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PASCAL TRIANGLES, CATALAN NUMBERS AND RENEWAL ARRAYS

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In response to some recent questions of L.W. Shapiro, we develop a theory of triangular arrays, called renewal arrays, which have arithmetic properties similar to those of Pascal's triangle. The Lagrange inversion formula has an important place in this theory and there is a close relation between it and the theory of renewal sequences. By way of illustration, we give several examples of renewal arrays of combinatorial interest, including complete generalizations of the familiar Pascal triangle and sequence of Catalan numbers.

1. Introduction

The binomial coefficients are as fundamental in combinatorial theory as they are ubiquitous, but Pascal's triangle is by no means the only such array of numbers which finds a place in that theory. Indeed, Shapiro has recently introduced another triangle of numbers $B_{n,k}$ defined recursively by $B_{1,1}=1$ and

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad n \geq 2, \tag{1a}$$

where

$$B_{n,k} = 0, \quad k<1 \text{ or } k>n;\ n \geq 1, \tag{1b}$$

and having many arithmetical properties in common with Pascal's triangle. Since

$$B_{n,1} = C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 1, \tag{2}$$

where C_n is the nth Catalan number [18, sequence 577]. Shapiro called this new triangle a Catalan triangle [17].

The numbers $B_{n,k}$ arise in a walk problem on the non-negative quadrant of the integral square lattice in two dimensional Euclidean space: they are the number of pairs of non-intersecting outward directed n-step paths issuing from the origin, the first coordinates of whose other ends differ by k. They have further interpretations in problems on random walks, dissections of polygons and relations on ordered sets.

Shapiro noticed, among other things, that

$$\sum_{n \geq k} B_{n,k} x^{n-k} = \left(\sum_{n \geq 1} B_{n,1} x^{n-1}\right)^k,$$

that is, the Catalan triangle is generated by its first column and accordingly asked: is there an arithmetic of arrays with this property?

We offer here some answers to this question in the form of the theory of renewal arrays which is developed in Section 2 and which is closely related to that of renewal sequences (Section 3). A number of examples which illustrate this theory are given (Sections 4, 5 and 6), beginning, in Section 4, with a family of arrays which generalizes the Pascal and Catalan triangles and which is associated with the higher Catalan sequences $\{c_t(n)\}$, $n \geq 0$, given by

$$c_t(n) = \frac{1}{tn+1}\binom{(t+1)n}{n}, \quad n \geq 0, \quad t \geq 0 \tag{3}$$

(so $C_n = c_1(n)$). These sequences also occur in a family of arrays arising from the enumeration of walks on the integral square lattice when they are restricted in various ways (Section 5). The sequences $\{c_t(n)\}$, $n \geq 0$, like the sequence $\{C_n\}$, $n \geq 0$, occur in a wide variety of combinatorial problems (for the C_n see, for example [5, 6, 8, 21] while for the $c_t(n)$ see, for example [24, pp. 13–14, 26–27, 168–194]). Other related sequences arise by allowing diagonal steps of various gradients in the walk. Further, more miscellaneous examples of the general theory are given in Section 6.

2. Renewal arrays

The convolution $f*g$ of two sequences $f=\{f_n\}$, $g=\{g_n\}$, $n \geq 0$, is defined by $h=f*g=\{h_n\}$, $n \geq 0$, where

$$h_n = \sum_{r=0}^{n} f_r g_{n-r}, \quad n \geq 0,$$

or, in terms of generating functions,

$$\sum_{n \geq 0} h_n x^n = \Big(\sum_{n \geq 0} f_n x^n\Big)\Big(\sum_{n \geq 0} g_n x^n\Big).$$

The r-fold convolution $f^{(r)} = \{f_n^{(r)}\}$, $n \geq 0$, $r \geq 0$, of f with itself may then be defined recursively by

$$f^{(r)} = f^{(r-1)} * f = f * f^{(r-1)}, \quad r \geq 1,$$

where $f_1^{(0)} = 1$ and $f_n^{(0)} = 0$, $n > 1$ (so $f^{(1)} = f$).

