,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c \left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{m,i,c}) \cdot x^j \right) \right) \cdot f_{n,i,c} - \\ & f_{m,i,c} \cdot \left(f_{n,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c \left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{n,i,c}) \cdot x^j \right) \right) \\ & = c \left(f_{m,i,c} \left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{n,i,c}) \cdot x^j \right) - \left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{m,i,c}) \cdot x^j \right) f_{n,i,c} \right) \\ & = c \sum_{j=0}^{i-1} (f_{m,i,c} \cdot x^j \cdot T_{i-1,j}(f_{n,i,c}) - f_{n,i,c} \cdot x^j \cdot T_{i-1,j}(f_{m,i,c})). \end{aligned}$$

By Lemma 2.8, this term is in the kernel of T . □

Lemma 2.12. $T(f_{m,i,c}(x)) = a_m$ whenever $0 \leq m \leq 2i$.

Proof. Suppose $m = 2i$. Since $f_{0,i,c} = 1$, and by applying Lemma 2.11 i times,

$$T(f_{2i,i,c}) = T(f_{2i,i,c} \cdot f_{0,i,c}) = T(f_{i,i,c} \cdot f_{i,i,c}).$$

By Lemma 2.10, $f_{i,i,c} = x^i$, so $(f_{i,i,c})^2 = x^{2i}$, and $T(x^{2i}) = a_{2i}$. The other cases follow similarly. \square

Lemma 2.13.

$$T(f_{2i+1,i,c}(x)) = a_{2i+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

Proof. If we expand the determinant on the right hand side of the equation along the last column, we get

$$\begin{aligned} & a_{2i+1} + ca_{2i}T_{i-1,i-1}(x^{i-1}) - c \sum_{j=0}^{i-1} T_{i-1,j}(x^i)a_{i+j} \\ &= T \left(x^{2i+1} + cx^{2i}T_{i-1,i-1}(x^{i-1}) - c \sum_{j=0}^{i-1} T_{i-1,j}(x^i)x^{i+j} \right) \\ &= T \left(x^i \left(x^i (x + cT_{i-1,i-1}(x^{i-1})) - c \sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c})x^j \right) \right) \\ &= T \left(x^i \left(f_{i,i,c} \cdot (x + cT_{i-1,i-1}(x^{i-1})) - c \sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c}) \cdot x^j \right) \right) \\ &= T(x^i \cdot f_{i+1,i,c}) = T(f_{i,i,c} \cdot f_{i+1,i,c}) = T(f_{2i+1,i,c}). \end{aligned}$$

Here, we have used Lemma 2.10 to identify x^i with $f_{i,i,c}$ and Lemma 2.11 for the last equality. \square

Now recall that A is the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Lemma 2.14. *The product $U^t A U$ is a Hankel matrix.*

Proof. We have $f_{r,i,c} \cdot f_{u,i,c} = \sum_{s,t} U_{s,r} U_{t,u} x^{s+t}$, so that

$$T(f_{r,i,c} \cdot f_{u,i,c}) = \sum_{s,t} U_{s,r} U_{t,u} a_{s+t} = \sum_{s,t} U_{s,r} a_{s+t} U_{t,u}.$$

This is precisely the (r, u) entry of $U^t A U$. Now the result follows from Lemma 2.11. \square

Recall from Definition 2.5 that $b(i, c)$ denotes the sequence whose Hankel matrix is U^tAU .

Corollary 2.15. *We have $b(i, c)_n = T(f_{n,i,c})$, and $b(i, c)$ is an HTP sequence preserving a_0 through a_{2i} . Moreover*

$$b(i, c)_{2i+1} = a_{2i+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

Proof. Since U^t and U are each triangular with 1s on the diagonal, it follows that the Hankel matrices of finite order associated to U^tAU have the same determinants as those of A . Thus the sequence of entries on the top row of U^tAU represents a transformation of A which preserves the Hankel transform. This is the same as the sequence of entries in the top row of AU since U^t preserves the top row. But it is easy to see that this sequence is given by $T(f_{n,i,c})$. The first $2i$ terms of the sequence are given by Lemma 2.12, while the $2i + 1$ term is given by Lemma 2.13. \square

3 Classifying all HTP sequences

Lemma 3.1. *For each integer $n \geq 0$, the determinant*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

is irreducible in R .

