A DISJOINT SYSTEM OF LINEAR RECURRING SEQUENCES GENERATED BY $u_{n+2} = u_{n+1} + u_n$ WHICH CONTAINS EVERY NATURAL NUMBER

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Burke and Bergum [1] called a (finite or infinite) family of n^{th}-order linear recurring sequences a (finite or infinite) *regular covering* if every natural number is contained in at least one of these sequences. If every natural number is contained in exactly one of these sequences, they called the family a (finite or infinite) *disjoint covering.* They gave examples of finite and infinite disjoint coverings generated by linear recurrences of every order n. In the case of the Fibonacci recurrence $u_{n+2} = u_{n+1} + u_n$, they constructed a regular covering which is not disjoint and asked whether a disjoint covering in this case exists as well. The following theorem answers this question.

Theorem: There is an infinite disjoint covering generated by the linear recurrence $u_{n+2} = u_{n+1} + u_n$.

We first state some easy properties of the Fibonacci numbers, $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, \ldots$. Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. We have

(1) $$\alpha < 1, \quad -1 < \beta < 0$$

and

(2) $$\alpha|\beta| = 1.$$

For the Fibonacci numbers, the Binet formula

(3) $$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad (n \in \mathbf{N})$$

holds.

For all $i \in \mathbf{N}$, let $u_{i,1}, u_{i,2} \in \mathbf{N}$ and the sequences $(u_{i,n})_{n\in\mathbf{N}}$ be defined by

(4) $$u_{i,n+2} = u_{i,n+1} + u_{i,n}.$$

Then we have

(5) $$u_{i,n} = F_{n-1}u_{i,2} + F_{n-2}u_{i,1}$$

and

(6) $$u_{i,n+1} = \alpha u_{i,n} + \beta^{n-2}(\beta u_{i,2} + u_{i,1})$$

for all $i, n \in \mathbf{N}, n \geq 2$.

Proof of the Theorem: We will construct sequences $(u_{i,n})_{n\in\mathbf{N}}$ of natural numbers for all $i \in \mathbf{N}$ generated by (4).

Start with $(u_{i,n})_{n\in\mathbf{N}} = (F_{n+1})_{n\in\mathbf{N}}$ and assume that $(u_{i,n})_{n\in\mathbf{N}}$ has been constructed for $i = 1, 2, \ldots, k-1$ for some $k \in \mathbf{N}$, $k \geq 2$, and that $u_{i,n} = u_{j,m}$ if and only if $m = n$ and $i = j$ $(i < k, j < k)$.

Now we construct $(u_{k,n})_{n\in\mathbf{N}}$ with the same property. Let $V_i = \{u_{j,n} | n \in \mathbf{N}, j = 1, 2, \ldots, i\}$.

By (1), (3), and (4), we have $\mathbf{N} \setminus V_{k-1} \neq \varnothing$. Thus, we can choose

(7) $$u_{k,1} = \min(\mathbf{N} \setminus V_{k-1}).$$

We will show that there are $u_{k,2} \in \mathbf{N}$ with

(8) $$u_{k,2} > u_{k,1}$$

and

(9) $$u_{k,2} > \max\{u_{i,2} | i = 1, 2, \ldots, k-1\},$$

such that the sequence $(u_{k,n})_{n\in\mathbf{N}}$ generated by (4) has the following property P:

(P) If $i < k$, then $u_{k,n} \neq u_{i,m}$ for all $n, m \in \mathbf{N}$.

Let $M_k = \max\{u_{k,1}, u_{1,2}, u_{2,2}, \ldots, u_{k-1,2}\}$. Then $u_{k,2} > M_k$ is equivalent to (8) and (9).

Let $S_k \in \mathbf{R}$ be sufficiently large. More precisely

(10) $$S_k \geq 4\alpha^{-1} u_{k,1} \ (>1)$$

and

(11) $$S_k > 5(k-1)\big((\log 4S_k)/\log\alpha\big)^2 + M_k$$
$$(\text{e.g.: } S_k = \big((5(k-1)/\log^2\alpha)^2 + 1\big)M_k\big).$$

To prove the existence of $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ has property (P), we first count the number of those integers $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ does not have property (P). For these $u_{k,2}$, there are $m, n \in \mathbf{N}$ and $i \in \{1, 2, \ldots, k-1\}$ with

(12) $$u_{k,n} = u_{i,m}.$$

From (7), (8), and (9), we get $n \geq 2$ and $m \geq 3$. By (5) we can write (12) as follows:

$$F_{n-1}u_{k,2} + F_{n-2}u_{k,1} = F_{m-1}u_{i,2} + F_{m-2}u_{i,1}.$$

We obtain

(13) $$n < m$$

and by (3) also

$$\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}u_{k,2} + \frac{\alpha^{n-2}-\beta^{n-2}}{\sqrt{5}}u_{k,1} = \frac{\alpha^{m-1}-\beta^{m-1}}{\sqrt{5}}u_{i,2} + \frac{\alpha^{m-2}-\beta^{m-2}}{\sqrt{5}}u_{i,1}.$$

