A NOTE ON A CLASS OF COMPUTATIONAL FORMULAS INVOLVING THE MULTIPLE SUM OF RECURRENCE SEQUENCES

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1. INTRODUCTION

The second-order linear recurrence sequence $U=\{U_n\}$, $n=0,1,2,\dots$, is defined by integers a, b, U_0, U_1 and by the recursion $U_{n+2}=bU_{n+1}+aU_n$ for $n\geq 0$. We suppose that $ab\neq 0$ and not both U_0 and U_1 are zero. If α and β denote the roots of the characteristic polynomial x^2-bx-a of the sequence U, then we have the Binet formula (see [1]):

$$U_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta},$$

where $A=U_1-U_0\beta$ and $B=U_1-U_0\alpha$. The generating function is

$$\sum_{n=0}^{\infty}U_nx^n=\frac{U_0+(U_1-U_0b)x}{1-bx-ax^2}.$$

If $U_0=0$, $U_1=1$, then the sequence $\mathcal{F}=\{U_n\}$ is called the generalized Fibonacci sequence, and $\mathcal{F}_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$.

In order to express our results, we denote by $\sigma_{i,j}(n,k)$ (i, j, and k are nonnegative integers) the summation of all products of choosing j elements from $n+2k-1$, $n+2k-2$, ..., $n+2k-i+1$ but not containing any two consecutive elements. We note that $\sigma_{i,j}(n,k)=0$ if $j<0$ or $j>[\frac{i}{2}]$, $\sigma_{i,0}(n,k)=1$ $(i\geq 0)$, $\sigma_{i,1}(n,k)=\frac{1}{2}(i-1)(2n+4k-i)$ $(i\geq 1)$. For example, when $i=6$, we have

$$\sigma_{6,0}(n,k)=1,$$
$$\sigma_{6,1}(n,k)=(n+2k-1)+(n+2k-2)+(n+2k-3)+(n+2k-4)+(n+2k-5),$$
$$\sigma_{6,2}(n,k)=(n+2k-1)(n+2k-3)+(n+2k-1)(n+2k-4)+(n+2k-1)(n+2k-5)$$
$$+(n+2k-2)(n+2k-4)+(n+2k-2)(n+2k-5)+(n+2k-3)(n+2k-5),$$
$$\sigma_{6,3}(n,k)=(n+2k-1)(n+2k-3)(n+2k-5).$$

It is easy to prove that

$$(n+2k-1)\sigma_{2k-2,k-1}(n,k-1)=\sigma_{2k,k}(n,k)\quad(k\geq 1)$$

and

$$(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)+\sigma_{k+i-1,i}(n+1,k-1)=\sigma_{k+i,i}(n,k)\quad(1\leq i\leq k,\ k\geq 2).$$

Recently, W. Zhang [2] obtained the following result: Let $U=(U_n\}$ be defined as above. If $U_0=0$, then for any positive integer $k\geq 2$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k} = \frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}[g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}],$$

where the summation is taken over all n-tuples with positive integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1+a_2+\cdots+a_k=n$, and he pointed out that $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$, their coefficients depending only on a, b, and k.

In this paper, we obtain

$$g_{k-1}(n)=\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,\,k-1)\mathcal{F}_{k-i} \quad (k\ge 1)$$

and

$$h_{k-1}(n)=a\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,\,k-1)\mathcal{F}_{k-i-1} \quad (k\ge 1),$$

where $\langle n\rangle_k = n(n+1)\cdots(n+k-1)$ with $\langle n\rangle_0=1$. We also give the congruence relation

$$g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}\equiv 0 \pmod{(k-1)!(b^2+4a)^{k-1}} \quad (k\ge 1),$$

which generalizes the results presented in [2].

2. THE RESULTS AND THEIR PROOFS

In this section, with $U_0=0$, let

$$G_k(x)=\left(\frac{U_1}{1-bx-ax^2}\right)^k=\sum_{n=0}^{\infty}U_n^{(k)}x^{n-1}.$$

Then

$$\sum_{a_1+a_2+\cdots+a_m=n} U_{a_1}^{(k_1)}U_{a_2}^{(k_2)}\cdots U_{a_m}^{(k_m)} = U_{n-m+1}^{(k_1+k_2+\cdots+k_m)}.$$

Taking $k_1=k_2=\cdots=k_m=1$, we have

Lemma 1: $\displaystyle\sum_{a_1+a_2+\cdots+a_m=n} U_{a_1}U_{a_2}\cdots U_{a_m} = U_{n-m+1}^{(m)}.$

Theorem 1: $\displaystyle U_n^{(k+1)}=\frac{U_1}{k(b^2+4a)}\{nbU_{n+1}^{(k)}+2a(n+2k-1)U_n^{(k)}\} \quad (k\ge 1).$

Proof:

$$\frac{d}{dx}(G_k(x)(b+2ax)^k)=G_k'(x)(b+2ax)^k+G_k(x)k(b+2ax)^{k-1}2a$$

and

$$\frac{d}{dx}(G_k(x)(b+2ax)^k)=\frac{d}{dx}\left(\frac{U_1(b+2ax)}{1-bx-ax^2}\right)^k$$

$$=kU_1\left(\frac{U_1(b+2ax)}{1-bx-ax^2}\right)^{k-1}\frac{2a(1-bx-ax^2)+(b+2ax)^2}{(1-bx-ax^2)^2}$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{2a^2x^2+2abx+b^2+2a}{(1-bx-ax^2)^2}$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{-2a(1-bx-ax^2)+b^2+4a}{(1-bx-ax^2)^2}.$$

