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COMBINATORIAL METHODS AND RECURRENCE RELATIONS WITH TWO INDICES

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Abstract

In this paper we solve several recurrence relations with two indices with using combinatorial methods. Moreover, we present several recurrence relations with two indices related to Dyck paths and Schröder paths.

Keywords: Recurrence relations with two indices, Dyck paths, Schröder paths.

AMS Mathematical Subject Classifications: 10A35, 65Q05, 05.10

1. Introduction

A *lattice path* of length n is a sequence of points $P_1, P_2, \ldots, P_n$ with $n \geq 1$ such that each point P_i belongs to the plane integer lattice and each two consecutive points P_i and P_{i+1} connect by a line segment. We will consider lattice paths in $\mathbb{Z}^2$ whose permitted step types are the up-step $U = (1, 1)$, the down-step $D = (1, -1)$, and the labeled horizontal-step $H_i = (2, 0)$ with label i. We will focus on paths that start from the origin and return to the x-axis, and that never pass below the x-axis. Let $\mathcal{D}_n$ denote the set of such paths of length $2n$ when only up-steps and down-steps are allowed, and let $\mathcal{S}_n^\ell$ denote the set of such paths of length n when all the three types are allowed such that the horizontal-step can be labeled with one of the labels $1, 2, \ldots, \ell$. It is well known that the cardinalities $|\mathcal{D}_n| = c_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see [2, A000108]), and $|\mathcal{S}_n^1| = s_n$, the n-the large Schröder number (see [2, A006318]). A lattice path is called an *ℓ-Schröder* path of length n if it belongs to the set $\mathcal{S}_n^\ell$ and there are no horizontal-steps labeled ℓ at level $0, 1, \ldots, \ell - 1$; in the case of $\ell = 1$ these paths are called just *Schröder paths*.

A *sequence with k indices* is a function $a : A^k \to B$, and denoted by $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k \in A}$ or $\{a_{\mathbf{n}}\}_{\mathbf{n} \in A^k}$, where $A \subseteq \mathbb{N}$. The element $a_{\mathbf{n}}$ of a sequence $\{a_{\mathbf{n}}\}_{\mathbf{n} \in A}$ is called the $\mathbf{n}$*th term*, and the vector $\mathbf{n}$ of integers is the *sequence vector of indices*. A *recurrence relation* is an equation which defines a sequence recursively, that is, each term of the sequence is defined as function of the preceding terms, together with *initial conditions*. The initial conditions are necessary to ensure an uniquely defined sequence. The aim of this paper is to study combinatorial methods to solve recurrence relations with two indices.

It well known there is no general procedure for solving recurrence relations, which is why it is an art. In this paper we present a combinatorial method for solving a special

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class of recurrence relations with two indices. More precisely, we study different types of recurrence relations with two indices where combinatorial methods provide a complete solution for these types of recurrence relations. To do that we need to define the following problem.

Hobby's problem. Let $\mathcal{A} = \{(n,m) \mid 0 \le m \le n\}$ be the second octant of the plane integer lattice $\mathbb{Z}^2$. Assume there is a rabbit, called Hobby, at his home at $O = (0,0) \in \mathcal{A}$ and his n bunnies are located at points $(j, n-1)$ for $j = 0, 1, \ldots, n-1$. Suppose that Hobby can not jump to the left, that is, it can jump from point $(i,j) \in \mathcal{A}$ to the point $(k, j+1) \in \mathcal{A}$ with $i \le k \le j+1$. *Hobby's problem* is to find the number of ways (lattice paths) for Hobby to get from his home to one of his n bunnies.

Our main goal is the use of the combinatorial methods and the kernel method (see [1]) to obtain an explicit solution for several types of recurrences relation with two indices. The paper is organized as follows. In Section 2 we deal with recurrence relation with two indices of the form

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j}, \qquad j = 0, 1, \ldots, n-1,$$

with the initial condition $a_{n,n} = \sum_{j=0}^{n-1} a_{n-1,j}$. This allows us to relate this recurrence relation to Dyck paths. In Section 3 we generalize our methods to study the recurrence relation

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j-1} + (\ell+1)a_{n-1,j}, \qquad j = 0, 1, \ldots, n-2,$$

with initial conditions $a_{n,n} = a_{n,n-1} = \ldots = a_{n,n-\ell} = \sum_{j=0}^{n-1} a_{n-1,j}$. This allow us to relate this recurrence relation to Schröder paths.