Proof. The statement is obvious if $n = 0$, so suppose $n \geq 1$. We make R into a graded ring by assigning $\deg(a_i) = 2n + 1 - i$. Then the determinant is a homogeneous polynomial of degree $(n + 1)^2$. As a polynomial in a_{2n} with coefficients in $\mathbb{Z}[a_0, a_1, \dots, a_{2n-1}]$, the determinant can be written

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} \cdot a_{2n} + \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

The coefficient of a_{2n} has degree $(n + 1)^2 - 1$, while the constant coefficient has degree $(n + 1)^2$. But no element in $\mathbb{Z}[a_0, a_1, \dots, a_{2n-1}]$ has degree 1. Therefore, the coefficient of a_{2n} does not divide the constant coefficient. By induction, the coefficient of a_{2n} is irreducible. The result follows from this. \square

Lemma 3.2. *Suppose that b is an HTP sequence which preserves a_0 through a_{2n-1} . Then $b_{2n} = a_{2n}$.*

Proof. In order to have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix},$$

we must have

$$(b_{2n} - a_{2n}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} = 0.$$

Thus, $b_{2n} - a_{2n}$ must be 0. □

Lemma 3.3. *Suppose b is an HTP sequence. Then either $b_1 - a_1$ or $b_1 + a_1$ is divisible by a_0 in R .*

Proof. Clearly, $b_0 = a_0$, since the 0th terms in the Hankel transforms must coincide. Now, in order to have

$$\begin{vmatrix} a_0 & b_1 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

we must have $b_2 a_0 - b_1^2 = a_2 a_0 - a_1^2$, so $a_0(b_2 - a_2) = b_1^2 - a_1^2 = (b_1 - a_1)(b_1 + a_1)$. Therefore, a_0 divides either $b_1 - a_1$ or $b_1 + a_1$. □

Lemma 3.4. *Fix an integer $n \geq 1$. Suppose b is an HTP sequence preserving a_0 through a_{2n} . Then $b_{2n+1} - a_{2n+1}$ is divisible by*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Proof. We have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix} \\ = b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & 0 \end{vmatrix}$$

The determinant on the right above can be written

$$\begin{aligned}
& \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix} \\
& + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.
\end{aligned}$$

(In each of the four above matrices, all rows except the last two are the same.) Now, we can write this as

$$\begin{aligned}
& = -b_{2n+1}^2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} - b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix} \\
& - b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.
\end{aligned}$$

The second and third terms in the above sum are equal, since the determinant of a matrix is equal to the determinant of its transpose.

Now suppose

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & a_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n+1} & a_{2n+2} \end{vmatrix}.$$

Expanding these determinants as above and setting them equal, we get

$$\begin{aligned}
b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - b_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix} = \\
a_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - a_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2a_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.
\end{aligned}$$

Thus, we must have

$$\begin{aligned}
& (b_{2n+2} - a_{2n+2}) \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} \\
= & (b_{2n+1}^2 - a_{2n+1}^2) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2(b_{2n+1} - a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.
\end{aligned}$$

By Lemma 3.1, either the claim is true or

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

must divide

$$(b_{2n+1} + a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

But in the quotient by the ideal I generated by $a_0, a_1, \dots, a_{n-1}, a_{n+2}, a_{n+3}, \dots, a_{2n+1}$, the first determinant becomes a_n^{n+1} (up to a sign), while the sum above becomes $2a_n^n * a_{n+1}$ (again up to a sign). Since a_n does not divide a_{n+1} in R/I , the claim must be true. \square

We now turn to showing that the set of *HTP* sequences forms a group under composition. For this, we first need the following Lemma.

Lemma 3.5. *Suppose that b and b' are two HTP sequences which each preserve a_0 through a_{2n} , $n \geq 1$. Also, suppose that $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$. Then $b \circ b'$ preserves a_0 through a_{2n+2} .*

Proof. Let $d = b \circ b'$. Clearly d preserves a_0 through a_{2n} . By Lemma 3.4, there is a c such that

$$b_{2n+1} = a_{2n+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$, we must have

$$b'_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since both b and b' preserve a_0 through a_{2n} , it follows that $d_{2n+1} = a_{2n+1}$. Now $d_{2n+2} = a_{2n+2}$ by Lemma 3.2. \square

