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Recurrence relations for polynomial sequences via Riordan matrices

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A B S T R A C T

We give recurrence relations for any family of generalized Appell polynomials unifying so some known recurrences for many classical sequences of polynomials. Our main tool to get our goal is the Riordan group. We use the product of Riordan matrices to interpret some relationships between different polynomial families. Moreover using the Hadamard product of series we get a general recurrence relation for the polynomial sequences associated to the so called generalized umbral calculus.

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1. Introduction

In this paper we obtain recurrence relations for a large class of polynomial sequences. In fact, we get this for any family of generalized Appell polynomials [2]. Our main tool to reach our goal is the so called Riordan group [5,12,16,17].

This work is a natural consequence of our previous papers [8–10], and then it can be also considered as a consequence of the well-known Banach's Fixed Point Theorem. We have also to say that some papers related to this one have recently appeared in the literature [4,18] but our approach is different from that in those papers because, our main result herein is the discovering of a general recurrence relation for sequences of polynomials associated, naturally, to Riordan matrices. In particular we get a characterization of Riordan arrays by rows.

The Riordan arrays are usually described by the generating function of their columns or, equivalently, by the induced action on any power series. In fact a Riordan array can be defined as an infinite matrix where the k-column is just the kth term of a geometric progression in $\mathbb{K}[[x]]$ with rate a power series of order one. To get a proper Riordan array, eventually an element of the Riordan group, [16], we also impose that the first term in the progression is a power series of order zero.

In [10, Section 3], the authors studied polynomial families associated to some particular Riordan arrays which appeared in an iterative process to calculate the reciprocal of a quadratic polynomial. There, we interpreted some products of Riordan matrices as changes of variables in the associated families of polynomials. This interpretation will be exploited herein. Earlier in [9] the authors approached Pascal triangle by a dynamical point of view using the Banach Fixed Point Theorem. This approach is suitable to construct any Riordan array. From this point of view it seems that our $T(f|g)$ notation for a Riordan array is adequate, where $f = \sum_{n\geqslant 0} f_n x^n,\ g = \sum_{n\geqslant 0} g_n x^n$ with $g_0 \neq 0$. The notation $T(f|g)$ represents the Riordan array of first term $\frac{f}{g}$ and rate $\frac{x}{g}$. So the Pascal triangle P is just $T(1|1-x)$. The action on a power series s is given by $T(f|g)(s(x)) = \frac{f(x)}{g(x)} s\left(\frac{x}{g(x)}\right)$. The mixture of the role of the parameters on the induced action allowed us to get the following algorithm of construction for $T(f|g)$ which is essential to get the results in this paper:

Algorithm 1. Construction of $T(f|g)$ $f = \sum_{n\geqslant 0} f_n x^n$, $g = \sum_{n\geqslant 0} g_n x^n$ with $g_0 \neq 0$, $T(f|g) = (d_{n,j})$ with $n, j \geqslant 0$, $\frac{f}{g} = \sum_{n\geqslant 0} d_n x^n$ and $d_{n,0} = d_n$

$$\left(\begin{array}{c|cccccc} f_0 & & & & & & \\ f_1 & d_{0,0} & d_{0,1} & d_{0,2} & d_{0,3} & d_{0,4} & \cdots \\ f_2 & d_{1,0} & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \cdots \\ f_3 & d_{2,0} & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ f_{n+1} & d_{n,0} & d_{n,1} & d_{n,2} & d_{n,3} & d_{n,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with $d_{n,j} = 0$ if $j > n$ and the following rules for $n \geqslant j$:

If $j > 0$

$$d_{n,j} = -\frac{g_1}{g_0} d_{n-1,j} - \frac{g_2}{g_0} d_{n-2,j} \cdots - \frac{g_n}{g_0} d_{0,j} + \frac{d_{n-1,j-1}}{g_0}$$

and if $j = 0$

$$d_{n,0} = -\frac{g_1}{g_0} d_{n-1,0} - \frac{g_2}{g_0} d_{n-2,0} \cdots - \frac{g_n}{g_0} d_{0,0} + \frac{f_n}{g_0}.$$

Note that $d_{0,0} = \frac{f_0}{g_0}$. Then, in the 0-column are just the coefficients of $\frac{f}{g}$, i.e. $d_{n,0} = d_n$.

