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EXPLICIT EXPRESSIONS FOR POWERS OF LINEAR RECURSIVE SEQUENCES

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1. DEFINITIONS

Van der Poorten [6] in a generalization of a result of Shannon and Horadam [8] has shown that (in my notation) if $\{w_n^{(i)}\}$ is a linear recursive sequence of orbitrary order i defined by the recurrence relation

(1.1) $$w_n^{(i)} = \sum_{j=1}^{i} P_{ij} w_{n-j}^{(i)}, \quad n \geq i,$$

where the P_{ij} are arbitrary integers, with suitable initial values $w_0^{(i)}, w_1^{(i)}, \cdots, w_{i-1}^{(i)}$, then the sequence of powers $\{w_n^{(i)r}\}$, for integers $r \geq 1$, satisfies a similar recurrence relation of order at most

$$\binom{r+i-1}{r}.$$

In other words, he has established the existence of generating functions

(1.2) $$w_r^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)r} x^n, \quad (w_n^{(i)r} \equiv (w_n^{(i)})^r).$$

The aim here is to find the recurrence relation for $\{w_n^{(i)r}\}$ and an explicit expression for $w_r^{(i)}(x)$. We shall concern ourselves with the non-degenerate case only; the degenerate case is no more difficult because the order of the recurrence relation for $\{w_n^{(i)r}\}$ is then lower than

$$\binom{r+i-1}{r}.$$

It is worth noting in passing that Marshall Hall [1] looked at the divisibility properites of a third-order sequence by a similar approach. From a second-order sequence with auxiliary equation roots α_1 and α_2 he formed a third-order sequence with auxiliary equation roots $\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2$.

2. RECURRENCE RELATION FOR SEQUENCE OF POWERS

Van der Poorten proved that if the auxiliary equation for $\{w_n^{(i)}\}$ is

(2.1) $$g(x) \equiv x^i - \sum_{j=1}^{i} P_{ij} x^{i-j} = \prod_{t=1}^{i} (x - \alpha_{it}) = 0,$$

then the sequence $\{w_n^{(i)r}\}$ satisfies a linear recurrence relation of order

$$\binom{r+i-1}{r}$$

with auxiliary equation

(2.2) $$g_r(x) \equiv \prod_{\Sigma\lambda_n = r} (x - \alpha_{i1}^{\lambda_1}\alpha_{i2}^{\lambda_2} \cdots \alpha_{ii}^{\lambda_i}) = 0,$$

the zeros of which are exactly the zeros of $g(x)$ taken r at a time.

We now set

(2.3) $$g_r(x) = x^u - \sum_{j=1}^{u} R_{uj}x^{u-j}, \qquad u = \binom{r+i-1}{r},$$

and we seek the R_{uj}.

Macmahon [5, p. 3] defines h_j, the homogeneous product sum of weight j of the quantities a_{jr}, as the sum of a number of symmetric functions, each of which is denoted by a partition of the number j. He showed that in our notation

$$h_j = \sum_{\Sigma n\lambda_n = j} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_j!} P_{i1}^{\lambda_1} P_{i2}^{\lambda_2} \cdots P_{ij}^{\lambda_j}.$$

The first three cases of h_j are

$$h_1 = P_{i1} = \Sigma a_{i1},$$

$$h_2 = P_{i1}^2 + P_{i2} = \Sigma a_{i1}^2 + \Sigma a_{i1}a_{i2},$$

$$h_3 = P_{i1}^3 + 2P_{i1}P_{i2} + P_{i3} = \Sigma a_{i1}^3 + \Sigma a_{i1}^2 a_{i2} + \Sigma a_{i1}a_{i2}a_{i3}.$$

Now $g_r(x) = 0$ is the equation whose zeros are the several terms of h_r with $a_{ij} = 0$ for $j > i$, since from its construction its zeros are a_{ij} taken r at a time; that is,

$$R_{u1} = h_r \quad \text{with} \quad a_{ij} = 0 \quad \text{for} \quad j > i,$$

since we have supposed that there are

$$\binom{r+i-1}{r} = u$$

distinct zeros of $g_r(x) = 0$.