Theorem 3.6. *The set of HTP sequences forms a group under composition.*

Proof. It suffices to show that any HTP sequence b has a left inverse. By Lemma 3.3, either $b_1 - a_1$ is divisible by a_0 or $b_1 + a_1$ is divisible by a_0 . We will assume first that $b_1 - a_1$ is divisible by a_0 . Now let n be the smallest number such that $b_{2n+1} \neq a_{2n+1}$ (so $n \geq 0$, since $b_0 = a_0$.) By Lemma 3.2, we know that $b_{2n} = a_{2n}$. Choose c_n as in Lemma 3.4 (or 3.3) so that

$$b_{2n+1} = a_{2n+1} - c_n \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Then by Lemma 3.5 and Corollary 2.15, the composition $b(n, c_n) \circ b$ preserves a_0 through a_{2n+2} .

Now, we inductively define a sequence c_0, c_1, c_2, \dots so that the composition

$$b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b$$

preserves a_0 through a_{2n+2} . Thus

$$(\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \text{id}.$$

We assumed above that $b_1 - a_1$ is divisible by a_0 . If $b_1 + a_1$ is divisible by a_0 , then we can reduce to the former case by replacing b by $b \circ \sigma$ (see Example 1.1). Then, as above, we can find a sequence c_0, c_1, c_2, \dots so that

$$\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b \circ \sigma = \text{id}.$$

Composing and precomposing both sides with σ (which satisfies $\sigma^2 = \text{id}$), we get

$$\sigma \circ (\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \text{id}.$$

\square

Lemma 3.7. *The inverse $b(n, c)^{-1}$ of $b(n, c)$ preserves a_0 through a_{2n} and satisfies*

$$b(n, c)^{-1}_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Proof. Since $b(n, c)$ preserves a_0 through a_{2n} by Corollary 2.15, it is clear that $b(n, c)^{-1}$ also preserves a_0 through a_{2n} . For the second part, $b(n, -c) = b(n, -c) \circ b(n, c) \circ b(n, c)^{-1}$. Since $b(n, -c) \circ b(n, c)$ preserves a_0 through a_{2n+2} by Lemma 3.5 and Corollary 2.15, it follows that $b(n, c)^{-1}$ must have the same $2n + 1$ term as $b(n, -c)$. \square

Remark 3.8. We do not know whether or not $b(n, c)^{-1} = b(n, -c)$ in general.

We now prove our main theorem.

Proof. As in Theorem 3.6, we can inductively define c_0, c_1, \dots such that either

$$b' := b \circ b(0, c_0)^{-1} \circ b(1, c_1)^{-1} \circ \dots \circ b(n, c_n)^{-1}$$

or

$$b' := b \circ \sigma \circ b(0, c_0)^{-1} \circ b(1, c_1)^{-1} \circ \dots \circ b(n, c_n)^{-1}$$

preserves a_0 through a_{2n+2} . Here, we use Lemma 3.7 together with Lemma 3.5 to complete the inductive step. Now either b or $b \circ \sigma$ can be written as

$$b' \circ b(n, c_n) \circ \dots \circ b(1, c_1) \circ b(0, c_0).$$

It follows that

$$\dots \circ b(n, c_n) \circ \dots \circ b(1, c_1) \circ b(0, c_0)$$

agrees on all terms with b or $b \circ \sigma$. In the latter case, we multiply both sides by σ . \square

References

- [1] M. Chamberland and C. French, [Generalized Catalan numbers and generalized Hankel transformations](#), *J. Integer Seq.* **10** (2007), Article 07.1.1.
- [2] R. Ehrenborg, The Hankel determinant of exponential polynomials, *Amer. Math. Monthly*, **107** (2000), 557–560.
- [3] J. Layman, [The Hankel transform and some of its properties](#), *J. Integer Seq.* **4** (2001), Article 01.1.5.
- [4] C. Radoux, Déterminant de Hankel construit sur des polynômes liés aux nombres de dérangements. *European J. Combin.* **12** (1991), 327–329.
- [5] M. Spivey and L. Steil, [The \$k\$ -binomial transforms and the Hankel transform](#), *J. Integer Seq.* **9** (2006), Article 06.1.1.
- [6] E. Weisstein, Binomial transform, from MathWorld – A Wolfram Web Resource, <http://mathworld.wolfram.com/BinomialTransform.html>.

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