2. Recurrence relation with two indices and Dyck paths

Let $a_{n,j}$ be a sequence with two indices satisfies the following recurrence relation

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j}, \qquad j = 0, 1, \ldots, n-1, \tag{2.1}$$

with the initial conditions $a_{n,n} = \sum_{j=0}^{n-1} a_{n-1,j}$ and $a_{0,0} = 1$. In this section we present two different methods for finding an explicit formula for the general term of the sequence $a_{n,j}$.

2.1. **The Kernel method.** The first of these methods can be described as follows. First, define $A_n(v) = \sum_{j=0}^{n} a_{n,j} v^j$ and $A(x;v) = \sum_{n \ge 0} A_n(v) x^n$. Multiplying (2.1) by v^j and summing over all $j = 0, 1, \ldots, n-1$ we arrive at

$$A_n(v) = \sum_{j=0}^{n} \frac{v^j - v^n}{1-v} a_{n-1,j} + v^n A_{n-1}(1) = \frac{1}{1-v}(A_{n-1}(v) - v^{n+1} A_{n-1}(1)).$$

Again, multiplying the above recurrence relation by x^n, summing over all $n \ge 1$ and using the initial condition $A_0(v) = 1$ we obtain the following functional equation

$$A(x;v) = 1 + \frac{x}{1-v}(A(x;v) - v^2 A(xv;1)),$$

which is equivalent to

$$\left(1 - \frac{x}{v(1-v)}\right) A(x/v;v) = 1 - \frac{xv}{1-v} A(x;1).$$

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This type of functional equation can be solved systematically using the *kernel method* (see [1]). In this case, if we assume that $v = \frac{1+\sqrt{1-4x}}{2x}$, then $A(x;1) = C^2(x)$, where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. Therefore,

$$A(x;v) = 1 + \frac{x(1-v^2C(xv))}{1-v-x} = \sum_{n\geq 0} x^n \sum_{j=0}^{n} v^j \sum_{i=0}^{n-j} (-1)^i \binom{n-j-i}{i} c_{n-i},$$

where c_n is the n-th Catalan number. Hence, the explicit formula for the (n,j)-th term of the sequence $a_{n,j}$ is given by $a_{n,j} = \sum_{i=0}^{n-j}(-1)^i\binom{n-j-i}{i}c_{n-i}$.

2.2. **Combinatorial method.** In this subsection we use a combinatorial method to solve recurrence relation (2.1). Assume that $a_{n,j}$ denotes the point (j,n) on the plane integer lattice $\mathbb{Z}^2$ as described in the following figure:

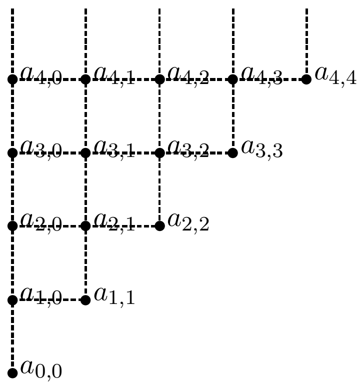

FIGURE 1. The terms of the sequence $a_{n,j}$.

Now, if we define another integer lattice with the property that two points (j,n) and (j',n') in the second octant are connect by a line segment if $n' = n-1$ and $0 \leq j' \leq j$, then we get that finding the $(n-1,j)$-th term of the sequence $a_{n,j}$ is equivalent to finding the number of Hobby's paths to get from his home to the $(j+1)$-st of his n bunnies. For example, Figure 2 presents all of Hobby's paths in the case $n = 2,3,4$.

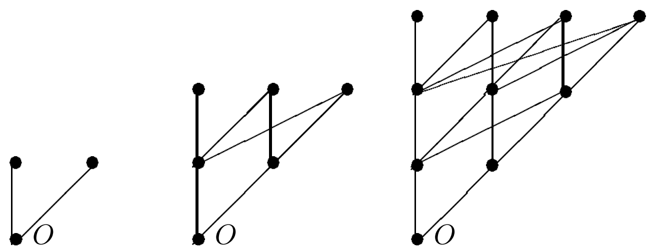

FIGURE 2. Hobby's paths to get from his home to one of his n bunnies, where $n = 2,3,4$.