Macmahon has proved [5, p. 19] that $H_{\substack{r\\j}}$ the homogeneous product sum, j together, of the whole of the terms of h_r, can be represented in terms of the symmetric functions (denoted by [ ]) of the roots of

$$x^i - h_1x^{i-1} + h_2x^{i-2} - \cdots = 0$$

by

(2.4) $$H_{\substack{r\\j}} = \sum_{\Sigma n\mu_n = j} (-1)^{r(3\mu_2+5\mu_4+\cdots)} \frac{[1^r]^{\mu_1}[2^r]^{\mu_2}[3^r]^{\mu_3}\cdots}{1^{\mu_1}\cdot 2^{\mu_2}\cdot 3^{\mu_3}\cdots \mu_1!\mu_2!\mu_3!\cdots}.$$

Some examples of $H_{\substack{r\\j}}$ are (with $a_{ij} = 0$ for $j > i$)

$$H_{\substack{2\\1}} = a_{21}^2 + a_{22}^2 + a_{21}a_{22},$$

$$H_{\substack{2\\2}} = a_{21}^4 + a_{22}^4 + 2a_{21}^2a_{22}^2 + a_{21}^3a_{22} + a_{21}a_{22}^3,$$

$$H_{\substack{2\\3}} = a_{21}^6 + a_{22}^6 + 2a_{21}^3a_{22}^3 + a_{21}^5a_{22} + a_{21}a_{22}^5 + 2a_{21}a_{22}^2 + 2a_{21}^2a_{22}^4,$$

$$H_{\substack{2\\1}}^2 = H_{\substack{2\\1}}H_{\substack{2\\1}} = a_{21}^4 + a_{22}^4 + 3a_{21}^2a_{22}^2 + 2a_{21}^3a_{22} + 2a_{21}a_{22}^3,$$

$$H_{\substack{2\\1}}^3 = a_{21}^6 + a_{22}^6 + 7a_{21}^3a_{22}^3 + 3a_{21}^5a_{22} + 3a_{21}a_{22}^5 + 6a_{21}^4a_{22}^2 + 6a_{21}^2a_{22}^4.$$

h_m is the homogeneous product sum of weight m of the terms of P_{i1}. $H_{\substack{r\\m}}$ is the homogeneous product sum of weight m of the terms of R_{u1}.

$(-1)^{j+1}P_{ij}$ is the product sum, j together, of the terms of P_{i1}.

$(-1)^{j+1} R_{uj}$ is the product sum, j together, of the terms of R_{u1}. It follows directly from Macmahon [5, p. 4] that

$$P_{ij} = \sum_{\Sigma n\lambda_n = j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_j!} \prod_{m=1}^{j} h_m^{\lambda_m} \quad ,$$

and so

$$R_{uj} = \sum_{\Sigma n\lambda_n = j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_j!} \prod_{m=1}^{j} H_{r_m}^{\lambda_m} \quad .$$

For example,

$$R_{31} = H_{2_1} = a_{21}^2 + a_{22}^2 + a_{21}a_{22} \quad ,$$

$$\begin{aligned} R_{32} = -H_{2_1}^2 + H_{2_2} &= -(\Sigma a_{21}^4 + 2\Sigma a_{21}^3 a_{22} + 3a_{21}^2 a_{22}^2) \\ &\quad + (\Sigma a_{21}^4 + \Sigma a_{21}^3 a_{22} + 2a_{21}^2 a_{22}^2) \\ &= -\Sigma a_{21}^3 a_{22} - \Sigma a_{21}^2 a_{22}^2 \quad , \end{aligned}$$

$$R_{33} = H_{2_1}^3 + H_{2_3} - 2H_{2_2}H_{2_1} = a_{21}^3 a_{22}^3 \quad .$$

We can verify these results by utilizing some of the properties of the generalized sequence of numbers $\{w_n^{(2)}\}$ developed by Horadam [3].

From Eq. (27) of Horadam's paper we have that

(2.5)
$$w_n^{(2)} w_{n-2}^{(2)} - w_{n-1}^{(2)^2} = (-P_{22})^{n-2} e \quad ,$$

where

$$e = P_{21} w_0^{(2)} w_1^{(2)} + P_{22} w_0^{(2)^2} - w_1^{(2)^2} \quad .$$

Thus

$$w_{n-1}^{(2)} w_{n-3}^{(2)} - w_{n-2}^{(2)^2} = (-P_{22})^{n-3} e$$

and

(2.6)
$$P_{22} w_{n-2}^{(2)^2} - P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = (-P_{22})^{n-2} e.$$

Subtracting (2.5) from (2.6), we get

(2.7)
$$P_{22} w_{n-2}^{(2)} + w_{n-1}^{(2)} = P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)} \quad .$$