The renewal array $\{b_{n,m}\}$, $0 \leq m \leq n$, generated by the sequence $b=\{b_n\}$, $n \geq 0$, is then the triangular array with

$$b_{n,m} = b_n^{(m+1)}, \quad 0 \leq m \leq n, \tag{4a}$$

$$b_{n,m} = 0, \quad m < 0 \text{ or } m > n, \tag{4b}$$

so that, on introducing the generating functions

$$B^{(m)}(x) = \sum_{n\geq 0} b_{n,m} x^{n-m}, \qquad B(x) = \sum_{n\geq 0} b_n x^n$$

we have

$$B^{(m)}(x) = (B(x))^{m+1}, \quad m \geq 1. \tag{5}$$

Given a renewal array $\{b_{r,m}\}$, we may obtain recursively a sequence $a = \{a_n\}$, $n \geq 0$, such that

$$B(x) = \sum_{r\geq 0} a_r (xB(x))^r, \tag{6}$$

which is important in the analysis of the array and which we refer to as the A-sequence associated with the array.

Conversely, given any sequence $a = \{a_n\}$, $n \geq 0$, we may define a triangular array $\{b_{n,m}\}$ recursively by (4b) and

$$b_{n,m} = \sum_{r\geq 0} a_r b_{n-1,m-1+r}. \tag{7}$$

Provided that $b_{0,0} = a_0$, it follows inductively that $\{b_{n,m}\}$ is the renewal array generated by the sequence $b = \{b_{n,0}\}$, $n \geq 0$, so that (5) holds and then, from (7), (6) holds as well and $a = \{a_r\}$, $n \geq 0$, is the A-sequence of the array. So, with a slight notational change, Shapiro's Catalan triangle is the case where $a_0 = 1 = a_2$, $a_1 = 2$ and $a_n = 0$, $n > 2$; and the theory of renewal arrays provides a generalization of his observations in this case.

From the way in which the definitions are framed, the correspondence between the sequence $\{b_n\}$ generating the array and the A-sequence $\{a_n\}$ of the array is biunique and, moreover, the A-sequence $\{ca_n\}$ then corresponds to the generating sequence $\{c^n b_n\}$ so that, without loss, we may take $b_0 = a_0 = 1$. What other properties does this correspondence have?

We may obtain from (6), in conjunction with Lagrange's inversion formula, an expression for the $b_{n,m}$ in terms of the a_r. Lagrange's inversion formula [23, pp. 132–133] states that if

$$y = a + xf(y), \quad y(0) = a,$$

then

$$g(y) = \sum_{n\geq 1} \frac{x}{n!} \frac{d^{n-1}}{dt^{n-1}} [g'(t)(f(t))^n]_{t=a}.$$

We may apply this either directly to (6) or, more easily, to the equivalent form

$$\tilde{B}(x) = x \sum_{r\geq 0} a_r (\tilde{B}(x))^r = xA(x\tilde{B}(x)), \tag{8}$$

where

$$\tilde{B}(x) = xB(x), \qquad A(x) = \sum_{n\geq 0} a_n x^n.$$

In the latter case, we have for $m \geq 1$,

$$(\tilde{B}(x))^m = \sum_{n \geq 1} \frac{x^n}{n!} \frac{d^{n-1}}{dt^{n-1}} [mt^{m-1}(A(t))^n]_{t=0}$$

leading to

$$b_{n,m} = \frac{m+1}{n+1} a_{n-m}^{(n+1)}, \quad 0 \leq m \leq n. \tag{9}$$

This last expression suggests a general construction of renewal arrays familiar from the theory of dams, queues and branching processes (see [22] and the references given there). Given a sequence $\{a_n\}$, $n \geq 0$, consider the renewal array $\{b_{n,m}\}$ generated by the sequence $b = \{b_n\}$, $n \geq 0$, with

$$b_n = \frac{1}{n+1} a_n^{(n+1)},$$

then

$$b_{n,m} = b_n^{(m+1)} = \frac{m+1}{n+1} a_{n-m}^{(n+1)}$$

and $\{a_n\}$, $n \geq 0$, is the A-sequence of the array. If, for example, $\{a_n\}$ is the probability distribution of the input of a discrete dam, whose content has distribution $g = \{g_n\}$, $n \geq 0$, where $g_m = 1$ and $g_n = 0$, $n \neq m$ and whose regime is that of unit release in unit time, then $b^{(m)}$ is the distribution of the time to first emptiness.