Now, we define a map ψ which maps the set of Hobbyś paths to get from his home to one of his n bunnies to the set of Dyck paths of length $2n$. Let $p = (0,0)(x_1,1)\ldots(x_{n-1},n-1)$ be Hobby's path of length n. We read the path p from left to right and successively

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generate the Dyck path. First we start the Dyck path by an up-step U. When $p_j = (x_j, j)$, $j \geq 1$, is read, then in the Dyck path we adjoin $x_j - x_{j-1}$ down-steps D followed by an up-step U. At the end of the Dyck path, we adjoin as many down-steps D as necessary to return to the x-axis. For example, let $p = (0,0)(0,1)(2,2)(2,3)(4,4)$ be

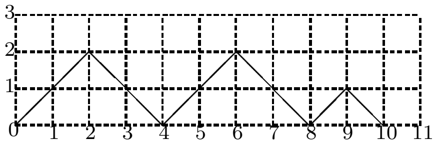

FIGURE 3. The Dyck path $\psi((0,0)(0,1)(2,2)(2,3)(4,4)) = UUDDUUDDUD$.

a Hobby's path of length 5 and let our Dyck path start with an up-step U. The first element to be read is $(0,1)$, therefore the Dyck path continues with one up-step U. Next $(2,2)$ is read, therefore the Dyck path continues with two down-steps D followed by an up-step U, etc. The complete Dyck path $\psi(p) = UUDDUUDDUD$ is shown in Figure 3.

Theorem 2.1. *The map ψ is a bijection between the set of Hobby's paths of length n and the set of Dyck paths of length $2n$. Moreover, the number of Hobby's paths of length n is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.*

Proof. Let $p = (0,0)(x_1,1)(x_2,2)\ldots(x_{n-1},n-1)$ be Hobby's path, so $x_j - x_{j-1} \geq 0$ and $j \geq x_j$ for all $j = 1,2,\ldots,n-1$. Thus it is possible always to adjoin the Dyck path by $x_j - x_{j-1}$ down-steps (formed by definition of ψ when considering (x_j, j)), hence $\psi(p)$ is a Dyck path of length $2n$. Conversely, given a Dyck path q of length $2n$, the inverse of ψ can be defined as follows. Assume that q can be decomposed as $uq^{(1)}q^{(2)}\ldots q^{(n-1)}dd\ldots d$ with $q^{(j)} = d\ldots du$ with $i_j \geq 0$ down-steps, for all $j = 1,2,\ldots,n-1$. Then we define the inverse of ψ as $\psi^{-1}(q) = p = (0,0)(i_1,1)(i_1+i_2,2)\ldots(i_1+\cdots+i_{n-1},n-1)$. So, from the definitions we get that p is a Hobby's path of length n, as required. $\square$

By the definition of the bijection ψ we have that the number of Hobby's paths to get from his home to the $(j+1)$-st of his n bunnies is the same as the number of Dyck paths of length $2n$ ending with exactly $n-1-j$ down-steps, and it is the same as the $(n-1,j)$-th term of the sequence $a_{n,j}$. Therefore we can state the following result.

Theorem 2.2. *Let $0 \leq j \leq n-1$. Then the number of Hobby's paths of length n to get from his home to the $(j+1)$-st of his n bunnies equals the $(n-1,j)$-th term of the sequence $a_{n,j}$ satisfying recurrence relation (2.1) and also equals the number of Dyck paths of length $2n$ ending with exactly $n-1-j$ down-steps which is given by*

$$\sum_{i=0}^{n-1-j} (-1)^i \binom{n-1-j-i}{i} C_{n-1-i}.$$

2.3. **Generalizations.** Here we present several directions to generalize the results of the previous section. The first of these directions is to consider the following recurrence relation, where the last term of (2.1) has been deleted:

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j-1}, \qquad j = 0,1,\ldots,n, \tag{2.2}$$

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with the initial condition $a_{n,0} = 0$ for all $n \geq 0$.

It is easy to see that the (n, j)-th term of the sequence $a_{n,j}$ that satisfies (2.2) equals the number of Hobby's paths with the restriction that Hobby does not jump from point (a, b) to point $(a, b + 1)$ with $a \geq 1$. Thus, from the definition of the bijection ψ we can be state the following result.

