

ON A CLASS OF GENERALIZED POLYNOMIALS

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1. INTRODUCTION

In a series of articles [1]-[3], André-Jeannin has recently defined the polynomials $U_n(p, q; x)$ and $V_n(p, q; x)$ by the recurrence relations (1) and (2), and has studied some of the combinatorial properties of the coefficients of U_n and V_n as well as some of the differential properties of these polynomials.

$$U_n = (x + p)U_{n-1} - qU_{n-2} \quad (n \geq 2), \quad U_0 = 0, U_1 = 1 \tag{1}$$

and

$$V_n = (x + p)V_{n-1} - qV_{n-2} \quad (n \geq 2), \quad V_0 = 2, V_1 = x + p. \tag{2}$$

The parameters p and q as well as the variable x are real numbers. If α and β are defined by

$$\alpha + \beta = x + p, \quad \alpha\beta = q, \tag{3}$$

then it is well known that [5]

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}}, \tag{4a}$$

and

$$V_n = \alpha^n + \beta^n, \tag{4b}$$

where

$$\Delta = (x + p)^2 - 4q. \tag{5}$$

The purpose of this article is to introduce and study some of the properties of the generalized polynomial $W_n(p, q; x)$ defined by

$$W_n = (x + p)W_{n-1} - qW_{n-2} \quad (n \geq 2), \tag{6}$$

where W_0 and W_1 are specified, as well as those of two other polynomials $u_n(p, q; x)$ and $v_n(p, q; x)$ that are very closely associated with U_n and V_n . We shall define these polynomials $u_n(p, q; x)$ and $v_n(p, q; x)$ to be

$$u_n = (x + p)u_{n-1} - qu_{n-2} \quad (n \geq 2), \quad u_0 = 1, u_1 = x + p - \sqrt{q} \tag{7}$$

and

$$v_n = (x + p)v_{n-1} - qv_{n-2} \quad (n \geq 2), \quad v_0 = 1, v_1 = x + p + \sqrt{q}. \tag{8}$$

2. SOME BASIC RELATIONS AMONG U_n, V_n, u_n AND v_n

Using the well-known properties of $W_n(a, b, p, q)$ introduced by Horadam [5], we may derive a number of relations between U_n and V_n . However, we shall not do so except to list a few of the important ones that will be required for the remainder of this article. It is easy to show that W_n as defined by (6) may be evaluated using the relation [5],

$$W_n = W_1 U_n - q W_0 U_{n-1} \quad (n \geq 1). \tag{9}$$

From (9) we can immediately derive the following relations:

$$V_n = U_{n+1} - q U_{n-1}, \tag{10}$$

$$u_n = U_{n+1} - \sqrt{q} U_n, \tag{11}$$

$$v_n = U_{n+1} + \sqrt{q} U_n, \tag{12}$$

$$V_n = u_n + \sqrt{q} u_{n-1} = v_n - \sqrt{q} v_{n-1}. \tag{13}$$

From the results in [5], we may also derive the following "Simson" formulas:

$$U_{n+1} U_{n-1} - U_n^2 = -q^{n-1}, \tag{14a}$$

$$V_{n+1} V_{n-1} - V_n^2 = q^{n-1} \Delta, \tag{14b}$$

$$u_{n+1} u_{n-1} - u_n^2 = q^{n-1/2} \Delta_u, \tag{14c}$$

$$v_{n+1} v_{n-1} - v_n^2 = -q^{n-1/2} \Delta_v, \tag{14d}$$

where

$$\Delta_u = x + p - 2\sqrt{q}, \tag{15a}$$

$$\Delta_v = x + p + 2\sqrt{q}, \tag{15b}$$

$$\Delta = \Delta_u \Delta_v. \tag{15c}$$

From (14a-14d), we have the interesting result that

$$q(U_{n+1} U_{n-1} - U_n^2)(V_{n+1} V_{n-1} - V_n^2) = (u_{n+1} u_{n-1} - u_n^2)(v_{n+1} v_{n-1} - v_n^2) = -q^{2n-1} \Delta. \tag{16}$$

