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ON SOME PROPERTIES OF A CLASS OF POLYNOMIALS SUGGESTED BY MITTAL

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Abstract

The object of this paper is to establish some generating relations by using operational formulae for a class of polynomials $T_{kn}^{(\alpha+s-1)}(x)$ *defined by Mittal. We have also derived finite summation formulae for (1.6) by employing operational techniques. In the end several special cases are discussed.*

Key Words : *Operational formulae; generating relations; finite sum formulae.*

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1. Introduction

Chak [1] defined a class of polynomials as:

$$G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n}e^x(x^kD)^n[x^\alpha e^{-x}] \tag{1.1}$$

where $D = \frac{d}{dx}$, k is constant and $n = 0, 1, 2, \ldots$.

Chatterjea [2] studied a class of polynomials for generalized Laguerre polynomial as:

$$T_{rn}^{(\alpha)}(x,p) = \frac{1}{n!}x^{-\alpha-n-1}\exp(px^r)(x^2D)^n[x^{\alpha+1}\exp(-px^r)]. \tag{1.2}$$

Gould and Hopper [3] introduced generalized Hermite polynomials as:

$$H_n^r(x,a,p) = (-1)^n x^{-a}\exp(px^r)D^n[x^a\exp(-px^r)]. \tag{1.3}$$

Singh [10] obtained generalized Truesdell polynomials by using Rodrigues formula, which is defined as:

$$T_n^{(\alpha)}(x,r,p) = x^{-\alpha}\exp(px^r)(xD)^n[x^\alpha\exp(-px^r)]. \tag{1.4}$$

In 1971, Mittal [5] proved the Rodrigues formula for a class of polynomials $T_{kn}^{(\alpha)}(x)$ as:

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}\exp\{p_k(x)\}D^n[x^{\alpha+n}\exp\{-p_k(x)\}] \tag{1.5}$$

where $p_k(x)$ is a polynomial in x of degree k.

Mittal [6] also proved the following relation for (1.5)

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!}\,x^{-\alpha-n}\exp\{p_k(x)\}\theta^n[x^\alpha\exp\{-p_k(x)\}] \tag{1.6}$$

and an operator $\theta \equiv x(s + xD)$, where s is constant.

The following well-known facts are prepared for studying (1.6).

Generalised Laguerre polynomials (Srivastava and Manocha[12]) defined as:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha-n-1}\,e^x}{n!}\,(x^2D)^n[x^{\alpha+1}\,e^{-x}]. \tag{1.7}$$

Hermite polynomials (Rainville [9]) defined as:

$$H_n(x) \ = \ (-1)^n \exp(x^2) \ D^n[\exp(-x^2)]. \tag{1.8}$$

Konhauser polynomials of first kind (Srivastava [11]) defined as:

$$Y_n^\alpha(x;k) \ = \ \frac{x^{-kn-\alpha-1} \ e^x}{k^n \ n!} \ (x^{k+1}D)^n[x^{\alpha+1} \ e^{-x}]. \tag{1.9}$$

Konhauser polynomials of second kind (Srivastava [11]) defined as:

$$Z_n^\alpha(x;k) \ = \ \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \tag{1.10}$$

where k is a positive integer.

Srivastava and Manocha [12] verified following result by using induction method,

$$(x^2D)^n\{f(x)\} \ = \ x^{n+1}D^n\{x^{n-1}f(x)\}. \tag{1.11}$$

2. Definitions and Notations

McBride [4] defined generating function as:

Let $G(x,t)$ be a function that can be expanded in powers of t such that

$G(x,t) = \sum\limits_{n=0}^{\infty} c_n f_n(x) t^n$, where c_n is a function of n that may contain the parameters of the set $\{f_n(x)\}$, but is independent of x and t. Then $G(x,t)$ is called a generating function of the set $\{f_n(x)\}$.

Remark: A set of functions may have more than one generating function.