The main recurrence relation obtained in this paper is

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0} \tag{1}$$

which is closely related to the algorithm. The coefficients of the polynomials $(p_n(x))$ are, in fact, the entries in the rows of the Riordan matrix $T(f|g)$.

Since our $T(f|g)$ notation for Riordan arrays is not the more usual one, it is convenient to translate the above recurrence to the notation $(d(x), h(x))$ with $h(0) = 0$ and $h'(0) \neq 0$. Another slightly different notation is used in [5,17]. Since the rule of conversion is $(d(x), h(x)) = T\left(\frac{xd}{h}\middle|\frac{x}{h}\right)$, then the coefficients (f_n) and (g_n) in (1) are defined by $\frac{xd}{h} = \sum_{n\geqslant 0} f_n x^n$ and $\frac{x}{h} = \sum_{n\geqslant 0} g_n x^n$. At the beginning of Section 3, for the reader convenience, we display some useful formulas relating our notation with the classical one.

The matrix notation used above in the algorithm will appear often along this work so it deserves some explanation: really the matrix $T(f|g)$ is what appears to the right of the vertical line. The additional column to the left of the line, whose elements are just the coefficients of the series f, is needed for the construction of the 0-column of the matrix $T(f|g)$. Observe that if we consider the whole matrix, ignoring the line, we get the Riordan matrix $T(fg|g)$. This explanation is to avoid repetitions along the text.

The paper is organized into four sections. In Section 2 we take the Pascal triangle as our first motivation. This example is given here to explain and to motivate the interpretation of Riordan matrices by rows. In fact, the known recurrence for combinatorial numbers is the key to pass from the columns interpretation to the rows interpretation and viceversa. In this sense our Algorithm 1 is a huge generalization of the rule $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. Later, we choose some classical sequences of polynomials: Fibonacci, Pell and Morgan-Voyce polynomials to point out how the structure of Riordan matrix is intrinsically in the known recurrence relations for these families. So we are going to associate to any of these classical families a Riordan matrix which determines completely the sequence of polynomials. Using the product in the Riordan group, i. e. the matrix product, we easily recover some known relationships between them.

In Section 3, we get our main recurrence relation (1) as a direct consequence of Algorithm 1. The theoretical framework so constructed extends strongly and explains easily the examples in Section 2 and some relationships between these families. We also recover the generating function of a family of polynomials by means of the action of $T(f|g)$ on a power series. Later on, we obtain the usual umbral composition of polynomial families simply as a translation of the matrix product in the Riordan group. At the end of this section we add a table with some classical families of polynomials and their associated Riordan arrays in the classical and our notation.

In Section 4, we obtain some general recurrence relations for any family of generalized Appell polynomials, as a consequence of our main recurrence (1), and then of Algorithm 1. In this way we get into the so called generalized Umbral Calculus, see [13,14]. We use the Hadamard product of series to pass from the Riordan framework to the more general framework of generalized Appell polynomials because the sequences of Riordan type are those generalized Appell sequences related to the geometric series $\frac{1}{1-x}$, which is the neutral element for the Hadamard product. We also relate in this section the Riordan group with the so called delta-operators introduced by Rota et al. [15].

In this paper $\mathbb{K}$ always represents a field of characteristic zero and $\mathbb{N}$ is the set of natural numbers including 0.

2. Some classical examples as motivation

The best known description of Pascal triangle is by rows. With the next first simple classical example we point out how to pass from the column-description to the row-description. To do this for any Riordan array is our main aim.

Example 2 (*Pascal's triangle*)**.** The starting point of the construction of Riordan arrays is the Pascal triangle. From this point of view, Pascal triangle (by columns) are the terms of the geometric progression, in