Theorem 2.3. *The (n, j)-th term of the sequence $a_{n,j}$ that satisfies (2.2) equals the number of Dyck paths that avoid DUU (that is, there is no down-step followed by two up-steps). Moreover, for all $n \geq 0$, $a_{n,0} = 1$ and $a_{n,j} = 2^{j-1}$ with $j = 1, 2, \ldots, n$.*

The second of these directions is to consider the following recurrence relation

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j}, \qquad j = 0, 1, \ldots, n, \tag{2.3}$$

with the initial condition $a_{n,0} = 1$ and $a_{n,j} = 0$ for all $n = 0, 1, \ldots, \ell$ and $j = 1, 2, \ldots, n - 1$. The problem of finding the (n, j)-th term of the above sequence that satisfies (2.3) is equivalent to finding the number of Hobby's paths where each path of Hobby starts with $(0, 0)(0, 1) \ldots (0, \ell)$. Thus, the bijection ψ maps these Hobby's paths to Dyck paths starting with $\ell + 1$ up-steps. Hence, we have the following result.

Corollary 2.4. *Let $n \geq \ell + 1$. The bijection ψ maps the set of Hobby's paths to get from his home to one of his n bunnies starting with $(0, 0)(0, 1) \ldots (0, \ell)$ to the set of Dyck paths of length $2n$ starting with ℓ up-steps. Moreover the number of such paths is given by $\frac{2+\ell}{2n-\ell}\binom{2n-\ell}{n+1}$.*

Proof. The first part of the corollary holds immediately from the bijection ψ. The second part can be checked easily by using the first return decomposition of Dyck paths. □

3. Recurrence relations with two indices and ℓ-Schröder paths

Let $a_{n,j} = a^{\ell}_{n,j}$ be a sequence with two indices satisfying the following recurrence relation

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \ldots + a_{n-1,j-1} + (1 + \ell)a_{n-1,j}, \qquad j = 0, 1, \ldots, n - 1 - \ell, \tag{3.1}$$

with the initial conditions $a_{n,n} = a_{n,n-1} = \cdots = a_{n,n-\ell} = \sum_{j=0}^{n-1} a_{n-1,j}$ and $a_{0,0} = 1$. In this section we present two different methods for finding an explicit formula for the general term of the the sequence $a_{n,j}$.

3.1. **The Kernel method.** We modify the definitions from Section 2.1 by defining $A^{\ell}_n(v) = \sum_{j=0}^{n} a^{\ell}_{n,j} v^j$ and $A_{\ell}(x; v) = \sum_{n\geq 0} A^{\ell}_n(v)x^n$. Multiplying recurrence relation (3.1) by v^j and summing over all $j = 0, 1, \ldots, n - 1 - \ell$ we arrive at

$$\begin{aligned} A^{\ell}_n(v) = \tfrac{v}{1-v}(A^{\ell}_{n-1}(v) - v^n A^{\ell}_{n-1}(1)) + (1 + \ell)A^{\ell}_{n-1}(v) \\ -\tfrac{v}{1-v}\left(\tfrac{v^{n-1-\ell}-v^n}{1-v} - (1 + \ell)v^{n-1}\right) A^{\ell}_{n-2}(1), \quad n \geq 1 + \ell. \end{aligned}$$

Again, multiplying the above recurrence relation by x^n, summing over all $n \geq 1$ and using the initial conditions $A^{\ell}_j(v) = j!\frac{1-v^{1+j}}{1-v}$, $j = 0, 1, \ldots, \ell$, we obtain the following

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functional equation

$$\begin{aligned}(3.2)\quad &\left(1-\tfrac{x}{1-v}-\tfrac{(1+\ell)x}{v}\right)\left(A_\ell(x/v;v)-\sum_{j=0}^{\ell-1} j!\tfrac{1-v^{1+j}}{1-v}\tfrac{x^j}{v^j}\right)\\ &=\ell!\tfrac{x^\ell}{v^\ell(1-v)}-\tfrac{xv}{1-v}\left(1-\tfrac{x}{1-v}-\tfrac{(1+\ell)x}{v}+\tfrac{x}{v^{1+\ell}(1-v)}\right)\left(A(x;1)-\sum_{j=0}^{\ell-2}(j+1)!x^j\right).\end{aligned}$$

This type of functional equation can be solved systematically using the *kernel method* [1]. In this case, if assume that $v=\frac{1+\ell x+\sqrt{(1-\ell x)^2-4x}}{2}$, then

$$A_\ell(x;1)=\sum_{j=0}^{\ell-2}(j+1)!x^j+\frac{\ell!x^{\ell-1}}{1-\ell x}C\left(\frac{x}{(1-\ell x)^2}\right),$$

where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. For example, if $\ell=0$ then we have that $A_0(x;1)=C^2(x)=\frac{(1-\sqrt{1-4x})^2}{4x^2}$ as shown in Theorem 2.1.