In our investigation we used the following properties of the differential operators;

$\theta \equiv x(s+xD)$ and $\theta_1 \equiv (1+xD)$, where $D \equiv \dfrac{d}{dx}$, (Mittal [7], Patil and Thakare [8]) which are useful to establish linear generating relations and finite sum formulae.

$$(2.1)\quad \theta^n = x^n(s+xD)(s+1+xD)(s+2+xD)\dots(s+(n-1)+xD)$$

$$(2.2)\quad \theta^n(x^\alpha) = (\alpha+s)_n\, x^{\alpha+n}$$

$$(2.3)\quad \theta^n(xuv) = x\sum_{m=0}^{\infty}\begin{pmatrix} n \\ m \end{pmatrix}\theta^{n-m}(v)\theta_1^m(u)$$

$$(2.4)\quad e^{t\theta}(x^\alpha) = x^\alpha(1-xt)^{-(\alpha+s)}$$

$$(2.5)\quad e^{t\theta}(xuv) = xe^{t\theta}(v)e^{t\theta_1}(u)$$

$$(2.6)\quad e^{t\theta}(x^\alpha f(x)) = x^\alpha(1-xt)^{-(\alpha+s)}\, f\Big[x(1-xt)^{-1}\Big]$$

$$(2.7)\quad e^{t\theta}(x^{\alpha-n} f(x)) = x^\alpha(1+t)^{-1+(\alpha+s)}\, f\Big[x(1+t)\Big]$$

$$(2.8)\quad (1-at)^{-\alpha/a} = (1-at)^{-\beta/a}\sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(at)^m}{m!}$$

3. Generating Relations

We obtained some generating relations of (1.6) as

$$(3.1)\quad \sum_{n=0}^{\infty} T_{kn}^{(\alpha+s-1)}(x)t^n = (1-t)^{-(\alpha+s)}\exp[p_k(x)-p_k\{x(1-t)^{-1}\}]$$

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x)t^n = (1+t)^{-1+(\alpha+s)} \exp[p_k(x) - p_k\{x(1+t)\}]$$

(3.2)

$$\sum_{m=0}^{\infty} \binom{m+n}{n} T_{k(n+m)}^{(\alpha+s-1)}(x)\, t^m$$
$$= (1-t)^{-(\alpha+s+n)} \exp\left[p_k(x) - p_k\left\{x(1-t)^{-1}\right\}\right] T_{kn}^{(\alpha+s-1)}\left\{x(1-t)^{-1}\right\}$$

(3.3)

$$\sum_{m=0}^{\infty} \binom{m+n}{n} T_{k(n+m)}^{(\alpha-m+s-1)}(x)\, t^m$$
$$= (1+t)^{\alpha+s-1} \exp\left[p_k(x) - p_k\{x(1+t)\}\right] T_{kn}^{(\alpha-m+s-1)}\{x(1+t)\}$$

(3.4)

Proof of (3.1). From (1.6), we consider

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha+s-1)}(x)t^n = x^{-\alpha} \exp\{p_k(x)\}\, e^{t\theta}\, [x^{\alpha} \exp\{-p_k(x)\}]$$

and using (2.6), above equation reduces to,

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha-s+1)}(x)t^n = x^{-\alpha} \exp\{p_k(x)\}\, x^{\alpha}\, (1-xt)^{-(\alpha+s)} \exp[-p_k\{x(1-xt)^{-1}\}]$$

$$= (1-xt)^{-(\alpha+s)} \exp[p_k(x) - p_k\{x(1-xt)^{-1}\}]$$

replacing t by t/x, which gives (3.1).