There are two other sequences of some combinatorial interest associated with the renewal array $\{b_{n,m}\}$. Firstly, the sequence $\{u_n\}$ of row sums given by

$$u_0 = 1; \qquad u_{n+1} = \sum_{m=0}^{n} b_{n,m}, \quad n \geq 0,$$

for which

$$\begin{aligned} U(x) = \sum_{n \geq 0} u_n x^n &= 1 + \sum_{r \geq 1} (xB(x))^r \\ &= 1 + xU(x)B(x) \end{aligned} \tag{10}$$

and, secondly, the so called Fibonacci sequence $\{b_n^*\}$ associated with the array given by

$$x^2 B^*(x) = x^2 \sum_{n \geq 0} b_n^* x^n = \sum_{r \geq 1} (x^2 B(x))^r$$

or equivalently

$$B^*(x) = B(x) + x^2 B^*(x) B(x). \tag{11}$$

Both sequences occur in the enumeration of objects which may be broken up into disjoint subobjects according to the first occurrence of some property the objects may or may not have. Two examples of the latter type of sequence are given in [17] (see also [13]). The row sums are familiar, from probabilistic contexts, in the form of renewal sequences, to which we now turn.

3. Renewal sequences

A sequence $\{u_n\}$, $n \geq 0$, for which

$$0 \leq u_n \leq u_0 = 1 \tag{12}$$

is a renewal sequence if for some non-negative sequence $\{f_n\}$, $n \geq 1$, we have

$$u_n = \sum_{r=1}^{n} f_r u_{n-r} \tag{13}$$

or

$$U(x) = \sum_{n\geq 0} u_n x^n = 1 + U(x)F(x), \qquad F(x) = \sum_{n\geq 1} f_n x^n. \tag{14}$$

and it then follows that

$$\sum_{n\geq 1} f_n \leq 1. \tag{15}$$

Conversely, if u_n is defined by (13), with $u_0 = 1$, for some non-negative sequence $\{f_n\}$, $n \geq 1$, satisfying (15), then $\{u_n\}$, $n \geq 0$, is a renewal sequence.

The equivalence of these two formulations arises from the representation of renewal sequences in terms of sequences of n-step transition probabilities in Markov chains [9, p.5]. For a Markov chain $X = \{X_n\}$, $n \geq 0$, and s a state of the chain we write:

$$u_n = p_{s,s}^{(n)} = \text{Prob}(X_n = s | X_0 = s), \qquad n \geq 0,$$
$$f_n = \text{Prob}(X_n = s, X_i = s, 0 < i < n | X_0 = s), \quad n \geq 1.$$

Then (13) holds; and conversely any renewal sequence arises in this way.

Comparing (12) and (13), it is apparent that $\{u_n\}$, $n \geq 0$, is the sequence of row sums of the renewal array generated by the sequence $\{f_{n+1}\}$, $n \geq 0$. Indeed we may regard the theory of renewal arrays as arising from that of renewal sequences by lifting the purely probabilistic requirement $u_n \leq 1$, or equivalently $\sum_{n\geq 1} f_n \leq 1$, the conditions $u_n \geq 0$ and $f_n \geq 0$ being more combinatorial in character. The sequence $\{u_n\}$, $n \geq 0$, without these probabilistic requirements is said to satisfy the renewal relation or decomposition (13) (the term "generalized renewal sequence" is also used); and many of the properties of renewal sequences, notably limit results [9, chapter 1], carry over to sequences satisfying a renewal relation. (See [16] for an application of this extension.)

The renewal array may itself be analysed further in terms of renewal relations since, if $a_0 = 1$ in (6), as in the examples below, we have the decompositions

$$B(x) = 1 + x\hat{B}_1(x)B(x),$$
$$\hat{B}(x) = \sum_{r\geq 1} a_r(xB(x))^{r-1}, \tag{16}$$
$$B^*(x) = 1 + x(x + \hat{B}(x))B^*(x).$$

4. Higher Catalan triangles

The non-zero coefficients a_n in the case of Shapiro's Catalan triangle form the second row below its apex of the Pascal triangle. This observation admits a ready generalization. Thus the ordinary Pascal triangle and the Catalan triangle are the first two members ($t=0$, 1) of a family of triangles $\{B_t(n,m)\}$ for which the A-sequence is given by

$$a_n = \begin{cases} \binom{t+1}{n}, & 0 \leq n \leq t+1, \\ 0, & n > t+1, \end{cases}$$

so (compare (7))

$$B_t(n, m) = \sum_{r=0}^{t+1} \binom{t+1}{n} B_t(n-1, m-1+r). \tag{17}$$

It follows, from (8), that

$$\tilde{B}_t(x) = x\sum_{n\geq 0} B_t(n, 0)x^n = x\sum_{r=0}^{t+1} \binom{t+1}{r}\tilde{B}_t(x)^r$$
$$= x(1 + \tilde{B}_t(x))^{t+1} \tag{18}$$

