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A GENERALIZATION OF THE "ALL OR NONE" DIVISIBILITY PROPERTY

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1. INTRODUCTION

In [6], Juan Pla proved the following interesting theorem.

Theorem 1.1: Let h_n be the general term of a given sequence of integers such that $h_{n+2}=h_{n+1}+h_n$, where h_0 and h_1 are arbitrary integers. Let c be an arbitrary integer other than -2, -1, 0, and 1. Let D be any divisor of c^2+c-1 other than 1. Then, the sequence $\{w_n\}$, where $w_n=ch_{n+1}-h_n$, for $n\geq 0$, is such that either (a) D divides every w_n or (b) D divides no w_n.

We would like to point out a more interesting fact that, essentially, the above theorem is the corollary of the following.

Theorem 1.2: Let $\{f_n\}$ be the Fibonacci sequence, that is, $f_0=0$, $f_1=1$, and $f_{n+2}=f_{n+1}+f_n$ for $n\geq 0$. Let $f(x)=x^2-x-1$. Then, for $n\in\mathbf{Z}$, we have

$$x^n\equiv f_nx+f_{n-1} \pmod{f(x)}. \tag{1.1}$$

Proof: Equation (1.1) holds for $n=0$ since $x^0=f_0x+f_{-1}$. Assume that (1.1) holds for $n=k$, $k\geq 0$, that is, $x^k\equiv f_kx+f_{k-1} \pmod{f(x)}$. Then $x^{k+1}\equiv f_kx^2+f_{k-1}x\equiv f_k(x+1)+f_{k-1}x=f_{k+1}x+f_k \pmod{f(x)}$. This means that (1.1) holds for all $n\geq 0$. Now assume that (1.1) holds for $n=-k$, $k\geq 0$, that is, $x^{-k}\equiv f_{-k}x+f_{-k-1} \pmod{f(x)}$. Then $x^{-k-1}\equiv f_{-k}+f_{-k-1}x^{-1} \pmod{f(x)}$. Since $x(x-1)\equiv 1 \pmod{f(x)}$, we have that $x^{-1}\equiv x-1 \pmod{f(x)}$, and so $x^{-k-1}\equiv f_{-k}+f_{-k-1}(x-1)=f_{-k-1}x+f_{-k-2} \pmod{f(x)}$. This means that (1.1) holds also for all $n<0$. □

Now we apply Theorem 1.2 to prove Theorem 1.1. We have $h_n=h_1f_n+h_0f_{n-1}$ and $h_{n+1}=h_1f_{n+1}+h_0f_n$ for $n\geq 0$ (see [2]), whence $w_n=-h_1(-f_{n+1}c+f_n)-h_0(-f_nc+f_{n-1})$. In (1.1), taking $x=-c$, we get $w_n\equiv -h_1(-c)^{n+1}-h_0(-c)^n=(-c)^n(ch_1-h_0)=(-c)^nw_0 \pmod{c^2+c-1}$. Since D divides c^2+c-1 and $D>1$, we have $\gcd(c,D)=1$. If D divides w_n for some $n\geq 0$, then D divides w_0. This leads to the fact that D divides w_n for all $n\geq 0$. □

In this paper we generalize the result of Theorem 1.2 to the case of k^{th}-order homogeneous recursion sequence with constant coefficients in Section 3. In Section 4 we generalize the interesting result of Theorem 1.1, correspondingly, i.e., we give and prove the main result of this paper. Some necessary preliminaries are given in Section 2.

2. PRELIMINARIES

Let the sequence $\{h_n\}=\{h_n\}_{n\geq 0}$ be defined by the recurrence relation

$$h_{n+k}=a_1h_{n+k-1}+\cdots+a_{k-1}h_{n+1}+a_kh_n, \tag{2.1}$$

and the initial condition

$$h_0 = c_0,\ h_1 = c_1, \ldots, h_{k-1} = c_{k-1}, \tag{2.2}$$

where $a_1, \ldots, a_k$ and $c_0, \ldots, c_{k-1}$ are constants. Then we call $\{h_n\}$ a k^{th}-order **Fibonacci-Lucas sequence** or simply an **F-L sequence**, and we call h_n the n^{th} **F-L number**. The polynomial

$$f(x) = x^k - a_1x^{k-1} - \cdots - a_{k-1}x - a_k \tag{2.3}$$

is called the **characteristic polynomial** of $\{h_n\}$. If $f(\theta) = 0$, then we call θ a **characteristic root** of $\{h_n\}$. The set of F-L sequences satisfying (2.1) is denoted by $\Omega(a_1, \ldots, a_k)$ and also by $\Omega(f(x))$.