3.2. **Combinatorial method.** In this subsection we use a combinatorial method to solve recurrence relation (3.1), as follows. Assume that $a_{n,j}$ denotes the point (j,n) on the plane integer lattice $\mathbb{Z}^2$ as described in Figure 1. Now, if we define another integer lattice with the property that two points (j,n) and (j',n') the second octant connected by

- one line segment if $n'=n-1$ and $0\le j'\le j-1$ with $n\ge 1+\ell$,
- $1+\ell$ line segments if $n'=n-1$ and $j'=j$ with $n\ge 1+\ell$,
- one line segment if $n'=n-1$ and $j'=0,1,\ldots,n-1$ with $n=1,2,\ldots,\ell$.

With this definition, finding the $(n-1,j)$-th term of the sequence $a_{n,j}$ is equivalent to finding the number of Hobby's paths in the ℓ-problem of Hobbydefined as follows: assume that Hobby stays at his home $O=(0,0)\in\mathcal{A}$ and his n bunnies stay at points $(j,n-1)$ for $j=0,1,\ldots,n-1$. Hobby can jump either diagonally or it has ℓ possibilities to jump upwards from the point (j,m), where $0\le j\le m-1-\ell\le n-2-\ell$. In this context, the ℓ possibilities are called $\gamma^1,\gamma^2,\ldots,\gamma^\ell$ *Hobby's decisions*. The ℓ-problem of Hobby is to find the number of paths to get from his home to one of his n bunniesas shown in Figure 4 for $n=2,3,4,5$ and $\ell=2$.

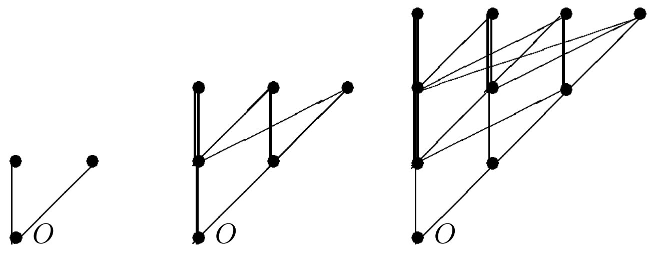

FIGURE 4. Hobby's paths to get from his home to one of his n bunnies with $\ell=2$, where $n=2,3,4,5$.

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Each of Hobby's paths to get from his home to one of his n bunnies can be coded as follows. Fix p to be the empty word and assume Hobby stays at his home. When Hobby jumps from point (a, b) to point

- $(a+c, b+1)$, $c \geq 1$, then in p we adjoin the letter u_c,
- $(a, b+1)$ with γ^j Hobby's decisions, then in p we adjoin the letter v_j.

For example, if $n = 2$ then Hobby has exactly six paths v_1v_1, v_1v_2, v_1u_1, v_1u_2, u_1v_1 and u_1u_1, see Figure 4.

Now, we define a map ϕ which maps the set of Hobby paths to get from his home to one of his n bunnies to the set of ℓ-Schröder paths of length $2n$. Let $p = p_1p_2 \dots p_n$ be Hobby's path of length n. We read the path p from left to right and successively generate the ℓ-Schröder path. First we start the ℓ-Schröder path by an up-step. When $p_j = u_{i_j}$ (resp. $p_j = v_1$, or $p_j = v_{i_j}$ with $i_j > 1$) is read, then in the ℓ-Schröder path we adjoin i_j down-steps followed by an up-step (resp. we adjoin one up-step, or a level-step labelled $i_j - 1$). At the end, in the ℓ-Schröder path we adjoin as many down-steps as necessary to return to x-axis. For example, let $p = u_0v_1u_1v_2$ be a Hobby's path of

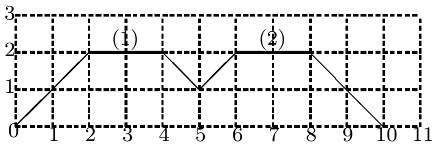

FIGURE 5. The 1-Schröder path $\phi(v_1v_2u_1v_2)$.

length 5 and let our ℓ-Schröder path start with an up-step. The first element to be read is u_0, therefore the ℓ-Schröder path continues with one up-step. Next v_1 is read, therefore the ℓ-Schröder path continues with one level-step labeled 1, etc. The complete ℓ-Schröder path $\phi(p)$ is shown in Figure 5. The reverse of the map ϕ is obvious, and we obtain the following result.