3. ZEROS AND ORTHOGONALITY PROPERTY OF $U_n, V_n, u_n,$ AND v_n

André-Jeannin ([1], [2]) has shown that

$$U_n = q^{(n-1)/2} \frac{\sin n\theta}{\sin \theta} \tag{17a}$$

and

$$V_n = 2q^{n/2} \cos n\theta, \tag{17b}$$

where $\cos \theta = (x + p) / 2\sqrt{q}$. Hence, from (11) and (17a), we get

$$u_n = q^{n/2} \frac{\cos(2n+1)\theta / 2}{\cos \theta / 2}. \tag{17c}$$

Similarly, from (12) and (17a), we have

$$v_n = q^{n/2} \frac{\sin(2n+1)\theta / 2}{\sin \theta / 2}. \tag{17d}$$

Hence, the zeros of $U_n, V_n, u_n,$ and v_n are given by

$$U_n: x_k = -p + 2\sqrt{q} \cos\left(\frac{k}{n} \cdot \pi\right), \quad k = 1, 2, \dots, n-1, \quad (18a)$$

$$V_n: x_k = -p + 2\sqrt{q} \cos\left(\frac{2k-1}{2n} \cdot \pi\right), \quad k = 1, 2, \dots, n, \quad (18b)$$

$$u_n: x_k = -p + 2\sqrt{q} \cos\left(\frac{2k-1}{2n+1} \cdot \pi\right), \quad k = 1, 2, \dots, n, \quad (18c)$$

$$v_n: x_k = -p + 2\sqrt{q} \cos\left(\frac{2k}{2n+1} \cdot \pi\right), \quad k = 1, 2, \dots, n. \quad (18d)$$

Of these, André-Jeannin ([1], [2]) has given the zeros for U_n and V_n . It should be observed that, if $p = 2$ and $q = 1$, then the above results correspond to the already known results for the zeros of $B_n(x)$, $C_n(x)$, $b_n(x)$, and $c_n(x)$ (see [6], [7], [4]).

André-Jeannin ([1], [2]) has shown further that U_n and V_n are orthogonal over the interval $(-p - 2\sqrt{q}, -p + 2\sqrt{q})$ with respect to the weight functions $w_U(x) = \sqrt{-\Delta}$ and $w_V(x) = 1/w_U(x)$, respectively. Using expressions (17c) and (17d), we may easily prove that u_n and v_n are also orthogonal over the same interval, but with respect to the weight functions $w_u(x) = \sqrt{-\Delta_u / \Delta}$, and $w_v(x) = 1/w_u(x)$, respectively.

4. Q-MATRIX AND FORMULAS FOR W_{nk-1} , W_{nk} AND W_{nk+1}

If we define the generating matrix Q to be

$$Q = \begin{bmatrix} x+p & -q \\ 1 & 0 \end{bmatrix}, \quad (19)$$

then it is straightforward to show by induction that

$$P = Q^k = \begin{bmatrix} U_{k+1} & -qU_k \\ U_k & -qU_{k-1} \end{bmatrix}. \quad (20)$$

The characteristic equation of P is given by

$$\lambda^2 - (U_{k+1} - qU_{k-1})\lambda + q(U_k^2 - U_{k+1}U_{k-1}) = 0.$$

Using relations (10) and (14a), we may reduce the above equation to

$$\lambda^2 - V_k\lambda + q^k = 0.$$

Hence, by the Cayley-Hamilton theorem, we have

$$P^2 = V_k P - q^k I. \quad (21)$$

Starting with (21), we may easily show by induction that

$$P^n(x) = \lambda_n(x)P(x) - q^k \lambda_{n-1}(x)I, \quad (22)$$

where $\lambda_n(x)$ satisfies the recurrence relation

$$\lambda_n(x) = V_k(x)\lambda_{n-1}(x) - q^k\lambda_{n-2}(x) \quad (n \geq 2), \quad \lambda_0 = 0, \lambda_1 = 1. \quad (23)$$