But

$$w_n^{(2)} - P_{22} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)} \quad ,$$

and

$$w_{n-1}^{(2)} - P_{22} w_{n-3}^{(2)} = P_{21} w_{n-2}^{(2)} \quad ,$$

so

$$w_n^{(2)} + P_{22}^2 w_{n-2}^{(2)^2} - 2P_{22} w_n^{(2)} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)^2} \quad ,$$

and

$$P_{22} w_{n-1}^{(2)^2} + P_{22} w_{n-3}^{(2)^2} - 2P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = P_{21}^2 P_{22} w_{n-2}^{(2)^2} \quad .$$

Adding the last two equations we obtain

$$w_n^{(2)^2} + P_{22} w_{n-1}^{(2)^2} + P_{22}^2 w_{n-2}^{(2)^2} + P_{22}^3 w_{n-3}^{(2)^2} - 2P_{22}(P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)^2}) = P_{21}^2 w_{n-1}^{(2)^2} + P_{21}^2 P_{22} w_{n-2}^{(2)^2} \quad .$$

Combining this with (2.7) we then have

(2.8) $$w_n^{(2)^2} = (P_{21}^2 + P_{22})w_{n-1}^{(2)^2} + (P_{22}^2 + P_{21}^2 P_{22})w_{n-2}^{(2)^2} + (-P_{22}^3)w_{n-3}^{(2)^2},$$

so

$$R_{31} = P_{21}^2 + P_{22} = a_{21}^2 + a_{22}^2 + a_{21}a_{22},$$

$$R_{32} = P_{22} + P_{21}P_{22} = -a_{21}^3 a_{22} - a_{21}a_{22}^3 - a_{21}^2 a_{22}^2,$$

$$R_{33} = -P_{22}^3 = a_{21}^3 a_{22}^3,$$

as required.

To obtain an expression for $\underset{m}{H_r}$ in terms of a_{ij}, we now use a result of Macmahon, namely,

$$[u^r] = (-1)^{r(u+1)}\sigma_u,$$

where σ_u denotes the sum of the u^{th} powers of the roots of $g_r(x) = 0$. It is sufficient for our purposes to state that Macmahon has shown that σ_u is the homogeneous product sum of order r of the quantities a_{ij}^u. It is thus given by

$$\sigma_u = \sum_{\Sigma t = r} \prod_m a_{im}^{ut_m}$$

by analogy with

$$h_r = \sum_{\Sigma t = r} \prod_m a_{im}^{t_m},$$

the homogeneous product sum of order r of the quantities a_{ij}. We now define σ_{iu}, the homogeneous product sum of order r of the quantities a_{ij}^u such that $a_{ij} = 0$ for $j > i$:

$$\sigma_{iu} = \sum_{\Sigma v = r} \prod_{j=1}^{i} a_{ij}^{uv_j},$$

and we introduce the term

$$\sigma_{iur} = (-1)^{r(u+1)}\sigma_{iu}.$$

We have thus established that for

$$w_n^{(i)^r} = \sum_{j=1}^{u} R_{uj} w_{n-j}^{(i)^r},$$

(2.9) $$R_{uj} = \sum_{\Sigma n\lambda_n = j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_j!} \prod_{m=1}^{j} \underset{m}{H_r}^{\lambda_m},$$

where

$$\underset{m}{H_r} = \sum_{\Sigma n\mu_n = m} (-1)^{r(3\mu_2 + 5\mu_4 + \cdots)} \prod_{v=1}^{m} \frac{(\sigma_{iur})^{\mu_v}}{v^{\mu_v}\cdot\mu_v},$$

and

$$\sigma_{iur} = (-1)^{r(u+1)} \sum_{\Sigma v = r} \prod_{j=1}^{i} a_{ij}^{uv_j},$$

and

$$u = \binom{i+r-1}{r}.$$

It is of interest to note that another formula for σ_{iur} can be given by

(2.9) $$\sigma_{iur} = (-1)^{r(u+1)} \sum_{j=1}^{i} a_{ij}^{(i+r-1)} \Big/ \prod_{j>k} (a_{ij}^u - a_{ik}^u).$$

We prove this by noting that

$$\sigma_{iu} = \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{uv_j} = (-1)^{r(u+1)}\sigma_{iur}$$

and defining

$$h'_r = \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{v_j}$$

and showing that

$$h'_r = \sum_{j=1}^{i} a_{ij}^{i+r-1} \Big/ \prod_{j>k} (a_{ij} - a_{ik}) .$$

It follows from Macmahon [5, p. 4] that h'_r satisfies a linear recurrence relation of order i given by

$$\begin{aligned} h'_r &= \sum_{n=1}^{i} P_{in} h'_{r-n}, & r > 0, \\ h'_r &= 1, & r = 0 \\ h'_r &= 0, & r < 0; \end{aligned}$$

the P_{ir} and a_{ir} are those of (2.1). We again assume that the a_{ir} are distinct so that from Jarden [4, p. 107]

