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# Linear Recurrences for $r$-Bell Polynomials

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**Abstract**

Letting $B_{n,r}$ be the $n$-$th$ $r$-Bell polynomial, it is well known that $B_n(x)$ admits specific integer coordinates in the two bases $\{x^i\}_i$ and $\{xB_i(x)\}_i$ according to, respectively, the Stirling numbers and the binomial coefficients. Our aim is to prove that the sequences $B_{n+m,r}(x)$ and $B_{n,r+s}(x)$ admit a binomial recurrence coefficient in different bases of the $\mathbb{Q}$-vector space formed by polynomials of $\mathbb{Q}[X]$.

## 1 Introduction

In different ways, Belbachir and Mihoubi [5] and Gould and Quaintance [10] showed that the Bell polynomial $B_{n+m}$ admits integer coordinates in the bases $\{x^iB_j(x)\}_{i,j}$. Xu and Cen [18]

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extended the latter in some particular cases of complete Bell polynomials. Also, the second author and Bencherif [2, 3] established that Chebyshev polynomials of first and second kind, and more generally bivariate polynomials associated with recurrence sequences of order two, including Jacobsthal polynomials, Vieta polynomials, Morgan-Voyce polynomials and others, admit remarkable integer coordinates in a specific bases. Some recurrence relations on Bell numbers and polynomials are given by Spivey [16] and some other relations by Sun and Wu [17]. What about $r$-Bell polynomials?

The $r$-Bell polynomials $\{B_{n,r}\}_{n\geq 0}$ are defined by their generating function

$$\sum_{n\geq 0} B_{n,r}(x)\frac{t^n}{n!} = \exp(x(e^t-1)+rt),$$

and satisfy the generalized Dobinsky formula

$$B_{n,r}(x) = \exp(-x)\sum_{i=0}^{\infty}\frac{(i+r)^n}{i!}x^i. \tag{1}$$

It is well known that $B_{n,r}(x)$ admits integer coordinates in the following two: bases $\{x^i\}_i$ and $\{B_i(x)\}_i$ as

$$B_{n,r}(x) = \sum_{i=0}^{n}\left\{\begin{matrix} n+r \\ i+r \end{matrix}\right\}_r x^i \text{ and } B_{n,r}(x) = \sum_{i=0}^{n}\binom{n}{i}r^{n-i}B_i(x), \tag{2}$$

according to, respectively, the $r$-Stirling numbers of the second kind and the binomial coefficients, see for example [11]. For a general overview of the $r$-Stirling numbers, one can see [6, 7, 8, 15]. An extension of $r$-Stirling numbers of the second kind and the $r$-Bell polynomials is given in [14]. In the sequel, we refer to [1, 4] for some properties and recurrence relations of $r$-Lah numbers.

Our aim is to prove that the polynomials $B_{n+m,r}$ and $B_{n,r+s}$ admit a binomial recurrence coefficient in the families

$$\{x^iB_{n,j+r}(x)\}_{i,j},\ \{x^iB_{n,i+r}(x)\}_i,\ \{x^iB_{j,r}(x)\}_j,\ \{B_{j,s}(x)\}_j \text{ and } \{x^iB_j(x)\},$$

of the basis of the $\mathbb{Q}$-vector space formed by polynomials of $\mathbb{Q}[X]$ .

# 2 Main results

Mező [11, Thm. 7.1] showed that the $r$-Bell polynomials satisfy the following recurrence relation

$$B_{n,r+1}(x) = \sum_{i=0}^{n}\binom{n}{i}B_{i,r}(x).$$

This can be generalized as follows.

**Theorem 1.** *Decomposition of $B_{n,r+s}(x)$ into the family of basis $\{B_{i,r}(x)\}_i$. For all nonnegative integers $n$, $r$ and $s$, we have*

$$B_{n,r+s}(x) = \sum_{i=0}^{n} \binom{n}{i} r^{n-i} B_{i,s}(x).$$

*Proof.* Use (1) to get

$$\frac{d^s}{dx^s}(\exp(x)B_{n,r}(x)) = \exp(x)B_{n,r+s}(x). \tag{3}$$

Using the following identity [11]

$$B_{n,r}(x) = \sum_{i=0}^{n} \binom{n}{i} r^{n-i} B_i(x), \tag{4}$$

we obtain

$$\frac{d^s}{dx^s}(\exp(x)B_{n,r}(x)) = \sum_{i=0}^{n} \binom{n}{i} r^{n-i} \frac{d^s}{dx^s}(\exp(x)B_i(x)),$$

and, applying property (3), we obtain the desired identity.