If $a_k \neq 0$, then (2.1) can be rewritten as

$$h_n = (h_{n+k} - a_1h_{n+k-1} - \cdots - a_{k-1}h_{n+1}) / a_k, \tag{2.4}$$

whence, from the given values of $h_0, h_1, \ldots, h_{k-1}$, we can calculate the values of $h_{-1}, h_{-2}, \ldots$. Therefore, in the case $a_k \neq 0$, we may consider $\{h_n\}$ as $\{h_n\}_{-\infty}^{+\infty}$. For convenience, we always assume that $a_k \neq 0$ whenever we refer to $\Omega(a_1, \ldots, a_k)$.

Obviously, $\Omega(a_1, \ldots, a_k)$ is a linear space [3] under the operations $\{h_n\} + \{w_n\} = \{h_n + w_n\}$ and $\lambda\{h_n\} = \{\lambda h_n\}$. Let $\{u_n^{(i)}\}$,. $0 \leq i \leq k-1$, be a sequence in $\Omega = \Omega(a_1, \ldots, a_k)$ with the initial condition $u_n^{(i)} = \delta_{ni}$ for $0 \leq n \leq k-1$, where δ is the Kronecker function. Then we call $\{u_n^{(i)}\}$ the i^{th} **basic sequence** in Ω. Construct a map, $\Omega \to \mathbf{R}^k$ such that each sequence $\{h_n\} \in \Omega$, with initial condition (2.2), corresponds to $(c_0, c_1, \ldots, c_{k-1})$. Clearly, this map is an isomorphism, and the basic sequences $\{u_n^{(0)}\}, \{u_n^{(1)}\}, \ldots, \{u_n^{(k-1)}\}$ form a base in Ω. Thus, we have the following lemmas.

Lemma 2.1: Let $\Omega = \Omega(a_1, \ldots, a_k)$. Let $\{u_n^{(i)}\}, 0 \leq i \leq k-1$, be the i^{th} basic sequence in Ω and let $\{h_n\}$ be an arbitrary sequence in Ω. Then $\{h_n\}$ can be represented uniquely by $\{u_n^{(0)}\}, \{u_n^{(1)}\}, \ldots, \{u_n^{(k-1)}\}$, as

$$h_n = \sum_{i=0}^{k-1} h_i u_n^{(i)} \text{ for } n \in \mathbf{Z}. \tag{2.5}$$

Lemma 2.2: Under the condition of Lemma 2.1, we have

$$h_{n+1} = (a_1h_{k-1} + a_2h_{k-2} + \cdots + a_kh_0)\, u_n^{(k-1)} + \sum_{i=0}^{k-2} h_{i+1}u_n^{(i)} \text{ for } n \in \mathbf{Z}. \tag{2.6}$$

Proof: Let $\{w_n\} = \{h_{n+1}\}$. Then $w_0 = h_1, \ldots, w_{k-2} = h_{k-1}$ and (2.1) implies $w_{k-1} = h_k = a_1h_{k-1} + a_2h_{k-2} + \cdots + a_kh_0$. Thus, the lemma is proved by Lemma 2.1. □

In (2.6), replacing $\{h_n\}$ by $\{u_n^{(0)}\}, \ldots, \{u_n^{(k-1)}\}$, respectively, we obtain

Lemma 2.3: Let $\{u_n^{(i)}\}, 0 \leq i \leq k-1$, be the i^{th} basic sequence in $\Omega(a_1, \ldots, a_k)$. Then, for $n \in \mathbf{Z}$, we have

$$u_{n+1}^{(0)} = a_k u_n^{(k-1)} \quad \text{and} \quad u_{n+1}^{(i)} = a_{k-i}u_n^{(k-1)} + u_n^{(i-1)} \text{ for } 1 \leq i \leq k-1. \tag{2.7}$$