(2.10) $$h'_r = \sum_{j=1}^{i} a_{ij}^r D_j / D ,$$

where D is the Vandermonde of the roots, given by

(2.11) $$D = \sum_{j=1}^{i} a_{ij}^{i-1} \prod_{\substack{j\neq n\neq m \\ n<m}} (a_{im} - a_{in}) = \prod_{j>n} (a_{ij} - a_{in}) \prod_{\substack{j\neq n\neq m \\ n<m}} (a_{im} - a_{in})$$

and D_j is the determinant of order i obtained from D on replacing its j^{th} column by the initial terms of the sequence, $h'_0, h'_1, \cdots h'_{i-1}$. It thus remains to prove that

(2.12) $$D_j = a_{ij}^{i-1} \prod_{\substack{j\neq n\neq m \\ n<m}} (a_{im} - a_{in}) = D a_{ij}^{i-1} \Big/ \prod_{j>n} (a_{ij} - a_{in}) .$$

We use the method of the contrapositive. If

$$D_j \neq a_{ij}^{i-1} \prod_{\substack{j\neq n\neq m \\ m>n}} (a_{im} - a_{in}) ,$$

then

$$D = \sum_{j=1}^{i} D_j$$

(from (2.10) with $n = 0$)

$$\neq \sum_{j=1}^{i} a_{ij}^{i-1} \prod_{\substack{j\neq n\neq m \\ m>n}} (a_{im} - a_{in})$$

which contradicts (2.11). This proves (2.12) and we have established that

$$h'_r = \sum_{j=1}^{i} a_{ij}^r D_j/D = \sum_{j=1}^{i} a_{ij}^{i+n-1} D_j/Da_{ij}^{i-1} = \sum_{j=1}^{i} a_{ij}^{i+r-1} \Big/ \prod_{j>n} (a_{ij} - a_{in}),$$

as required.

3. GENERATING FUNCTION FOR SEQUENCE OF POWERS

Van der Poorten [6] further proved that if

$$w^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)} x^n = f(x)/x^i g(x^{-1}), \tag{3.1}$$

then there exists a polynomial $f_r(x)$ of degree at most $u-1$, such that

$$w_r^{(i)}(x) = f_r(x)/x^u g_r(x^{-1}), \qquad u = \binom{r+i-1}{r} . \tag{3.2}$$

We first seek an expression for $f_r(x)$.

$$w_r^{(i)}(x) = w_0^{(i)^r} + w_1^{(i)^r} x + w_2^{(i)^r} x^2 + \cdots + w_{u-1}^{(i)^r} x^{u-1} + w_u^{(i)^r} x^u + \cdots$$

$$-R_{u1} x w_r^{(i)}(x) = -R_{u1} w_0^{(i)^r} x - R_{u1} w_1^{(i)^r} x^2 - \cdots - R_{u1} w_{n-2}^{(i)^r} x^{u-1} - R_{u1} w_{u-1}^{(i)^r} x^u - \cdots$$

$$-R_{u2} x^2 w_r^{(i)}(x) = -R_{u2} w_0^{(i)^r} x^2 - \cdots - R_{u2} w_{n-3}^{(i)^r} x^{u-1} - R_{u2} w_{u-1}^{(i)^r} x^u - \cdots$$

$$\vdots$$

$$-R_{u,u-1} x^{u-1} w_r^{(i)}(x) = -R_{u,u-1} w_0^{(i)^r} x^{u-1} - R_{u,u-1} w_1^{(i)^r} x^u - \cdots$$

$$-R_{uu} x^u w_r^{(i)}(x) = -R_{uu} w_0^{(i)^r} x^u - \cdots$$

We then sum both sides of these equations. On the left we have

$$w_r^{(i)}(x)\left(1 - \sum_{j=1}^{u} R_{uj} x^j\right) = w_r^{(i)}(x) x^u \left(x^{-u} - \sum_{j=1}^{u} R_{uj} x^{-(u-j)}\right) = w_r^{(i)}(x) x^u g_r(x^{-1}),$$

as in van der Poorten.