Hence,

$$G_k'(x)(b+2ax)^k + G_k(x)k(b+2ax)^{k-1}2a$$

$$= k(b+2ax)^{k-1}U_1\left(\frac{U_1}{1-bx-ax^2}\right)^{k-1}\frac{-2a(1-bx-ax^2)+b^2+4a}{(1-bx-ax^2)^2}.$$

Therefore,

$$G_k'(x)U_1(b+2ax)+2akU_1G_k(x) = -2akU_1G_k(x)+(b^2+4a)kG_{k+1}(x).$$

This concludes the proof of Theorem 1. □

Theorem 2: $U_n^{(k+1)} = \dfrac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(k)U_{n+k-i} \quad (k \ge 0).$

Proof: This theorem can be proved by induction. When $k=0$, the theorem is trivial. When $k=1$, the theorem is true by applying Theorem 1. Assume the theorem is true for a positive integer $k-1$, then

$$U_n^{(k+1)} = \frac{U_1}{k(b^2+4a)}\{nbU_{n+1}^{(k)}+2a(n+2k-1)U_n^{(k)}\}$$

$$= \frac{U_1}{k(b^2+4a)}\left\{nb\frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n+1\rangle_{k-i-1,i}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+2a(n+2k-1)\frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n\rangle_{k-i-1}\sigma_{k+i-1,i}(n,k-1)U_{n+k-i-1}\right\}$$

$$= \frac{U_1^k}{k!(b^2+4a)^k}\left\{\sum_{i=0}^{k-1}(2a)^i b^{k-n}\langle n+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+\sum_{i=0}^{k-1}(2a)^{i+1}b^{k-i-1}\langle n\rangle_{k-i-1}(n+2k-1)\sigma_{k+i-1,i}(n,k-1)U_{n+k-i-1}\right\}$$

$$= \frac{U_1^k}{k!(b^2+4a)^k}\left\{\sum_{i=0}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i-1,i}(n+1,k-1)U_{n+k-i}\right.$$

$$\left.+\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)U_{n+k-i}\right\}$$

$$= \frac{U_1^k}{k!(b^2+4a)^k}\left\{b^k\langle n\rangle_k\sigma_{k-1,0}(n+1,k-1)U_{n+k}+\sum_{i=1}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}U_{n+k-i}[\sigma_{k+i-1,i}(n,k-1)\right.$$

$$\left.+(n+2k-1)\sigma_{k+i-2,i-1}(n,k-1)]+(2a)^k(n+2k-1)\sigma_{2k-2,k-1}(n,k-1)U_n\right\}$$

$$= \frac{U_1^k}{k!(b^2+4a)^k}\left\{b^k\langle n\rangle_k\sigma_{k,0}(n+1,k)U_{n+k} + \sum_{i=1}^{k-1}(2a)^i b^{k-i}\langle n\rangle_{k-i}U_{n+k-i}\sigma_{k+i,i}(n,k) + (2a)^k\sigma_{2k,k}(n,k)U_n\right\}$$

$$= \frac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(n,k)U_{n+k-i}.$$

That is, the theorem is also true for k. This proves the Theorem 2. □

Lemma 2: $U_{m+k} = \mathcal{F}_{k+1}U_m + a\mathcal{F}_k U_{m-1}$ $(k \geq 0, m \geq 1)$.

Proof: Use Binet's formula. □

Theorem 3: $U_n^{(k+1)} = \frac{U_1^k}{k!(b^2+4a)^k}\sum_{i=0}^{k}(2a)^i b^{k-i}\langle n\rangle_{k-i}\sigma_{k+i,i}(n,k)(\mathcal{F}_{k-i+1}U_n + a\mathcal{F}_{k-i}U_{n-1})$ $(k \geq 0)$.

Proof: Use Theorem 2 and Lemma 2. □

Theorem 4:
$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k}$$

$$= \frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i}\right]U_{n-k+1}\right.$$
$$\left. + a\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i-1}\right]U_{n-k}\right\} \quad (k \geq 1).$$

Proof: Noting Lemma 1 and Theorem 3, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k} = U_{n-k+1}^{(k)}$$

$$= \frac{U_1^{k-1}}{(k-1)!(b^2+4a)^{k-1}}\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-1-i}\sigma_{k-1+i,i}(n-k+1,k-1)$$
$$\times(\mathcal{F}_{k-i}U_{n-k+1} + a\mathcal{F}_{k-i-1}U_{n-k})$$

$$= \frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}\left\{\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i}\right]U_{n-k+1}\right.$$
$$\left. + a\left[\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i-1}\right]U_{n-k}\right\}. \quad □$$

From this theorem, we can get the expression of $g_{k-1}(n)$ and $h_{k-1}(n)$, namely,

$$g_{k-1}(n) = \sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i} \quad (k \geq 1)$$

and

$$h_{k-1}(n) = a\sum_{i=0}^{k-1}(2a)^i b^{k-i-1}\langle n-k+1\rangle_{k-i-1}\sigma_{k+i-1,i}(n-k+1,k-1)\mathcal{F}_{k-i-1} \quad (k \geq 1)..$$

Theorem 5: $g_{k-1}(n)U_{n-k+1} + h_{k-1}(n)U_{n-k} \equiv 0 \pmod{(k-1)!(b^2+4a)^{k-1}}$ $(k \geq 1)$.

This result is a generalization of Corollary 2 of [2]. When $U_1 = a = b = 1$ and $k = 1, 2, 3$, respectively, this result becomes (i)-(iii) of Corollary 2 of [2].

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THE PASSING OF THREE FIBONACCI ASSOCIATION FRIENDS

We were all deeply saddened to learn of the recent deaths of **Joe Arkin, Daniel Fielder** and **Peter Kiss**. These three long-time members of the Fibonacci Association will be greatly missed.