Hence, from (20) and (22), we have

$$Q^{nk}(x) = \lambda_n(x)Q^k(x) - q^k\lambda_{n-1}(x)I. \quad (24)$$

Therefore, we have

$$U_{nk}(x) = \lambda_n(x)U_k(x), \quad (25a)$$

$$U_{nk+1}(x) = \lambda_n(x)U_{k+1}(x) - q^k\lambda_{n-1}(x), \quad (25b)$$

and

$$U_{nk-1}(x) = \lambda_n(x)U_{k-1}(x) + q^{k-1}\lambda_{n-1}(x). \quad (25c)$$

We now derive similar results for the polynomial W , and thus for the polynomials V , u , and v . Consider the matrix

$$R = \begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix}.$$

Using relation (9), we may rewrite R as

$$\begin{aligned} R &= W_1 \begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix} - qW_0 \begin{bmatrix} U_{nk} & -qU_{nk-1} \\ U_{nk-1} & -qU_{nk-2} \end{bmatrix} \\ &= W_1 Q^{nk} - qW_0 Q^{nk-1}, \quad \text{using (20),} \\ &= Q^{nk} (W_1 I - qW_0 Q^{-1}). \end{aligned}$$

Hence,

$$\begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix} = \begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix} \begin{bmatrix} W_1 & -qW_0 \\ W_0 & W_1 - (x+p)W_0 \end{bmatrix}.$$

From the above identity, we may derive the following relations after some manipulations using (9) and (25a-25c):

$$W_{nk} = \lambda_n W_k - q^k W_0 \lambda_{n-1}, \quad (26a)$$

$$W_{nk+1} = \lambda_n W_{k+1} - q^k W_1 \lambda_{n-1}, \quad (26b)$$

$$W_{nk-1} = \lambda_n W_{k-1} + q^{k-1} \lambda_{n-1} \{W_1 - (x+p)W_0\}. \quad (26c)$$

Using appropriate values for W_0 and W_1 in (26a-26c), we may now derive the following relations for the polynomials V , u , and v :

$$V_{nk} = \lambda_n V_k - 2q^k \lambda_{n-1}, \quad (27a)$$

$$V_{nk+1} = \lambda_n V_{k+1} - q^k (x+p) \lambda_{n-1}, \quad (27b)$$

$$V_{nk-1} = \lambda_n V_{k-1} - q^{k-1} (x+p) \lambda_{n-1}; \quad (27c)$$

$$u_{nk} = \lambda_n u_k - q^k \lambda_{n-1}, \quad (28a)$$

$$u_{nk+1} = \lambda_n u_{k+1} - q^k (x+p - \sqrt{q}) \lambda_{n-1}, \quad (28b)$$

$$u_{nk-1} = \lambda_n u_{k-1} - q^{k-1/2} \lambda_{n-1}; \quad (28c)$$

$$v_{nk} = \lambda_n v_k - q^k \lambda_{n-1}, \tag{29a}$$

$$v_{nk+1} = \lambda_n v_{k+1} - q^k (x + p + \sqrt{q}) \lambda_{n-1}, \tag{29b}$$

$$v_{nk-1} = \lambda_n v_{k-1} + q^{k-1/2} \lambda_{n-1}. \tag{29c}$$

It is clear from (23) that, if $V_k | \lambda_{n-2}$, then $V_k | \lambda_n$ also. However, $V_k | \lambda_2$ since $\lambda_2 = V_k$. Hence, by induction, it follows that $V_k | \lambda_n$ when n is even. Thus, we see from (25a) that $V_k | U_{kn}$ for even n , while $U_k | U_{kn}$ for all n . Further, we see from (27a) that $V_k | V_{kn}$ for odd n . Thus, we have the following results:

$$U_k | U_{kn} \quad \text{for all } n; \tag{30a}$$

$$V_k | U_{kn} \quad \text{for even } n; \tag{30b}$$

$$V_k | V_{kn} \quad \text{for odd } n. \tag{30c}$$

It is evident that similar results hold for Fibonacci and Lucas polynomials, Pell and Pell-Lucas polynomials, etc., since these polynomials are special cases of U_n and V_n . In particular, for the Fibonacci, Lucas, Pell, and Pell-Lucas numbers $F_n, L_n, P_n,$ and Q_n , we obtain from (30) the already known results:

$$F_k | F_{kn}, \quad P_k | P_{kn}, \quad \text{for all } n; \tag{31a}$$

$$L_k | F_{kn}, \quad Q_k | P_{kn}, \quad \text{for even } n; \tag{31b}$$

$$L_k | L_{kn}, \quad Q_k | P_{kn}, \quad \text{for odd } n. \tag{31c}$$

5. SPECIAL CASE WHEN $q = 1$

This corresponds to a modified version of the Morgan-Voyce polynomials, where $x + 2$ is replaced by $x + p$ in the difference equations. We shall denote the modified Morgan-Voyce polynomials by $\tilde{B}_n(x), \tilde{b}_n(x), \tilde{C}_n(x),$ and $\tilde{c}_n(x)$, where

$$\tilde{B}_n(x) = U_{n+1}(p, 1; x), \tag{32a}$$

$$\tilde{C}_n(x) = V_n(p, 1; x), \tag{32b}$$

$$\tilde{b}_n(x) = u_n(p, 1; x), \tag{32c}$$

$$\tilde{c}_n(x) = v_n(p, 1; x). \tag{32d}$$

Hence, from (14a-14d), we have the "Simson" formulas:

$$\tilde{B}_{n+1} \tilde{B}_{n-1} - \tilde{B}_n^2 = -1; \tag{33a}$$

$$\tilde{C}_{n+1} \tilde{C}_{n-1} - \tilde{C}_n^2 = (x + p)^2 - 4 = \Delta = \Delta_b \Delta_c; \tag{33b}$$

$$\tilde{b}_{n+1} \tilde{b}_{n-1} - \tilde{b}_n^2 = x + p - 2 = \Delta_b; \tag{33c}$$

$$\tilde{c}_{n+1} \tilde{c}_{n-1} - \tilde{c}_n^2 = -(x + p + 2) = -\Delta_c. \tag{33d}$$

André-Jeannin [3] has shown that $\tilde{B}_n^{(k)}(x)$ and $\tilde{C}_n^{(k)}(x)$, $k = 0, 1, 2, \dots$, where k stands for the k^{th} derivative, satisfy the following second-order differential equations:

$$\tilde{B}_n^{(k)}(x): \Delta y'' + (2k+3)(x+p)y' + \{(k+1)^2 - (n+1)^2\}y = 0, \quad (34a)$$

$$\tilde{C}_n^{(k)}(x): \Delta y'' + (2k+1)(x+p)y' + (k^2 - n^2)y = 0, \quad (34b)$$

where

$$\Delta = (x+p)^2 - 4. \quad (34c)$$

We will now derive similar results for $\tilde{b}_n^{(k)}(x)$ and $\tilde{c}_n^{(k)}(x)$. It is already known (see [6]) that $b_n(x)$ satisfies the differential equation

$$x(x+4)b_n''(x) + 2(x+1)b_n'(x) - n(n+1)b_n(x) = 0. \quad (35)$$

Changing x to $x+p-2$ and noting that $\tilde{b}_n(x) = b_n(x+p-2)$, we find that equation (35) reduces to

$$\Delta \tilde{b}_n''(x) + 2(x+p-1)\tilde{b}_n'(x) - n(n+1)\tilde{b}_n(x) = 0, \quad (36)$$

where Δ is given by (34c). Differentiating (36) k times and using the Leibniz rule, we can show that $\tilde{b}_n^{(k)}(x)$ satisfies the differential equation

$$\tilde{b}_n^{(k)}(x): \Delta y'' + 2\{(k+1)(x+p)-1\}y' + \{k(k+1) - n(n+1)\}y = 0. \quad (37a)$$

Similarly, we can show that $\tilde{c}_n^{(k)}(x)$ satisfies the equation

$$\tilde{c}_n^{(k)}(x): \Delta y'' + 2\{(k+1)(x+p)+1\}y' + \{k(k+1) - n(n+1)\}y = 0. \quad (37b)$$

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