Lemma 2.4: Under the condition of Lemma 2.3, we have

$$u_n^{(i)} = \sum_{j=0}^{i} a_{k-i+j} u_{n-1-j}^{(k-1)},\ i = 0, \ldots, k-1,\ n \in \mathbf{Z}. \tag{2.8}$$

Proof: From (2.7), (2.8) holds for $i=0$. Assume (2.8) holds for i, $0 \le i < k-1$. Then (2.7) and the induction hypothesis imply that

$$u_n^{(i+1)} = a_{k-i-1}u_{n-1}^{(k-1)} + u_{n-1}^{(i)} = a_{k-i-1}u_{n-1}^{(k-1)} + \sum_{j=0}^{i} a_{k-i+j}u_{n-2-j}^{(k-1)}$$
$$= \sum_{j=0}^{i+1} a_{k-(i+1)+j}u_{n-1-j}^{(k-1)},$$

and we are done. □

From (2.7) and (2.8), we observe that the $(k+1)^{\text{th}}$ basic sequence in $\Omega(a_1, \dots, a_k)$ plays an important role, so that we call it the **principal sequence** in Ω and denote it by $\{u_n^{(k-1)}\} = \{u_n\}$.

Now, substituting (2.8) into (2.5), we get

Lemma 2.5: Let $\{u_n\}$ be the principal sequence in $\Omega = \Omega(a_1, \dots, a_k)$. Let $\{h_n\}$ be an arbitrary sequence in Ω. Then

$$h_n = \sum_{i=0}^{k-1} b_{k-1-i}u_{n-i} \quad \text{for } n \in \mathbb{Z}. \tag{2.9}$$

where

$$b_{k-1} = h_{k-1} \quad \text{and} \quad b_{k-1-i} = \sum_{j=0}^{k-1-i} a_{i+1+j}h_{k-2-j} \quad \text{for } 1 \le i \le k-1. \tag{2.10}$$

3. A PROPERTY OF THE CHARACTERISTIC POLYNOMIAL OF A k^{th}-ORDER F-L SEQUENCE

Theorem 3.1: Let $\{u_n^{(i)}\}$, $0 \le i \le k-1$, be the i^{th} basic sequence in $\Omega(f(x))$, where $f(x)$ is denoted by (2.3). Then

(a) $x^n \equiv \sum_{i=0}^{k-1} u_n^{(i)}x^i \pmod{f(x)}$ for $n \in \mathbb{Z}$. (3.1)

(b) If, besides (3.1), we have $x^n \equiv \sum_{i=0}^{k-1} v_n^{(i)}x^i \pmod{f(x)}$, where each of the $v_n^{(i)}$'s ($i = 0, \dots, k-1$) is independent of x, then $u_n^{(i)} = v_n^{(i)}$, $i = 0, \dots, k-1$.

Proof: Part (b) is proved by the uniqueness of the remainder of x^n over $f(x)$. Now we must prove part (a). By the definition of $\{u_n^{(i)}\}$, $i = 0, \dots, k-1$, (3.1) holds for $n=0$. Assume that (3.1) holds for $n = m$, $m \ge 0$. Then, from the induction hypothesis and (2.7), we have

$$x^{m+1} \equiv x\sum_{i=0}^{k-1} u_m^{(i)}x^i = u_m^{(k-1)}x^k + \sum_{i=0}^{k-2} u_m^{(i)}x^{i+1}$$
$$\equiv u_m^{(k-1)}(a_1x^{k-1} + \cdots + a_{k-1}x + a_k) + \sum_{i=0}^{k-2} u_m^{(i)}x^{i+1}$$
$$= a_ku_m^{(k-1)} + \sum_{i=1}^{k-1}(a_{k-i}u_m^{(k-1)} + u_m^{(i-1)})x^i = \sum_{i=0}^{k-1} u_{m+1}^{(i)}x^i \pmod{f(x)}.$$

This implies that (3.1) holds for all $n \ge 0$.