$\mathbb{K}[[x]]$, of first term $\frac{1}{1-x}$ and rate $\frac{x}{1-x}$. So Pascal triangle P is, by columns, $P = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}, \frac{x^2}{(1-x)^3}, \ldots, \frac{x^n}{(1-x)^{n+1}}, \ldots,\right)$. Of course it is not the way to introduce Pascal triangle, or Tartaglia triangle, for the first time to students, because in particular it requires some understanding of the abstraction of infinity and order both on the *number* of columns and on the elements in any column. On the contrary, the non-null elements in any row of Pascal triangle form a finite set of data. Usually Pascal triangle is introduced by rows as the coefficients of the sequence of polynomials $p_n(x) = (1+x)^n$. The Newton formula $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ allows us to say that the nth row of Pascal triangle is, by increasing order of power of x, $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$. Using algebra, $(1+x)^{n+1} = (x+1)(1+x)^n$, or combinatorics, counting subsets, we see that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. This means that the Pascal triangle $P = (p_{n,k})_{n,k\in\mathbb{N}}$ follows the rule: $p_{n,0} = 1$ for every $n \in \mathbb{N}$, because $\binom{n}{0} = 1$ and $p_{n+1,k} = p_{n,k} + p_{n,k-1}$ for $1 \leqslant k \leqslant n$. Using for example the combinatorial interpretation of $\binom{n}{k}$ we see at once that $\binom{n}{k} = 0$ if $k > n$. What is the same, the Pascal triangle $(p_{n,k})_{n,k\in\mathbb{N}}$ is totally determined by the following recurrence relation: If we consider $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$ then $p_0(x) = 1$ and $p_{n+1}(x) = (x+1)p_n(x)$, $\forall n \geqslant 0$. It is obvious because the above relations means that $p_n(x) = (1+x)^n$.

Example 3 (*The Fibonacci polynomials, The Pell polynomials, The Morgan-Voyce polynomials*)**.** The Fibonacci polynomials are the polynomials defined by, $F_0(x) = 1$, $F_1(x) = x$ and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geqslant 2$. If we consider the sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ given by $g_0 = 1$, $g_1 = 0$, $g_2 = -1$, $g_n = 0$, $\forall n \geqslant 3$ and $f_0 = 1$, $f_n = 0$ $\forall n \geqslant 1$, we can unify the recurrence relation with the initial conditions because if we write

$$F_n(x) = \left(\frac{x-g_1}{g_0}\right) F_{n-1}(x) - \frac{g_2}{g_0} F_{n-2}(x) - \cdots - \frac{g_n}{g_0} F_0(x) + \frac{f_n}{g_0}$$

for $n \geqslant 0$ we obtain both: the recurrence relation and the initial conditions. Note that the above recurrence for Fibonacci polynomials fits the main recurrence relation (1).

If we consider the Riordan matrix, $T(f|g)$ for $f = 1$ and $g = 1 - x^2$, $T(1|1-x^2) = (d_{n,k})$ then the polynomials associated to $T(1|1-x^2)$ are just the Fibonacci polynomials. Using Algorithm 1, the rule of construction is: $d_{n,k} = d_{n-2,k} + d_{n-1,k-1}$, for $k > 0$, $d_{n,0} = d_{n-2,0}$ for $n \geqslant 2$, $d_{0,0} = 1$ and $d_{1,0} = 0$. The few first rows are:

$$\left(\begin{array}{c|cccccc}
1 & & & & & & \\
0 & 1 & & & & & \\
0 & 0 & 1 & & & & \\
0 & 1 & 0 & 1 & & & \\
0 & 0 & 2 & 0 & 1 & & \\
0 & 1 & 0 & 3 & 0 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Consequently the first associated polynomials (look at the rows of the matrix) are $F_0(x) = 1$, $F_1(x) = x$, $F_2(x) = 1 + x^2$, $F_3(x) = 2x + x^3$, $F_4(x) = 1 + 3x^2 + x^4, \ldots$ which are the Fibonacci polynomials. Using the induced action of $T(1|1-x^2)$ we get the generating function of this sequence

$$\sum_{n \geqslant 0} F_n(t)x^n = T\left(1|1-x^2\right)\left(\frac{1}{1-xt}\right) = \frac{1}{1-x^2-xt}.$$

The Pell polynomials are related to the Fibonacci polynomials. Consider $P_0(x) = 1$ and $P_1(x) = 2x$ with the polynomial recurrence $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$. So $\frac{x-g_1}{g_0} = 2x$, $\frac{-g_2}{g_0} = 1$ then $g(x) = \frac{1}{2}$