We give now a combinatorial proof: let $x$ be a positive integer (a number of colors). By the definition of the $r$-Bell numbers, $B_{n,r+s}(x)$ gives the number of partitions of an $(n + r + s)$-element set, with the restriction that the first $r + s$ elements are in distinct subsets (these are called distinguished elements from now on). Moreover, the blocks not containing distinguished elements are colored with one of the $x$ colors.

We can construct such partitions in the following way: from the $n$ non-distinguished elements we put $n - i$ into the blocks of $r$ distinguished elements. To do this, we have $\binom{n}{n-i} = \binom{n}{i}$ possibilities choosing those $n-i$ elements. Then, we put these elements into the above mentioned blocks, which can happen on $r^{n-i}$ ways. Then the remaining $n+s-(n-i) = s + i$ elements have to form a partition in which $s$ elements go to different blocks and the other blocks are colored with one of the $x$ colors. The number of these possibilities is exactly $B_{i,s}(x)$. The left and right hand sides coincide for any positive integer $x$, so they coincide for any $x \in \mathbb{R}$. □

**Corollary 2.** *For all nonnegative integers $n$, $k$, $r$ and $s$, we have*

$$\begin{Bmatrix} n+r+s \\ k+r+s \end{Bmatrix}_{r+s} = \frac{1}{k!} \sum_{j=0}^{n-k} \binom{s}{j} \begin{Bmatrix} n+r \\ j+k+r \end{Bmatrix}_r (j+k)!, \tag{5}$$

$$\begin{Bmatrix} n+r+s \\ k+r+s \end{Bmatrix}_{r+s} = \sum_{i=k}^{n} \binom{n}{i} \begin{Bmatrix} i+r \\ k+r \end{Bmatrix}_r s^{n-i}. \tag{6}$$

*Proof.* From the definition of $B_{n,r}(x)$ given by (2), we have

$$\frac{d^s}{dx^s}(\exp(x)B_{n,r}(x)) = \sum_{i=0}^{n} \begin{Bmatrix} n+r \\ i+r \end{Bmatrix}_r \frac{d^s}{dx^s}(x^i \exp(x)),$$

and upon using the Leibniz formula, one obtains

$$\begin{aligned} B_{n,r+s}(x) &= \sum_{i=0}^{n}\sum_{k=0}^{i}\binom{s}{k}\frac{i!}{(i-k)!}{n+r \brace i+r}_r x^{i-k} \\ &= \sum_{i=0}^{n}\sum_{l=0}^{i}\binom{s}{i-l}\frac{i!}{l!}{n+r \brace i+r}_r x^{l} \\ &= \sum_{l=0}^{n} x^l \sum_{i=l}^{n}\binom{s}{i-l}\frac{i!}{l!}{n+r \brace i+r}_r \end{aligned}$$

The identity (5) follows by identification using the definition of $B_{n,r+s}(x)$, and the fact that the elements $1, 2, \ldots, r+s$ are in different parts.

We have a combinatorial interpretation as follows: for $j = 0, \ldots, s$, there are $\binom{s}{s-j} = \binom{s}{j}$ ways to form $s-j$ singletons using the elements in $\{1, \ldots, s\}$ and there are ${n+r \brace k+r+j}_r$ ways to partition the set $\{s+1, \ldots, n+r+s\}$ into $(k+r+s)-(s-j) = k+r+j$ subsets such that the elements of the set $\{s+1, \ldots, s+r\}$ are in different subsets. The $j$ elements of the set $\{1, \ldots, s\}$ not already used can be inserted in the $(k+r+j)-r = k+j$ subsets in

$$(k+j)\cdots((k+j)-j+1) = \frac{(k+j)!}{k!}$$

ways. Then the number of partitions of the set $\{1, \ldots, n+r+s\}$ into $k+r+s$ subsets such that the elements of the set $\{1, \ldots, r+s\}$ are in different subsets is

$${n+r+s \brace k+r+s}_{r+s} = \sum_{j=0}^{s}\binom{s}{j}{n+r \brace k+r+j}_r \frac{(k+j)!}{k!}.$$