Now assume that (3.1) holds for $n = -m$, $m \ge 0$. Then

$$x^{-m-1} \equiv x^{-1}\left(\sum_{i=0}^{k-1} u_{-m}^{(i)} x^i\right) = \sum_{i=1}^{k-1} u_{-m}^{(i)} x^{i-1} + u_{-m}^{(0)} x^{-1} \pmod{f(x)}. \tag{3.2}$$

From $x(x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) \equiv a_k \pmod{f(x)}$ and $a_k \neq 0$, we obtain

$$x^{-1} \equiv (x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) / a_k \pmod{f(x)}. \tag{3.3}$$

Substituting (3.3) into (3.2) and noting that $u_{-m}^{(0)} / a_k = u_{-m-1}^{(k-1)}$ we get, by (2.7)

$$\begin{aligned} x^{-m-1} &\equiv \sum_{i=1}^{k-1} u_{-m}^{(i)} x^{i-1} + u_{-m-1}^{(k-1)}(x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) \\ &= u_{-m-1}^{(k-1)} x^{k-1} + \sum_{i=1}^{k-1} (u_{-m}^{(i)} - a_{k-i} u_{-m-1}^{(k-1)}) x^{i-1} \\ &= u_{-m-1}^{(k-1)} x^{k-1} + \sum_{i=1}^{k-1} u_{-m-1}^{(i-1)} x^{i-1} = \sum_{i=0}^{k-1} u_{-m-1}^{(i)} x^i \pmod{f(x)}. \end{aligned}$$

This implies that (3.1) holds also for $n < 0$. □

Corollary: Under the condition of Theorem 3.1, if $f(\theta) = 0$, then

$$\theta^n = \sum_{i=0}^{k-1} u_n^{(i)} \theta^i \quad \text{for } n \in \mathbf{Z}. \tag{3.4}$$

It can be observed that the results in [1], [4], and [5] may be obtained easily by using (3.4).

4. A GENERALIZATION OF THE "ALL OR NONE" DIVISIBILITY PROPERTY

Theorem 4.1: Let $\{h_n\}$ be an arbitrary sequence in $\Omega(a_1, \ldots, a_k) = \Omega(f(x))$, where $a_1, \ldots, a_k$ are integers and $f(x)$ is denoted by (2.3). Let $c \in \mathbf{Z}$, $f(c) \neq \pm 1$. Let D be a divisor of $f(c)$ other than 1, and $\gcd(c, D) = 1$. Suppose that

$$w_n = \sum_{i=0}^{k-1} g_{k-1-i}(c) h_{n+k-1-i}, \tag{4.1}$$

where

$$g_{k-1}(x) = x^{k-1} \quad \text{and} \quad g_{k-1-i}(x) = \sum_{j=0}^{k-1-i} a_{i+1+j} x^{k-2-j} \quad \text{for } 1 \leq i \leq k-1. \tag{4.2}$$

Then either D divides w_n for all $n \geq 0$ or D divides no w_n.

To prove the theorem, we need the following lemmas.

Lemma 4.1:

$$\sum_{i=0}^{k-1} g_{k-1-i}(x) u_{n-i} \equiv x^n \pmod{f(x)}, \tag{4.3}$$

where $\{u_n\}$ is the principal sequence in $\Omega(f(x))$.

Proof: Let $\{u_n^{(i)}\}$, $0 \leq i \leq k-1$, be the i^{th} basic sequence. From Theorem 3.1 and Lemma 2.4, we have

$$\begin{aligned} x^n &\equiv \sum_{t=0}^{k-1} u_n^{(t)} x^t = x^{k-1}u_n + \sum_{t=0}^{k-2} x^t \sum_{i=0}^{t} a_{k-t+i} u_{n-1-i} \\ &= x^{k-1}u_n + \sum_{i=0}^{k-2} u_{n-1-i} \sum_{t=i}^{k-2} a_{k-t+i} x^t = \sum_{i=0}^{k-2} u_{n-1-i} \sum_{j=0}^{k-2-i} a_{i+2+j} x^{k-2-j} + u_n x^{k-1} \\ &= \sum_{i=0}^{k-2} u_{n-1-i} g_{k-2-i}(x) + g_{k-1}(x) u_n = \sum_{i=0}^{k-1} g_{k-1-i}(x) u_{n-i} \pmod{f(x)}. \ \square \end{aligned}$$

Lemma 4.2:

$$\sum_{i=0}^{k-1} b_i c^i = w_0, \tag{4.4}$$

where b_i $(0 \le i \le k-1)$ is denoted by (2.10).

