# THE GENERATION OF HIGHER-ORDER LINEAR RECURRENCES FROM SECOND-ORDER LINEAR RECURRENCES

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Let $\{u_n\}$ be a Lucas sequence of the first kind defined by the second-order recursion relation

$$u_{n+2} = au_{n+1} + bu_n,$$

where $a$ and $b$ are integers and $u_0 = 0$, $u_1 = 1$. By the Binet formulas

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

where $\alpha$ and $\beta$ are roots of the characteristic polynomial

$$x^2 - ax - b.$$

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of the characteristic polynomial of $\{u_n\}$. We shall prove the following theorem which demonstrates that the quotients of specified terms of the second-order recurrence $\{u_n\}$ satisfy a higher-order relation.

*Theorem 1:* Consider the sequence

$$\{w_n\}_{n=1}^{\infty} = \{u_{nk}/u_n\}_{n=1}^{\infty},$$

where $k$ is a fixed positive integer, $\alpha\beta \neq 0$, and $\alpha/\beta$ is not a root of unity. Then $\{w_n\}$ satisfies a $k^{\text{th}}$-order linear integral recursion relation. Further, the order $k$ is minimal.

Along the lines of this theorem, Selmer [1] has shown how one can form a higher-order linear recurrence consisting of the term-wise products of two other linear recurrences. In particular, let $\{s_n\}$ be an $m^{\text{th}}$-order and $\{t_n\}$ be a $p^{\text{th}}$-order linear integral recurrence with the associated polynomials $s(x)$ and $t(x)$, respectively. Let $\alpha_i$, $i = 1, 2, \ldots, m$, and $\beta_j$, $j = 1, 2, \ldots, p$, be the roots of the polynomials $s(x)$ and $t(x)$, respectively, and assume that each polynomial has no repeated roots. Then, the sequence

$$\{h_n\} = \{s_n t_n\}$$

satisfies a linear integral recurrence of order at most $mp$, whose characteristic polynomial $h(x)$ has roots consisting of the $r$ distinct elements of the set $\{\alpha_i\beta_j\}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$. Note that the coefficients of $h(x)$ are integral because they are symmetric in the conjugate algebraic integers $\alpha_i\beta_j$. However, $\{h_n\}$ may satisfy a recursion relation of order lower than $r$.

Selmer's proof depends on the fact that the recurrences $\{s_n\}$ and $\{t_n\}$ can be expressed in terms of their characteristic roots by means of the formulas

$$s_n = \sum_{i=1}^{m} \gamma_i \alpha_i^n, \quad t_n = \sum_{j=1}^{p} \delta_j \beta_j^n. \tag{1}$$

This follows from the fact that the sequences $\{\alpha_i^n\}$, $1 \leqslant i \leqslant m$, and $\{\beta_j^n\}$, $1 \leqslant j \leqslant p$, satisfy the same recursion relations as $\{s_n\}$ and $\{t_n\}$, respectively. Further, a linear combination of sequences satisfying the same linear recursion relation also satisfies that linear recursion relation. By means of Cramer's rule, one can then solve (1) for $s_n$, $1 \leqslant n \leqslant m$, and $t_n$, $1 \leqslant n \leqslant p$, in terms of $\alpha_i^n$, $1 \leqslant i \leqslant m$, and $\beta_j^n$, $1 \leqslant j \leqslant p$, respectively. The fact that the roots $\alpha_i$, $1 \leqslant i \leqslant m$, and $\beta_j$, $1 \leqslant j \leqslant p$, are distinct guarantees unique solutions in terms of $\alpha_i^n$ and $\beta_j^n$. Now,

$$h_n = s_n t_n = \left(\sum_{i=1}^{m} \gamma_i \alpha_i^n\right)\left(\sum_{j=1}^{p} \delta_j \beta_j^n\right) = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant p}} \gamma_i \delta_j (\alpha_i \beta_j)^n,$$

and each $\alpha_i \beta_j$ is a root of the polynomial $h(x)$.

Before proving our main result, we will need the following lemma. A proof of this lemma is given by Willett [2].

*Lemma 1:* Consider the sequence $\{s_n\}$. We introduce the determinant

$$D_r(t) = \begin{vmatrix} s_t & s_{t+1} & \cdots & s_{t+r-1} \\ s_{t+1} & s_{t+2} & & s_{t+r} \\ \cdots & \cdots & \cdots & \cdots \\ s_{t+r-1} & s_{t+r} & & s_{t+2r-2} \end{vmatrix}$$

Then $\{s_n\}$ satisfies a recursion relation of minimal order $k$ if and only if

$$D_k(0) \neq 0$$

and

$$D_r(0) = 0 \text{ for } r > k.$$

We are now ready for the proof of the main result. The first part of the proof will show that $\{w_n\}$ satisfies a $k^{\text{th}}$-order linear integral recursion relation. The second part of the proof will establish the minimality of $k$. The simple proof of minimality was suggested by Professor Ernst S. Selmer.