Proof of (3.2). From (1.6) we consider,

$$T_{kn}^{(\alpha-n+s-1)}(x) = \frac{1}{n!}\, x^{-(\alpha-n)-n} \exp\{p_k(x)\}\, \theta^n \,[x^{\alpha-n} \exp\{-p_k(x)\}]$$

or

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x) t^n = (x)^{-\alpha} \exp\{p_k(x)\}\, e^{t\theta}\, [x^{\alpha-n} \exp(-p_k(x))]$$

by using (2.7), we get

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n+s-1)}(x) t^n = x^{-\alpha} \exp\{p_k(x)\}\, x^{\alpha} (1+t)^{-1+(\alpha+s)} \exp\{-p_k\{x(1+t)\}]$$

$$= (1+t)^{-1+(\alpha+s)} \exp[p_k(x) - p_k\{x(1+t)\}].$$

Proof of (3.3). Again from (1.6) we consider,

$$\theta^n [x^{\alpha} \exp\{-p_k(x)\}] = n!\; x^{\alpha+n} \exp\{-p_k(x)\}\, T_{kn}^{(\alpha+s-1)}(x)$$

or

$$e^{t\theta}(\theta^n [x^{\alpha} \exp\{-p_k(x)\}]) = n!\; e^{t\theta}\, [x^{\alpha+n} \exp\{-p_k(x)\}\, T_{kn}^{(\alpha+s-1)}(x)]$$

using (2.6) we get,

$$\sum_{m=0}^{\infty} \frac{t^m\; \theta^{m+n}}{m!}\, [x^{\alpha} \exp\{-p_k(x)\}]$$

$$= n!\; x^{\alpha+n} (1-xt)^{-(\alpha+s+n)} \exp[-p_k\{x(1-xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1-xt)^{-1}\}$$

therefore, we get

$$\sum_{m=0}^{\infty} \frac{1}{m!\ n!}(m+n)!\, x^{\alpha+m+n}\ \exp\{-p_k(x)\}\, T_{k(m+n)}^{(\alpha+s-1)}(x) t^m$$

$$= \ x^{\alpha+n}(1-xt)^{-(\alpha+s+n)}\ \exp[-p_k\{x(1-xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1-xt)^{-1}\}$$

hence above equation reduces to,

$$\sum_{m=0}^{\infty} x^m \begin{pmatrix} m+n \\ n \end{pmatrix} T_{k(m+n)}^{(\alpha+s-1)}(x) t^m$$

$$= \ (1-xt)^{-(\alpha+s+n)}\ \exp[p_k(x) - p_k\{x(1-xt)^{-1}\}] T_{kn}^{(\alpha+s-1)}\{x(1-xt)^{-1}\}$$

replacing t by t/x, which gives (3.3).

Proof of (3.4). Again from (1.6) we consider,

$$\theta^n[x^\alpha \exp\{-p_k(x)\}] \ = \ n!\ x^{\alpha+n} \exp\{-p_k(x)\}\, T_{kn}^{(\alpha+s-1)}(x)$$

replacing α by $\alpha - m$, we get

$$\theta^n[x^{\alpha-m} \exp\{-p_k(x)\}] \ = \ n!\, x^{\alpha-m+n}\ \exp\{-p_k(x)\}\, T_{kn}^{(\alpha-m+s-1)}(x)$$

or

$$e^{t\theta}(\theta^n[x^{\alpha-m} E_\alpha\{-p_k(x)\}]) \ = \ n!\, e^{t\theta}[x^{(\alpha+n)-m}\ \exp\{-p_k(x)\}\, T_{kn}^{(\alpha-m+s-1)}(x)]$$

using (2.7) we get,

$$\sum_{n=0}^{\infty} \frac{t^m\ \theta^{m+n}}{m!}\, [x^{\alpha-m}\ \exp\{-p_k(x)\}]$$

$$= \ n!\ x^{\alpha+n}(1+t)^{\alpha+s-1} \exp[-p_k\{x(1+t)\}] T_{kn}^{(\alpha-m+s-1)}\{x(1+t)\}$$

therefore, we get

$$\sum_{m=0}^{\infty} \frac{1}{m!\ n!}\,(m+n)!\,x^{\alpha-m+m+n}\ \exp\{-p_k(x)\}\,T_{k(m+n)}^{(\alpha-m+s-1)}(x)t^m$$

$$= \ x^{\alpha+n}(1+t)^{\alpha+s-1}\ \exp[-p_k\{x(1+t)\}]T_{kn}^{(\alpha-m+s-1)}\{x(1+t)\}$$

which reduces to (3.4).