from which it follows in turn that (compare (9) and [17, 2.1])

$$B_{t-1}(n,m) = \frac{m+1}{n+1}\binom{t(n+1)}{n-m}, \quad 0 \leq m \leq n, t \geq 1$$

and, in particular, that

$$B_t(n, 0) = c_t(n+1), \quad n \geq 0, t \geq 0.$$

5. Walks on the integral square lattice

Now, starting from the recurrence relation generating the ordinary Pascal triangle which we write for ease of interpretation in the form

$$w(n, m) = w(n-1, m) + w(n, m-1), \quad n, m \geq 0, (n, m) \neq (0, 0), \tag{19a}$$

we obtain, on iteration,

$$w(n, m) = \sum_{s=0}^{t+1} \binom{t+1}{s} w(n-s, m+s-t-1)$$

which, resembling (19), suggests a link between the two. Taking $w(0,0)=1$ and

$$w(n, m) = 0, \quad n<0 \quad \text{or} \quad m<0, \tag{19b}$$

so

$$w(n, m) = \binom{n+m}{m}, \quad n, m \geq 0, \tag{19c}$$

then $w(n, m)$ is just the number of outward directed walks on the non-negative quadrant of the integral square lattice from the origin to the point (n, m), (19a) reflecting the edge structure of the lattice.

Similarly if for t, a non-negative integer, $w_t(n, m)$ is given by $w_t(0, 0) = 1$ and

$$w_t(n, m) = w_t(n-1, m) + w_t(n, m-1), \quad n, m \geq 0, (n, m) \neq (0, 0),$$
$$w_t(n, m) = 0, \quad m<0 \text{ or } n<tm,$$

then $w_0(n, m)$ and for $t \geq 1$, $w_t(n, m)$ is the number of outward directed walks on the non-negative quadrant of the integral square lattice from the origen to the point (n, m) (19a) reflecting the edge structure of the lattice.

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$$w_t(n, m) = 0, \quad m<0 \text{ or } n<tm,$$

then $w_0(n, m)$ and for $t \geq 1$, $w_t(n, m)$ is the number of outward directed walks on the non-negative quadrant of the integral square lattice from the origen to the point (n, m) which remain on or below the line $ty = x$. This interpretation allows us to deduce several results. For example, considering the last time, if ever, such a walk from the origen to the point $(tn + m, n)$ visits the line $ty = x$, we find

$$w_t(tn+m, n) = \sum_{r=0}^{n+m-1} w_t(tr, r) w_t(t(n-r)+m-1, n-r), \quad n \geq 0, m>0, \tag{20a}$$

or

$$w_t(x; m) = \sum_{n \geq 0} w_t(tn+m, n) x^t = W_t(x) W_t(x; m-1), \quad m>1.$$

where

$$W_t(x) = W_t(x; 0),$$

so

$$W_t(x; m) = (W_t(x))^{m+1}, \quad m \geq 0. \tag{20b}$$

Again, considering the first time, if ever, such an outward directed walk from the origen to (tn, n) visits the line $ty = x$ we find (compare (16))

$$W_t(x) = 1 + x\hat{W}_t(x)W_t(x), \qquad \hat{W}_t(x) = (W_t(x))^t,$$

so (compare (18))

$$W_t(x) = 1 + x(W_t(x))^{t+1}, \quad t \geq 0, \tag{21}$$

and then, inductively (compare (6, 8))

$$W_t(x) = W_{t-1}(xW_t(x)), \quad t \geq 1. \tag{22}$$

Applying Lagrange's inversion formula to (21) leads to

$$w_t(tn, n) = c_t(n), \tag{23}$$

which also follows from (19c) together with

$$w_t(n, m) = w(n, m) - tw(n+1, m-1), \quad 0 \leq tm \leq n.$$

Writing

$$b_t(n+k, m) = w_t(tn+m, n)$$

it follows from (20), (23) that $\{b_t(n, m)\}$, $0 \leq m \leq n$, $t \geq 0$, is the renewal array generated by the sequence $\{c_t(n)\}$, $n \geq 0$. Moreover, from (22), for $t \geq 1$, the A-sequence of the array $\{b_t(n, m)\}$ is $\{c_{t-1}(n)\}$, $n \geq 0$. The link between the two arrays $\{B_t(n, m)\}$ and $\{b_t(n, m)\}$ is provided by (compare (18) and (21))

$$\tilde{B}_t(x) = xB_t(x) = x(W_t(x))^{t+1} = W_t(x) - 1.$$

Further, writing

$$A_n(r, s+1) = w_s(sn + r - 1, n)$$

we obtain, from (20a), the "Vandermonde" convolution identity in the form

$$A_n(a+c, b) = \sum_{m=0}^{n} A_m(a, b)\, A_n(c, b),$$

which has been studied by Gould [7], among others (See [19] for further interpretations and references.)