Proof:

$$\begin{aligned} \sum_{i=0}^{k-1} b_i c^i &= h_{k-1}c^{k-1} + \sum_{i=0}^{k-2} b_{k-1-(k-1-i)} c^t \\ &= h_{k-1}c^{k-1} + \sum_{i=0}^{k-2} c^i \sum_{j=0}^{i} a_{k-i+j} h_{k-2-j} = h_{k-1}c^{k-1} + \sum_{j=0}^{k-2} h_{k-2-j} \sum_{i=j}^{k-2} a_{k-i+j} c^i \\ &= h_{k-1}c^{k-1} + \sum_{j=0}^{k-2} h_{k-2-j} \sum_{i=0}^{k-2-j} a_{2+j+i} c^{k-2-i} = h_{k-1} g_{k-1}(c) + \sum_{j=0}^{k-2} h_{k-2-j} g_{k-2-j}(c) \\ &= \sum_{j=0}^{k-1} g_{k-1-j}(c) h_{k-1-j} = w_0, \end{aligned}$$

by (4.2) and (4.1). $\square$

Proof of Theorem 4.1: From (4.1) and Lemma 2.5, we have

$$\begin{aligned} w_n &= \sum_{j=0}^{k-1} g_{k-1-j}(c) h_{n+k-1-j} = \sum_{j=0}^{k-1} g_{k-1-j}(c) \sum_{i=0}^{k-1} b_{k-1-i} u_{n+k-1-j-i} \\ &= \sum_{i=0}^{k-1} b_{k-1-i} \sum_{j=0}^{k-1} g_{k-1-j}(c) u_{n+k-1-j-i}. \end{aligned}$$

In Lemma 4.1, taking $x = c$, we get

$$\sum_{j=0}^{k-1} g_{k-1-j}(c) u_{n+k-1-j-i} \equiv c^{n+k-1-i} \pmod{f(c)},$$

whence, from Lemma 4.2,

$$w_n \equiv \sum_{i=0}^{k-1} b_{k-1-i} c^{n+k-1-i} = c^n \sum_{i=0}^{k-1} b_{k-1-i} c^{k-1-i} = c^n w_0 \pmod{f(c)}.$$

Because $\gcd(c, D) = 1$, if D divides w_n for some $n \ge 0$, then D must divide w_0, so D divides w_n for all $n \ge 0$. $\square$

Example 1: Let $f(x)=x^3-x^2-x-1$, then $k=3$, $a_1=a_2=a_3=1$. Let $c=-2$, then $f(c)=-11$. Take $D=11$, then $\gcd(c,D)=1$. Assume that $\{h_n\}\in\Omega(f(x))$ and $h_0=0, h_1=h_2=1$. From (4.2) and (4.1), we have $g_2(c)=(-2)^2=4$, $g_1(c)=1\times(-2)+1=-1$, $g_0(c)=1\times(-2)=-2$, and $w_n=4h_{n+2}-h_{n+1}-2h_n$, respectively. Since $w_0=4h_2-h_1-2h_0=3$ and 11 does not divide 3, thus 11 divides no w_n.

Example 2: Let $f(x)=x^3-x^2+2x-3$, then $k=3, a_1=1, a_2=-2, a_3=3$. Let $c=3$, then $f(c)=21$. Take $D=7$, then $\gcd(c,D)=1$. Assume that $\{h_n\}\in\Omega(f(x))$ and that $h_0=h_2=1$, $h_1=-1$. From (4.2) and (4.1), we have $g_2(c)=3^2=9$, $g_1(c)=(-2)\times 3+3=-3$, $g_0(c)=3\times 3=9$, and $w_n=9h_{n+2}-3h_{n+1}+9h_n$, respectively. Since $w_0=9h_2-3h_1+9h_0=21$ and 7 divides 21, thus 7 divides w_n for all $n\geq 0$.

Concluding Remark: Theorem 3.1 can be seen in [7], which was published in Chinese in 1993. Some other applications of Theorem 3.1 and its corollary to the identities involving F-L numbers, congruence relations, modular periodicities, divisibilities, etc., are also stated in [7].

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