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ON THE GENERAL LINEAR RECURRENCE RELATION

Ray Melham
School of Mathematical Sciences, University of Technology, Sydney
Broadway N.S.W., 2007, Australia

Derek Jennings
Department of Mathematics, University of Southampton, Hampshire, S09 5NH, England
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The general m^{th}-order linear recurrence relation can be written as

$$R_n = \sum_{i=1}^{m} a_i R_{n-i}, \quad \text{for } m \geq 2, \tag{1}$$

where the a_i's are any complex numbers, with $a_m \neq 0$. If suitable initial values $R_{-(m-2)}$, $R_{-(m-3)}$, $\dots, R_0, R_1$ are specified, the sequence $\{R_n\}$ is uniquely determined for all integral n.

The auxiliary equation of (1) is

$$x^m = \sum_{i=1}^{m} a_i x^{m-i}. \tag{2}$$

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be the m roots, assumed distinct, of (2) and define $\overline{\alpha}_j$ by

$$\overline{\alpha}_j = \prod_{\substack{i=1 \\ i \neq j}}^{m} (\alpha_j - \alpha_i).$$

Then the *fundamental* $\{U_n\}$ and *primordial* $\{V_n\}$ sequences that satisfy (1) are given by the following Binet formulas [1]. For any integer n, we have

$$U_n = \sum_{j=1}^{m} \frac{\alpha_j^{n+m-2}}{\overline{\alpha}_j} \quad \text{and} \quad V_n = \sum_{j=1}^{m} \alpha_j^n, \tag{3}$$

so that $U_{-(m-2)} = U_{-(m-3)} = \cdots = U_{-1} = U_0 = 0$ and $U_1 = 1$. Also $V_1 = a_1$ and

$$V_i = a_1 V_{i-1} + \cdots + a_{i-1} V_1 + i a_i, \quad \text{for } 1 \leq i \leq m. \tag{4}$$

In this paper we answer a question of Jarden, who in his book [2] (p. 88), see also [1], asked for the value of $U_{2n} - U_n V_n$ for the m^{th}-order linear recurrence relation. For example, when $m = 2$, where $a_1 = a_2 = 1$, $\{U_n\}$ and $\{V_n\}$ are the Fibonacci and Lucas sequences, respectively. In this case, we have

$$U_{2n} - U_n V_n = 0.$$

For the general third- and fourth-order linear recurrence relations we have, respectively,

$$U_{2n} - U_n V_n = a_3^n U_{-n} \quad \text{and} \quad U_{2n} - U_n V_n = (-1)^n a_4^n \{U_{-n} V_{-n} - U_{-2n}\}.$$

For the general m^{th}-order linear recurrence relation, we have the following, very appealing theorem.

Theorem: For any integer n, and $m \geq 2$, we have

$$U_{2n} - U_n V_n = (-1)^{(m+1)(n+1)} a_m^n \sum_{i=0}^{m-2} \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-n}^{k_1} V_{-2n}^{k_2} \cdots V_{-in}^{k_i} U_{-(m-2-i)n},$$

where a_m is the constant term in the auxiliary equation and the inner summation is taken over all partitions of $i = 1k_1 + 2k_2 + \cdots + ik_i$ so that k_j is the number of parts of size j. Here, $k = k_1 + k_2 + \cdots + k_i$ is the total number of parts in the partition. The coefficient of $U_{-(m-2-i)n}$, inside the second summation sign, is taken to be 1 when $i = 0$.

In order to prove the above theorem, we use the following lemma.

Lemma: Using the above notation, we have

$$\sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \ldots V_{-i}^{k_i} = \frac{a_{m-i}}{a_m} \quad \text{for } 0 \leq i \leq (m-1),$$
$$= -\frac{1}{a_m} \quad \text{for } i = m.$$

Proof of Lemma: First, we note that

$$\begin{aligned}&\exp\left\{-\left(\frac{V_{-1}}{1}x + \frac{V_{-2}}{2}x^2 + \frac{V_{-3}}{3}x^3 + \cdots\right)\right\}\\ &= \sum_{i=0}^{\infty} x^i \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \ldots V_{-i}^{k_i}.\end{aligned} \tag{5}$$

Therefore, we need to evaluate the function,

$$f(x) = \sum_{i=1}^{\infty} \frac{V_{-i}}{i} x^i.$$

Using the fact that $\{V_n\}$ satisfies the recurrence relation (1), with the help of (4) it is not hard to see that the generating function $g(x) = \sum_{n=0}^{\infty} V_{-n} x^n$, for V_{-n}, is given by

$$g(x) = \frac{ma_m + (m-1)a_{m-1}x + (m-2)a_{m-2}x^2 + \cdots + 2a_2x^{m-2} + a_1x^{m-1}}{a_m + a_{m-1}x + \cdots + a_1x^{m-1} - x^m}. \tag{6}$$

Letting

$$h(x) = 1 + \frac{a_{m-1}}{a_m}x + \frac{a_{m-2}}{a_m}x^2 + \cdots + \frac{a_1}{a_m}x^{m-1} - \frac{1}{a_m}x^m, \tag{7}$$

from (6) and (7) we have

$$g(x) = m - \frac{h'(x)}{h(x)}x. \tag{8}$$

Now, since $V_0 = m$, from (8) we have

$$-\sum_{n=1}^{\infty} V_{-n} x^{n-1} = \frac{m - g(x)}{x} = \frac{h'(x)}{h(x)}.$$