For the identity (6), using the definition of $B_{n,r}(x)$ and Theorem 1 gives

$$\begin{aligned} \sum_{k=0}^{n}{n+r+s \brace k+r+s}_{r+s} x^k &= B_{n,r+s}(x) \\ &= \sum_{i=0}^{n}\binom{n}{i}s^{n-i}B_{i,r}(x) \\ &= \sum_{i=0}^{n}\binom{n}{i}s^{n-i}\sum_{k=0}^{i}{i+r \brace k+r}_r x^k \\ &= \sum_{k=0}^{n}x^k\sum_{i=k}^{n}\binom{n}{i}{i+r \brace k+r}_r s^{n-i}. \end{aligned}$$

Then, by identification, we obtain the identity (6) of the corollary.

We also give a combinatorial proof for this identity: from the $n$ non-distinguished elements $i$ go to the $k+r$ blocks which contain the first $r$ distinguished elements: $\binom{n}{i}\left\{{i+r \atop k+r}\right\}_r$ possibilities. The remaining $n-i$ elements go to the $s$ additional distinguished blocks, in $s^{n-i}$ ways. (So the $k+r+s$ blocks are guaranteed). Finally we sum the $i$ disjoint cases. $\square$

We note that the formula (6) is immediate from [6, Lemma 13] with appropriate substitutions.

In different ways, Belbachir and Mihoubi [5] and Gould and Quaintance [10] showed that $B_{n+m}(x)$ admits a recurrence relation according to the family $\{x^i B_j(x)\}$ as follows:

$$B_{n+m}(x) = \sum_{k=0}^{n}\sum_{j=0}^{m} \left\{{m \atop j}\right\} \binom{n}{k} j^{n-k} x^j B_k(x), \tag{7}$$

In [11], Mező cited the Carlitz identities [7, eq. (3.22–3.23)] given by

$$B_{n+m,r} = \sum_{k=0}^{m} \left\{{m+r \atop k+r}\right\}_r B_{n,k+r} \text{ and } B_{n,r+s} = \sum_{k=0}^{s} \left[{s+r \atop k+r}\right]_r (-1)^{s-k} B_{n+k,r},$$

and established [13], by a combinatorial proof, the following identity

$$B_{n+m,r} = \sum_{k=0}^{n}\sum_{j=0}^{m} \left\{{m+r \atop j+r}\right\}_r \binom{n}{k} (j+r)^{n-k} B_k,$$

where $B_n = B_n(1)$ is the number of ways to partition a set of $n$ elements into non-empty subsets, $B_{n,r} = B_{n,r}(1)$ is the number of ways to partition a set of $n+r$ elements into non-empty subsets such that the first $r$ elements are in different subsets and $\left\{{n \atop k}\right\}_r$ is an $r$-Stirling number of the second kind; see [6, 7, 8]. The following theorem generalizes these results.

**Theorem 3.** *Decomposing $B_{n+m,r}(x)$ into the family of the basis $\{x^k B_{n,k+r}(x)\}_k$, $\{x^j B_{k,r}(x)\}_{j,k}$ and $\{x^j B_k(x)\}_{j,k}$ : for all nonnegative integers $n$, $m$, $r$ and $s$, we have*

$$B_{n+m,r}(x) = \sum_{k=0}^{m} \left\{{m+r \atop k+r}\right\}_r x^k B_{n,k+r}(x) \tag{8}$$

$$B_{n+m,r}(x) = \sum_{k=0}^{n}\sum_{j=0}^{m} \left\{{m+r \atop j+r}\right\}_r \binom{n}{k} j^{n-k} x^j B_{k,r}(x) \tag{9}$$

$$B_{n+m,r}(x) = \sum_{k=0}^{n}\sum_{j=0}^{m} \left\{{m+r \atop j+r}\right\}_r \binom{n}{k} (j+r)^{n-k} x^j B_k(x) \tag{10}$$

*Also, we have*

$$x^s B_{n,r+s}(x) = \sum_{k=0}^{s} \left[{s+r \atop k+r}\right]_r (-1)^{s-k} B_{n+k,r}(x). \tag{11}$$