Since $u_{k,2} \leq S_k$ and $|\beta| < \alpha/2$, we get

$$S_k \geq u_{k,2} \geq \frac{\alpha^{m-1}-|\beta|^{m-1}}{\alpha^{n-1}+|\beta|^{n-1}}u_{i,2} + \frac{\alpha^{m-2}-|\beta|^{m-2}}{\alpha^{n-1}+|\beta|^{n-1}}u_{i,1} - \frac{\alpha^{n-2}+|\beta|^{n-2}}{\alpha^{n-1}-|\beta|^{n-1}}u_{k,1}$$
$$\geq \frac{\frac{1}{2}\alpha^{m-1}}{2\alpha^{n-1}}u_{i,2} + \frac{\frac{1}{2}\alpha^{m-2}}{2\alpha^{n-1}}u_{i,1} - 4\alpha^{-1}u_{k,1}.$$

Observing (10), this implies

$$8S_k+4(S_k+4\alpha^{-1}u_{k,1})\geq\alpha^{m-n-1}(\alpha u_{i,2}+u_{i,1})>4g\alpha^{m-n-1}$$
$$2S_k>\alpha^{m-n-1}$$
$$\frac{\log 2S_k}{\log\alpha}>m-n-1. \tag{14}$$

We have $u_{k,n+1}\neq u_{i,m+1}$. Otherwise we would get from (12) and (4) that $u_{k,\ell}=u_{i,m-n+\ell}$ for all $\ell\in\mathbf{N}$. In particular, $u_{k,1}=u_{i,m-n+1}$ would contradict (7).

Using this and (6), (12), (1), (13), (8), (9), (2), and $u_{k,2}\leq S_k$, we get

$$\begin{aligned}1&\leq|u_{k,n+1}-u_{i,m+1}|\\&=|\alpha u_{k,n}+\beta^{n-2}(\beta u_{k,2}+u_{k,1})-\alpha u_{i,m}-\beta^{m-2}(\beta u_{i,2}+u_{i,1})|\\&\leq|\beta|^{n-2}|\beta u_{k,2}+u_{k,1}|+|\beta|^{m-2}|\beta u_{i,2}+u_{i,1}|\\&\leq|\beta|^{n-2}(|\beta u_{k,2}|+|u_{k,1}|+|\beta u_{i,2}|+|u_{i,1}|)\\&\leq|\beta|^{n-2}4u_{k,2}\leq\alpha^{-(n-2)}4S_k\end{aligned}$$
$$\alpha^{n-2}\leq 4S_k$$
$$n\leq\frac{\log 4S_k}{\log\alpha}+2.$$

Combining this with (14), we obtain

$$m<\frac{\log 2S_k}{\log\alpha}+n+1\leq\frac{\log 2S_k}{\log\alpha}+\frac{\log 4S_k}{\log\alpha}+3\leq 3\frac{\log 4S_k}{\log\alpha}. \tag{15}$$

Now we will give an upper bound for the number of triples (n,m,i) such that $u_{k,n}=u_{i,m}$, $1\leq i\leq k-1$. In this case (15) holds. First, fix i and m.

Since $2\leq n<m$, there are at most $m-2$ possible values for n. Since

$$3\leq m<(3\log 4S_k)/\log\alpha,\qquad \text{for fixed } i,$$

there are at most

$$\frac{1}{2}\left(\frac{3\log 4S_k}{\log\alpha}-1\right)\left(\frac{3\log 4S_k}{\log\alpha}-2\right)\leq 5\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

possible pairs (m,n).

Finally, since $1\leq i\leq k-1$, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

possible triples (n,m,i). To each triple such that $u_{k,n}=u_{i,m}$, $1\leq i\leq k-1$ belongs exactly one $u_{k,2}\in(M_k,S_k]\cap\mathbf{N}$, because for two different values of $u_{k,2}$ and the fixed value of $u_{k,1}$, the recurrence (4) would give two different values of $u_{k,n}$, both of which cannot be equal to $u_{i,n}$. Consequently, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

values of $u_{k,2} \in (M_k, S_k]\cap \mathrm{N}$ such that $u_{k,n} = u_{i,m}$ for some $n, m, 1 \le i \le k-1$. Therefore, the number of values $u_{k,2} \in (M_k, S_k]\cap \mathrm{N}$ such that $u_{k,n} \ne u_{i,m}$ for all $n, m, 1 \le i \le k-1$ is at least

$$S_k - M_k - 5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2,$$

which is positive by (11), and hence the choice of such an $u_{k,2}$ is possible.

This induction on k shows that there are infinitely many sequences $(u_{k,n})_{n\in\mathrm{N}}$. Every natural number occurs in one of these sequences by (7). It occurs exactly once by property (P) which holds for these sequences.

REFERENCE

1. J. R. Burke & G. E. Bergum. "Covering the Integers with Linear Recurrences." In *Applications of Fibonacci Numbers*, vol 2, pp. 143-47. Ed. A. N. Philippou et al. Dordrecht: Kluwer Academic Publishers, 1988.

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