Theorem 3.1. *The map ϕ is a bijection between the set of ℓ-Hobby's paths of length n and the set of ℓ-Schröder paths of length $2n$. Moreover, the ordinary generating function for the number of Hobby's paths of length n is given by*

$$\sum_{j=0}^{\ell-2}(j+1)!x^j + \frac{\ell! x^{\ell-1}}{1-\ell x} C\left(\frac{x}{1-\ell x}\right),$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers.

In particular, the number of Hobby's paths of length n with $\ell = 0$ is given by the n-th Catalan number, and the number of Hobby's paths of length n with $\ell = 1$ is given by the n-th Schröder number.

3.3. **Generalizations.** Let us consider the following recurrence relation

$$a_{n,j} = a_{n-1,0} + a_{n-1,1} + \dots + a_{n-1,j+1}, \qquad j = 0, 1, \dots, n-2, \tag{3.3}$$

with the initial conditions $a_{0,0} = 0$ and $a_{n,n} = a_{n,n-1} = \sum_{j=0}^{n-1} a_{n-1,j}$. Here, the (n, j)-th term of the above sequence equals to the number of Hobby's paths of length n such

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that it can jump from point (b, j) to point $(b+1, j+1)$ for all $1 \leq j \leq b \leq n$; in this context this step is called the *back step* of Hobby at level b, see Figure 6 where the back step is shown with by a dashed line.

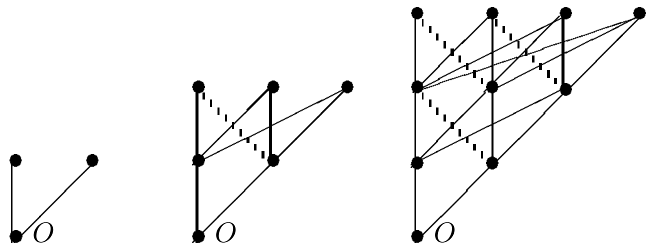

FIGURE 6. Hobby's paths of length n with back step, where $n = 2, 3, 4$.

Now, we define a map ϕ' which maps the set of Hobby's paths of length n with back steps into the set of Schröder paths of length $2n$. Let $p = (x_0, 0)(x_1, 1)(x_2, 2) \ldots (x_{n-1}, n-1)$ be Hobby's path of length n. We read the path p from left to right and successively generate the Schröder path. First we start the Schröder path with one up-step. When $p_j = (x_j, j)$ is read, then in the Schröder path we adjoin $x_j - x_{j-1}$ down-steps D followed by an up-step U (resp. horizontal-step H) if $x_j - x_{j-1} \geq 0$ (resp. $x_j - x_{j-1} = -1$). At the end, in the Schröder path we adjoin as many down-steps as necessary to return to the x-axis. Since the reverse map of ϕ' is obvious we get the following result.

Theorem 3.2. *The map ϕ' is a bijection between the set of Hobby's paths of length n with back steps to the set of Schröder paths of length $2n$.*

The above theorem gives the following corollary.

Corollary 3.3. *The bijection $\phi' : R \mapsto S$ between the set of Hobby's paths of length n with back steps and the set of Schröder paths of length $2n$ satisfies that the number of back steps of Hobby equals the number of horizontal-steps in the corresponding Schröder path.*

REFERENCES

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. Formal power series and algebraic combinatorics (Barcelona, 1999). *Discr. Math.* **246:1-3** (2002) 29–55.
[2] N.J.A. SLOANE AND S. PLOUFFE, The Encyclopedia of Integer Sequences, Academic Press, New York (1995).
[3] R. STANLEY, Enumerative Combinatorics, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, xi + 306 pages; second printing, Cambridge University Press, Cambridge, 1996.
[4] H. Wilf, *Generatingfunctionology*, Academic Press, New York, 1990.

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