4. Finite Summation Formulae

We obtained finite summation formula for (1.6) as

(4.1) $$T_{kn}^{(\alpha+s-1)}(x) \ = \ \sum_{m=0}^{n}(m!)^{-1}\,(\alpha-\beta)_m\ T_{k(n-m)}^{(\beta+s-1)}(x)$$

(4.2) $$T_{kn}^{(\alpha+s-1)}(x) \ = \ \sum_{m=0}^{n}\frac{1}{m!}\ (\alpha)_m\ T_{k(n-m)}^{(s-1)}(x)$$

Proof of (4.1). We can write (1.6) as,

$$\sum_{n=0}^{\infty} x^n\,T_{kn}^{(\alpha+s-1)}(x)t^n \ = \ x^{-\alpha}\exp\{p_k(x)\}\,e^{t\theta}[x^{\alpha}\exp\{-p_k(x)\}]$$

by using (2.6), we write

$$\sum_{n=0}^{\infty} x^n T_{kn}^{(\alpha+s-1)}\,(x)\,t^n$$

$$= \ x^{-\alpha}\exp\{p_k(x)\}\,x^{\alpha}(1-xt)^{-(\alpha+s)}\exp[-p_k\{x(1-xt)^{-1}\}]$$

$$= \ (1-xt)^{-(\alpha+s)}\exp[p_k(x)-p_k\{x(1-xt)^{-1}\}]$$

applying (2.8), which yields

$$\sum_{n=0}^{\infty} x^n\,T_{kn}^{(\alpha+s-1)}(x)t^n$$

$$= (1-xt)^{-(\beta+s)} \sum_{m=0}^{\infty} (\alpha-\beta)_m \frac{(xt)^m}{m!} \exp[p_k(x) - p_k\{x(1-xt)^{-1}\}]$$

$$= \sum_{n=0}^{\infty} (\alpha-\beta)_m \frac{x^m\, t^m}{m!} \exp\{p_k(x)\}(1-xt)^{-(\beta+s)} \exp[-p_k\{x(1-xt)^{-1}\}]$$

using (3.1), above equation reduces to,

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha+s-1)}(x) t^n =$$

$$= \sum_{m=0}^{\infty} (\alpha-\beta)_m \frac{x^m\ t^m}{m!} \exp\{p_k(x)\} x^{-\beta}\, e^{t\theta} [x^{\beta} \exp\{-p_k(x)\}]$$

$$= \sum_{m,n=0}^{\infty} (\alpha-\beta)_m \frac{x^m\ t^{n+m}}{m!\ n!} \exp\{p_k(x)\} x^{-\beta}\, \theta^n [x^{\beta} \exp(-p_k(x))]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!} (\alpha-\beta)_m \frac{x^{-\beta+m}}{(n-m)!} \exp\{p_k(x)\}\, \theta^{n-m} [x^{\beta} \exp\{-p_k(x)\}] t^n$$

equating the coefficients of t^n, we get

$$x^n T_{kn}^{(\alpha+s-1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\alpha-\beta)_m \frac{x^{-\beta+m}}{(n-m)!} \exp\{p_k(x)\}\, \theta^{n-m} [x^{\beta} \exp\{-p_k(x)\}]$$

Therefore, we obtain

$$T_{kn}^{(\alpha+s-1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\alpha-\beta)_m \frac{x^{-\beta(-n-m)}}{(n-m)!} \exp\{p_k(x)\}\, \theta^{n-m} [x^{\beta} \exp\{-p_k(x)\}]$$

and applying (1.6) then above equation immediately leads to (4.1).