Results, similar to those above for the integral square lattice, hold for other lattices. For example, if for some fixed integral t, k with $t \geq 1$, $k \geq 0$ and all integral n, m we take the lattice points to be $[n/(k+1), m/(k+1)]$ and, in addition to unit horizontal and vertical steps, allow diagonal steps from the lattice point $[(n-t)/(k+1), (m-1)/(k+1)]$ to $[n/(k+1), m/(k+1)]$, we obtain [16] a family of sequences closely related to those of (3). In particular, if $t = k \geq 1$, the associated renewal arrays are generated by the higher Motzkin sequences $\{m_t(n)\}$, $n \geq 0$, [18, sequence 456; 3, 20], given by

$$m_t(n) = \sum_{i=0}^{[n/(t+1)]} \binom{n}{(t+1)i} c_t(i), \quad n \geqslant 0, t \geqslant 1,$$

while if $t \geqslant 1$, $k = 0$, the arrays are generated by the higher Schröder $\{r_t(u)\}$, $n \geqslant 0$, [18, sequence 1163 (also 1170, where there is a misprint); 15] given by

$$r_t(n) = \sum_{i=0}^{n} \binom{n + t(n-i)}{i} c_t(n-i), \quad n \geqslant 0, t \geqslant 1.$$

These sequences provide generalizations of the more familiar Morzkin and Schröder sequences $\{m_1(n)\}$ and $\{r_1(n)\}$, $n \geqslant 0$, respectively in the same way as the sequences $\{c_t(n)\}$, $n \geqslant 0$, generalize the Catalan sequence $\{C_n\}$, $n \geqslant 0$.

6. Further examples

A further family of arrays $\{t_k(n, m)\}$, $0 \leqslant m \leqslant n$, $k \geqslant 1$, generated by the sequences $\{t_k(n)\}$, $n \geqslant 0$, satisfying (compare (8))

$$T_k(x) = x \sum_{n \geqslant 0} t_k(n) x^n = x \sum_{r=0}^{k} (T_k(x))^r \tag{24}$$

appears in [10] with the interpretation that $t_k(n)$ is the number of planted planar trees with n edges all of whose vertices, except the root, having valence at most $(k+1)$, the root having valence at most k. From (24), the A-sequence of the array $\{t_k(n, m)\}$ is

$$a_n = \begin{cases} 1, & 0 \leqslant n \leqslant k, \\ 0, & n > k, \end{cases}$$

so that the array $\{b_1(n, m)\}$ may be regarded as the limiting case where k is infinite. The case $k = 1$ is again the ordinary Pascal triangle. The case $k = 2$, which also appears in [1], is the Motzkin triangle, [3] generated by the sequence $\{m_1(n)\}$, $n \geqslant 0$. A combinatorial analysis leading to the sequence $\{b_n^*\}$, $n \geqslant 0$, of (11) for this array appears in [14].

Another source of examples is the theory of partitions and Pascal's triangle may itself be seen in this context. As a further example, if $p(n, m)$ is the number of ways of partitioning the non-negative integer n into non-negative integers of which $m \geqslant 0$ are 1's, then $\{p(n, m)\}$, $0 \leqslant m \leqslant n$, is the renewal array generated by the usual Fibonacci sequence $\{x_{n-2}\}$, $n \geqslant 0$, given recursively by

$$x_n = x_{n-1} + x_{n-2}, \quad n \geqslant 0$$

starting with $x_{-1} = 0$ and $x_{-2} = 1$.

L. Carlitz has pointed out (private communication) that both kinds of Stirling numbers, as well as the associated Stirling numbers of both kinds (for definitions of these see [11]) and other numbers defined in terms of them (see [2] and the references given there) may be arranged, after some notational changes into renewal arrays. So, for example, the generating sequences $\{b_n\}$ for the renewal arrays associated with the first and second kind of Stirling numbers have generating functions $B(x) = -\log_e(1-x)$ and $B(x) = e^x - 1$ respectively.

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