*Proof.* For the identity (8) we proceed as follows: the identity given in [5] and [16] can be written as follows

$$B_{n+m}(x) = \sum_{i=0}^{n}\sum_{j=0}^{m} \left\{ {m \atop j} \right\} \binom{n}{i} j^{n-i} x^j B_i(x) = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} x^j \sum_{i=0}^{n} \binom{n}{i} j^{n-i} B_{i,0}(x).$$

From Theorem 1, we have $\sum_{i=0}^{n} \binom{n}{i} j^{n-i} B_{i,s}(x) = B_{n,j+s}(x)$, $s \geq 0$, then

$$B_{n+m}(x) = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} x^j B_{n,j}(x),$$

and therefore

$$\frac{d^r}{dx^r}(\exp(x) B_{n+m}(x)) = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \frac{d^r}{dx^r}(x^j \exp(x) B_{n,j}(x)). \tag{12}$$

Now, using (1), we get

$$\frac{d^r}{dx^r}(\exp(x) B_n(x)) = \exp(x) B_{n,r}(x) \tag{13}$$

and using (13) and the Leibniz formula in (12), we state that

$$\begin{aligned} B_{n+m,r}(x) &= \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \sum_{i=0}^{j} \binom{r}{i} \frac{j!}{(j-i)!} x^{j-i} B_{n,j-i+r}(x) \\ &= \sum_{k=0}^{m} x^k B_{n,k+r}(x) \sum_{j=k}^{m} \left\{ {m \atop j} \right\} \binom{r}{j-k} \frac{j!}{k!}. \end{aligned}$$

Let

$$a(m,k,r) = \sum_{j=k}^{m} \left\{ {m \atop j} \right\} \binom{r}{j-k} \frac{j!}{k!}.$$

Then

$$\begin{aligned} \sum_{m\geq 0} a(m,k,r) \frac{t^m}{m!} &= \sum_{j\geq k} \binom{r}{j-k} \frac{j!}{k!} \sum_{m\geq j} \left\{ {m \atop j} \right\} \frac{t^m}{m!} \\ &= \frac{1}{k!} \sum_{j\geq k} \binom{r}{j-k} (\exp(t)-1)^j \\ &= \frac{(\exp(t)-1)^k}{k!} \sum_{j\geq 0} \binom{r}{j} (\exp(t)-1)^j \\ &= \frac{(\exp(t)-1)^k}{k!} \exp(rt), \end{aligned}$$

which means that $a(m,k,r) = \left\{ {m+r \atop k+r} \right\}_r$ and $B_{n+m,r}(x) = \sum_{k=0}^{m} \left\{ {m+r \atop k+r} \right\}_r x^k B_{n,k+r}(x)$.

For a combinatorial proof, we consider that there are $n+m$ non-distinguished elements. From these we put $m$ and the $r$ distinguished elements into $k+r$ blocks, such that the $r$ distinguished elements are separated: there are $\left\{ {m+r \atop k+r} \right\}_r$ cases. We have to color the $k$ blocks not containing distinguished elements, and this can happen $x^k$ ways. Then $n$ items remain. We can put these elements into the already constructed blocks or into new blocks. We can handle the already constructed blocks as distinguished elements. So we have $n+k+r$ elements, of which $k+r$ are distinguished. In addition, we have to color the non-distinguished blocks. To do this, we have $B_{n,k+r}(x)$ possibilities. Altogether, if $k$ is fixed, we have $\left\{ {m+r \atop k+r} \right\}_r x^k B_{n,k+r}(x)$ cases. We can sum over $k$.

For the identity (9), use Theorem 1 to replace $B_{n,k+r}(x)$ by $\sum_{j=0}^{n} \binom{n}{j} k^{n-j} B_{j,r}(x)$.

For the identity (10), use relation (4) to replace $B_{n,k+r}(x)$ by $\sum_{j=0}^{n} \binom{n}{j} (k+r)^{n-j} B_j(x)$.