On the right we obtain

$$f_r(x) = \sum_{j=0}^{u-1} T_{uj} x^j , \tag{3.3}$$

where

$$T_{uj} = w_j^{(i)^r} - \sum_{m=0}^{j} R_{um} w_{j-m}^{(i)^r}, \qquad R_{u0} \equiv 0,$$

since

$$w_n^{(i)^r} x^n = \sum_{j=1}^{u} R_{uj} w_{n-j}^{(i)^r} x^n .$$

Thus we have

$$w_r^{(i)}(x) = \left(\sum_{j=0}^{u-1} \left\{ w_j^{(i)^r} - \sum_{m=1}^{j} R_{um} w_{j-m}^{(i)^r} \right\} x^j \right) \Big/ x^u g_r(x^{-1}). \tag{3.4}$$

We now show how (3.4) agrees with Eq. (33) of Horadam [3] when $i=2$ and $r=2$. We first multiply each side of the equation by $x^3 g_2(x^{-1})$.

The left-hand side of (3.4) is then

$$\begin{aligned} x^3 g_2(x^{-1}) w_2^{(2)}(x) &= (-1(P_{21}^2 + P_{22})x - (P_{22}^2 + P_{21}^2 P_{22})x^2 + P_{22}^3 x^3) w_2^{(2)}(x) \\ &= (1 + P_{22}x)(1 - (P_{21}^2 + 2P_{22})x + P_{22}^2 x^2) w_2^{(2)}(x) . \end{aligned}$$

When $i=2$, the right-hand side of (3.4) is

$$\sum_{j=0}^{2}\left\{w_j^{(2)^2}-\sum_{m=1}^{j}R_{3m}w_{j-m}^{(2)^2}\right\}x^j = w_0^{(2)^2}+w_1^{(2)^2}x+w_2^{(2)^2}x^2-R_{31}w_0^{(2)}x^2-R_{31}w_1^{(2)}x^2-R_{32}w_0^{(2)^2}x^2$$

$$= w_0^{(2)^2}+w_1^{(2)^2}x+P_{21}w_1^{(2)^2}x^2+P_{22}^2w_0^{(2)^2}x^2+2P_{21}P_{22}w_0^{(2)}w_1^{(2)}x^2$$

$$-P_{21}w_0^{(2)^2}x-P_{22}w_0^{(2)^2}x-P_{21}w_1^{(2)^2}x^2-P_{22}w_1^{(2)^2}x^2$$

$$-P_{22}^2w_0^{(2)^2}x^2-P_{21}^2P_{22}w_0^{(2)^2}x^2$$

$$= (1+P_{22}x)w_0^{(2)^2}-(1+P_{22}x)(P_{21}w_0^{(2)}-w_1^{(2)})^2x$$

$$-2x(P_{21}w_0^{(2)}w_1^{(2)}+P_{22}w_0^{(2)^2}-w_1^{(2)^2})\frac{(1+P_{22}x)}{(1+P_{22}x)}$$

$$= (1+P_{22}x)(w_0^{(2)^2}-x(P_{21}w_0^{(2)}-w_1^{(2)})^2-2xe\,w_0^{(2)}(-P_{22}x))$$

(since $w_0(-P_{22}x)$ $= (1+P_{22}x)^{-1}$).

This agrees with Horadam's Eq. (33) if we multiply that equation through by $(1+P_{22}x)$ and note that $a_{21}^2+a_{22} = P_{21}^2+2P_{22}$. When $r=1$, we get $u=i$, $R_{im}=P_{im}$. If we consider the special case of $\{w_n^{(i)}\}$:

$$w_n^{(i)} = 0, \qquad n<0$$

$$w_n^{(i)} = 1, \qquad n=0$$

$$w_n^{(i)} = \sum_{r=1}^{i}P_{ir}w_{n-r}^{(i)}, \qquad n>0,$$

then $\{w_n^{(i)}\} \equiv \{u_n^{(i)}\}$, the fundamental sequence discussed by Shannon [7], and (3.4) becomes

$$u^{(i)}(x) = \left\{\sum_{j=0}^{n-1}\left\{u_j^{(i)}-\sum_{m=1}^{j}P_{im}u_{j-m}^{(i)}\right\}x^j\right\} \Big/ x^ig(x^{-1}) = \left\{u_0^{(i)}+\sum_{j=1}^{n-1}\left(u_j^{(i)}-u_j^{(i)}\right)x^j\right\}\Big/x^ig(x^{-1})$$

$$= 1/x^ig(x^{-1}), \qquad \text{where} \qquad n = \binom{i+r-1}{r},$$

which is effectively Eq. (1) of Hoggatt and Lind [2]. (Equation (2) of Hoggatt and Lind [2] is essentially the same as Eq. (2.4) of Shannon.)

Thus in (2.9) we have found the coefficients in the recurrence relation for $\{w_n^{(i)^r}\}$ and in (3.4) an explicit expression for the generating function for $\{w_n^{(i)r}\}$.

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