Proof of (4.2). We can write (1.6) as,

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} x^{-\alpha-n} \exp\{p_k(x)\}\, \theta^n [x x^{\alpha-1} \exp\{-p_k(x)\}]$$

using (2.3) we get,

and by using (2.1) which yields,

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!}\, x^{-\alpha-n} \exp\{p_k(x)\}\, x \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}$$

$$\times\, x^{n-m}[(s+xD)(s+1+xD)(s+2+xD)\ldots(s+(n-m-1)+xD)]\exp\{-p_k(x)\}$$

$$\times\, x^{m}[(1+xD)(2+xD)(3+xD)\ldots(m+xD)]x^{\alpha-1}$$

$$T_{kn}^{(\alpha+s-1)}(x) = \exp\{p_k(x)\} \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!} \prod_{i=0}^{n-m-1} (s+i+xD)\exp\{-p_k(x)\}(\alpha)_m$$

(4.3)

Putting $\alpha = 0$ and replacing n by $n-m$ in (1.6) which reduces to

$$T_{k(n-m)}^{(s-1)}(x) = \frac{1}{(n-m)!}\, x^{-(n-m)} \exp\{p_k(x)\}\, \theta^{n-m}[\exp\{-p_k(x)\}]$$

thus, we have

$$\frac{1}{(n-m)!}\, \theta^{n-m}[\exp\{-p_k(x)\}] = \frac{x^{n-m}}{\exp\{p_k(x)\}}\, T_{k(n-m)}^{(s-1)}(x)$$

using (2.1), we get

$$\frac{1}{(n-m)!} \prod_{i=0}^{n-m-1} (s+i+xD)[\exp\{-p_k(x)\}] = \frac{1}{\exp\{p_k(x)\}}\, T_{k(n-m)}^{(s-1)}(x).$$

(4.4)

use of (4.4) and (4.3), gives complete proof of (4.2).

5. Concluding Remarks

Some special cases of $T_{kn}^{(\alpha+s-1)}(x)$ polynomials are given below:

If we replace α by $\alpha+1$, $p_k(x) = p_1(x) = x$ and $s = 0$ in (1.6), then this equation reduces to

$$T_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) = Z_n^{\alpha}(x;1) = Y_n^{\alpha}(x;1). \tag{5.1}$$

Again replacing α by $\alpha+1$, $p_k(x) = px^r$ and $s = 0$ in (1.6), which gives

$$T_{rn}^{(\alpha)}(x) = T_{rn}^{(\alpha)}(x,p). \tag{5.2}$$

Substituting $\alpha = 1-n$, $p_k(x) = x^2$, $s = 0$ in (1.6) and using (1.11), which yields

$$T_{2n}^{(1-n)}(x) = \frac{(-x)^n}{n!} H_n(x). \tag{5.3}$$

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References

[1] A. M.Chak, *A class of polynomials and generalization of stirling numbers*, Duke J. Math. **23** pp. 45–55, (1956).

[2] S. K. Charttejea, *On generalization of Laguerre polynomials*, Rend. Mat. Univ. Padova **34**, pp. 180–190, (1964).

[3] H. W. Gould and A. T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. J. **29**, pp. 51–63, (1962).

[4] E. B. McBride, *Obtaining Generating Functions*, Springer-Verlag New York, (1971).

[5] H. B. Mittal, *A generalization of Laguerre polynomials*, Publ. Math. Debrecen, **18**, pp. 53–88, (1971).

[6] H. B. Mittal, *Operational representations for the generalized Laggure polynomial*, Glasnik Mat. Ser III, **6**(26), pp. 45–53, (1971).

[7] H. B. Mittal, *Bilinear and bilateral generating relations*, American J. Math. **99**, pp. 23–45, (1977).

[8] K. R. Patil and N.K. Thakre, *Operational formulas for a function defined by a generalized Rodrigues formula-II*, Sci. J. Shivaji Univ. **15**, pp. 1–10, (1975).

[9] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, (1960).

[10] R. P. Singh, *On generalized Truesdell polynomials*, Rivista de Mathematica, **8**, pp. 345–353, (1968).

[11] H. M. Srivastava, *Some biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math. **98**(1), pp. 235–250, (1982).

[12] H. M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Limited, Chichester, (1984).

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