Integrating, and using $h(0) = 1$ to eliminate the constant of integration, we have

$$-\sum_{n=1}^{\infty} \frac{V_{-n}}{n} x^n = \log h(x).$$

Therefore,

$$\exp\left\{-\sum_{n=1}^{\infty} \frac{V_{-n}}{n} x^n\right\} = h(x). \tag{9}$$

So, from (5) and (9) we have

$$h(x) = \sum_{i=0}^{\infty} x^i \sum \frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \dots V_{-i}^{k_i}. \tag{10}$$

Using the expression for $h(x)$ given by (7), we can equate the coefficients of x in (10) to complete the proof of the lemma. □

Proof of Theorem: From the Binet formulas (3) for U_n and V_n, we have

$$\begin{aligned} U_{2n} - U_n V_n &= \left(\frac{\alpha_1^{2n+m-2}}{\overline{\alpha}_1} + \frac{\alpha_2^{2n+m-2}}{\overline{\alpha}_2} + \cdots + \frac{\alpha_m^{2n+m-2}}{\overline{\alpha}_m}\right) \\ &\quad - \left(\frac{\alpha_1^{n+m-2}}{\overline{\alpha}_1} + \frac{\alpha_2^{n+m-2}}{\overline{\alpha}_2} + \cdots + \frac{\alpha_m^{n+m-2}}{\overline{\alpha}_m}\right)(\alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n) \\ &= -\sum_{i \neq j} \frac{\alpha_j^{n+m-2}\alpha_i^n}{\overline{\alpha}_j}, \end{aligned} \tag{11}$$

where the summation is taken over all $1 \le i, j \le m$, such that $i \ne j$. Therefore, to prove the theorem, we need to show that the right-hand side of the theorem is given by the right-hand side of (11). First, we require some new notation. The a_i in (2) are given by

$$a_i = (-1)^{i+1} \sum \alpha_1 \alpha_2 \dots \alpha_i,$$

where α_i are the roots of (2) and the summation is taken over all possible distinct products of i distinct α_j's. Now define $a_i(n)$ and $c_i(n)$ by

$$a_i(n) = (-1)^{i+1} \sum \alpha_1^n \alpha_2^n \dots \alpha_i^n \quad \text{and} \quad c_i(n) = \sum \alpha_1^n \alpha_2^n \dots \alpha_i^n,$$

so that $a_i(n) = (-1)^{i+1} c_i(n)$. Now, by the lemma, for any integer n, we have

$$\begin{aligned} \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}} V_{-n}^{k_1} V_{-2n}^{k_2} \dots V_{-in}^{k_i} &= \frac{a_{m-i}(n)}{a_m(n)} \quad \text{for } 0 \le i \le (m-1), \\ &= -\frac{1}{a_m(n)} \quad \text{for } i = m. \end{aligned} \tag{12}$$

Using (12), we can rewrite the theorem as

$$U_{2n} - U_n V_n = (-1)^{(m+1)(n+1)} a_m^n \sum_{i=0}^{m-2} \frac{a_{m-i}(n)}{a_m(n)} U_{-(m-2-i)n}. \tag{13}$$

Since

$$a_m^n = (-1)^{(m+1)n} c_m(n), \qquad a_{m-i}(n) = (-1)^{m+i+1} c_{m-i}(n), \tag{14}$$

and

$$a_m(n) = (-1)^{m+1} c_m(n),$$

we have, from (13) and (14),

$$U_{2n} - U_n V_n = (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) U_{-(m-2-i)n}. \tag{15}$$

By the Binet formula,

$$U_{-(m-2-i)n} = \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j},$$

which, when inserted into (15), gives

$$\begin{aligned} U_{2n} - U_n V_n &= (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j} \\ &= (-1)^{m+1} \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n}. \end{aligned} \tag{16}$$

Now we note that

$$\begin{aligned} \left(x + \frac{1}{\alpha_1^n}\right)\left(x + \frac{1}{\alpha_2^n}\right) \cdots \left(x + \frac{1}{\alpha_m^n}\right) &= \sum_{i=0}^{m} \frac{c_i(n)}{c_m(n)} x^i \\ &= \sum_{i=0}^{m} \frac{c_{m-i}(n)}{c_m(n)} x^{m-i} \end{aligned} \tag{17}$$

So if we let $x = -1/\alpha_j^n$ in (17), for any $j = 1, 2, \ldots, m$, we have

$$\sum_{i=0}^{m} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = 0. \tag{18}$$

From (18), we easily obtain

$$(-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = -c_1(n) \alpha_j^{-n} + c_0(n). \tag{19}$$

Now we note that $c_0(n) = 1$ and $c_1(n) = \sum_{i=1}^{m} \alpha_i^n$. Therefore, using (19) in (16), we have

$$U_{2n} - U_n V_n = \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \left\{ -\sum_{i=1}^{m} \alpha_i^n \alpha_j^{-n} + 1 \right\} = -\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j} + \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} = -\sum_{i \neq j} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j}.$$

Which agrees with the right-hand side of (11). Hence, the theorem is proved. □

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