As a combinatorial proof, we can argue as follows: from the $n$ elements we choose $k$ elements in $\binom{n}{k}$ ways and separate them. The remaining $m+r$ elements go to $j+r$ blocks, but $r$ elements stay in disjoint sets. This can happen in $\left\{ {m+r \atop j+r} \right\}_r$ ways. We have to color the $j$ blocks; this is why the factor $x^j$ appears. The non-separated $n-k$ elements go to these blocks. This means $(j+r)^{n-k}$ cases. Finally, the above $k$ separated items go to separated and colored blocks; this is what $B_k(x)$ represents. We sum over the possible values of $j$ and $k$. Again, the left- and right-hand sides coincide for any positive integer $x$, so they coincide for any $x \in \mathbb{R}$.

For the identity (11) using (1) and the following identity (see [6])

$$\sum_{k=0}^{m} \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r x^k = (x+r)(x+r+1)\cdots(x+r+m-1),$$

we can write

$$\begin{aligned} \sum_{k=0}^{s} \begin{bmatrix} s+r \\ k+r \end{bmatrix}_r (-1)^{s-k} B_{n+k,r}(x) &= (-1)^s \exp(-x) \sum_{i=0}^{\infty} (i+r)^n \frac{x^i}{i!} \sum_{k=0}^{s} \begin{bmatrix} s+r \\ k+r \end{bmatrix}_r (-i-r)^k \\ &= (-1)^s \exp(-x) \sum_{i=0}^{\infty} (-i)(-i+1)\cdots(-i+s-1)(i+r)^n \frac{x^i}{i!} \end{aligned}$$

and this can be written as

$$\begin{aligned} \exp(-x) \sum_{i=0}^{\infty} i(i-1)\cdots(i-s+1)(i+r)^n \frac{x^i}{i!} &= x^s \exp(-x) \sum_{i=s}^{\infty} (i+r)^n \frac{x^{i-s}}{(i-s)!} \\ &= x^s \exp(-x) \sum_{i=0}^{\infty} (i+r+s)^n \frac{x^i}{i!} \\ &= x^s B_{n,r+s}(x). \end{aligned}$$

$\square$

**Corollary 4.** *For all nonnegative integers $n$, $m$, $k$, $r$ and $s$, we have*

$$\begin{Bmatrix} n+m+r \\ k+r \end{Bmatrix}_r = \sum_{j=0}^{\min(m,k)} \begin{Bmatrix} m+r \\ j+r \end{Bmatrix}_r \begin{Bmatrix} n+j+r \\ k+r \end{Bmatrix}_{j+r}, \tag{14}$$

$$\begin{Bmatrix} n+r+s \\ k+r+s \end{Bmatrix}_{r+s} = \sum_{j=0}^{s} (-1)^{s-j} \begin{bmatrix} s+r \\ j+r \end{bmatrix}_r \begin{Bmatrix} n+j+r \\ k+s+r \end{Bmatrix}_r. \tag{15}$$

*Proof.* For the identity (14), we have from Theorem 3

$$B_{n+m,r}(x) = \sum_{j=0}^{m} \begin{Bmatrix} m+r \\ j+r \end{Bmatrix}_r x^j B_{n,j+r}(x).$$

Upon using (2) to replace $B_{n,j+r}(x)$ by $\sum_{i=0}^{n} \left\{ {n+j+r \atop i+j+r} \right\}_{j+r} x^i$, we can write

$$\begin{aligned} B_{n+m,r}(x) &= \sum_{j=0}^{m} \begin{Bmatrix} m+r \\ j+r \end{Bmatrix}_r \sum_{i=0}^{n} \begin{Bmatrix} n+j+r \\ i+j+r \end{Bmatrix}_{j+r} x^{i+j} \\ &= \sum_{k=0}^{n+m} x^k \sum_{j=0}^{\min(m,k)} \begin{Bmatrix} m+r \\ j+r \end{Bmatrix}_r \begin{Bmatrix} n+j+r \\ k+r \end{Bmatrix}_{j+r}, \end{aligned}$$

and using the definition $B_{n+m,r}(x) = \sum_{k=0}^{n+m} \left\{ {n+m+r \atop k+r} \right\}_r x^k$, the first identity follows by identification. The identity (15) follows by the same way upon using the fourth identity of Theorem 3. □

*Remark* 5. One can proceed similarly, as in the proof of the Spivey's identity [16] to obtain a combinatorial proof for the identity (9) when $x$ is a positive integer.

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