*Proof of Theorem 1:* First, we claim that $u_n \neq 0$ for $n \geqslant 1$ and $\{w_n\}$ is well-defined. If $u_n = 0$, then $\alpha^n - \beta^n = 0$ and $(\alpha/\beta)^n = 1$, since $\beta \neq 0$. This is impossible because $\alpha/\beta$ is not a root of unity. Note that

$$w_n = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n} \cdot \beta^{in}.$$

The $k$ algebraic integers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, are all distinct because $\alpha/\beta$ is not a root of unity. If $\alpha$ and $\beta$ are rational integers, then the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, certainly satisfy a monic polynomial of degree $k$ over the rational integers. If $\alpha$ and $\beta$ are irrational, then $\alpha$ and $\beta$ are conjugate in the algebraic number field $K = Q(\alpha, \beta) = Q(\alpha)$, where $Q$ denotes the rational numbers. Thus, $\alpha^{k-1-i}\beta^i$ and $\alpha^i\beta^{k-1-i}$ are conjugate in $K$. Hence, the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, satisfy a polynomial of degree $k$ which is a product of monic, integral quadratic polynomials and at most one monic, integral linear polynomial. By our discussion preceding the statement of Lemma 1, the sequences $\{(\alpha^{k-1-i}\beta^i)^n\}_{n=1}^{\infty}$, $0 \leqslant i \leqslant k-1$, all satisfy the same linear integral recursion relation of order $k$. Thus, $\{w_n\}_{n=1}^{\infty}$, the sum of these $k$ sequences, also satisfies this same recursion relation.

To prove the minimality of $k$, we first note that $\{w_n\}$ may also be defined for $n = 0$ if we put $w_0 = k$. Replacing $D_r(t)$ of Lemma 1 by $D_r(s_n, t)$, the minimality will follow if we can show that $D_k(w_n, 0) \neq 0$. To illustrate the method, let us take $k = 3$ as an example, when

$$D_k(w_n, 0) = \begin{vmatrix} 3 & \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 \\ \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 \\ \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 & \alpha^8 + \alpha^4\beta^4 + \beta^8 \end{vmatrix}.$$

The corresponding matrix may be written as the product

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha\beta & \beta^2 \\ \alpha^4 & \alpha^2\beta^2 & \beta^4 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{pmatrix}$$

Thus, $D_k(w_n, 0)$ is the square of a Vandermonde determinant:

$$D_k(w_n, 0) = \begin{vmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{vmatrix}^2 = [(\alpha\beta - \alpha^2)(\beta^2 - \alpha^2)(\beta^2 - \alpha\beta)]^2.$$

Since we assume $\alpha\beta \neq 0$ and $\alpha/\beta$ is not a root of unity, we have $D_k(w_n, 0) \neq 0$, as required.

In the general case, we similarly get

$$D_k(w_n, 0) = \begin{vmatrix} 1 & \alpha^{k-1} & (\alpha^{k-1})^2 & \cdots & (\alpha^{k-1})^{k-1} \\ 1 & \alpha^{k-2}\beta & (\alpha^{k-2}\beta)^2 & \cdots & (\alpha^{k-2}\beta)^{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \beta^{k-1} & (\beta^{k-1})^2 & \cdots & (\beta^{k-1})^{k-1} \end{vmatrix}^2 \neq 0,$$

and the proof of the minimality is completed.

As a final remark, we note the condition for $\alpha/\beta$ not to be a root of unity. When $\alpha\beta = -b \neq 0$, then $z = \alpha/\beta$ is the root of a quadratic equation

$$p(z) = z^2 + \left(\frac{a^2}{b} + 2\right)z + 1 = 0.$$

If $\alpha/\beta$ shall not be a root of unity, we must have $z \neq \pm 1$, and $p(z)$ cannot be one of the quadratic cyclotomic polynomials $z^2 + 1$, $z^2 \pm z + 1$. Hence, we must demand that

$$\frac{a^2}{b} + 2 \neq \pm 2, 0, \pm 1 \quad \text{or} \quad -a^2 \neq 0, b, 2b, 3b, 4b.$$

## REFERENCES

1. E. S. Selmer. "Linear Recurrence Relations over Finite Fields." Lecture notes; Department of Mathematics, University of Bergen, Norway, 1966.
2. M. Willett. "On a Theorem of Kronecker." *The Fibonacci Quarterly* 14, no 1 (1976):27-29.

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