$-\frac{1}{2}x^2$ and $f(x)=\frac{1}{2}$. Hence the involved Riordan matrix is $T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right)$ with the rule of construction: $d_{n,k}=d_{n-2,k}+2d_{n-1,k-1},\ k>0$ and with generating function

$$\sum_{n\geqslant 0}P_n(t)x^n=T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right)\left(\frac{1}{1-xt}\right)=\frac{1}{1-x^2-2xt}.$$

We note that:

$$T\left(\frac{1}{2}\middle|1\right)T(1|1-x^2)T\left(1\middle|\frac{1}{2}\right)=T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right).$$

So, following Proposition 14 in [10], we get that $P_n(x)=F_n(2x)$ which is a known relation between Pell and Fibonacci polynomials.

Another related families of polynomials that we can treat using these techniques are the Morgan-Voyce families of polynomials. Consider now the Riordan matrices $T(1|(1-x)^2)$ and $T(1-x|(1-x)^2)$. These triangles have the same rule of construction $d_{n,k}=2d_{n-1,k}-d_{n-2,k}+d_{n-1,k-1}$ but different initial conditions. In fact they are:

$$\left(\begin{array}{c|cccccc}1&&&&&&\\0&1&&&&&\\0&2&1&&&&\\0&3&4&1&&&\\0&4&10&6&1&&\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\end{array}\right)\qquad\left(\begin{array}{c|cccccc}1&&&&&&\\-1&1&&&&&\\0&1&1&&&&\\0&1&3&1&&&\\0&1&6&5&1&&\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\end{array}\right)$$

where

$B_0(x)=1$	$b_0(x)=1$
$B_1(x)=2+x$	$b_1(x)=1+x$
$B_2(x)=3+4x+x^2$	$b_2(x)=1+3x+x^2$
$B_3(x)=4+10x+6x^2+x^3$	$b_3(x)=1+6x+5x^2+x^3$
In general	
$B_n(x)=(x+2)B_{n-1}(x)-B_{n-2}(x)$	$b_n(x)=(x+2)b_{n-1}(x)-b_{n-2}(x)$

with generating functions:

$$\sum_{n\geqslant 0}B_n(t)x^n=T(1|(1-x)^2)\left(\frac{1}{1-xt}\right)=\frac{1}{1-(2+t)x+x^2}$$
$$\sum_{n\geqslant 0}b_n(t)x^n=T(1-x|(1-x)^2)\left(\frac{1}{1-xt}\right)=\frac{1-x}{1-(2+t)x+x^2}$$

On the other hand it is known that the sequences $(B_n(x))_{n\in\mathbb{N}}$ and $(b_n(x))_{n\in\mathbb{N}}$ are related by means of the equalities:

$$B_n(x)=(x+1)B_{n-1}(x)+b_{n-1}(x),$$
$$b_n(x)=xB_{n-1}(x)+b_{n-1}(x).$$

Or equivalently

$$B_n(x)-B_{n-1}(x)=b_n(x), \tag{2}$$
$$b_n(x)-b_{n-1}(x)=xB_{n-1}(x). \tag{3}$$

These equalities can be interpreted by means of the product of adequate Riordan arrays. The first of them, (2), is

$$T(1-x|1)T(1|(1-x)^2)=T(1-x|(1-x)^2).$$

For the equality (3) we consider the product of matrices

$$T(1-x|1)T(1-x|(1-x)^2) = T((1-x)^2|(1-x)^2).$$

3. Polynomial sequences associated to Riordan matrices and its recurrence relations

In this section we are going to obtain the basic main result in this paper as a consequence of our algorithm in [9] and stated again in Section 1 as Algorithm 1. We use [9,10] for notation and basic results. Since our notation is not the usual one, for the convenience of the reader we present some relations between the classical and our notation for Riordan arrays.

Usually a Riordan array, D, is represented, up the name of the indeterminate, by $D = (d(x), h(x))$ or $D = \mathcal{R}(d(x), h(x))$, where $d(x)$ is a power series and the power series $h(x)$ is such that $h(0) = 0, h'(0) \neq 0$. If in addition $d(0) \neq 0$ then the Riordan array D is called proper. In our notation $D = T(f|g)$, with $g(0) \neq 0$, represents the Riordan array such that the generating function of the j-column is $\frac{x^j f}{g^{j+1}}$, beginning at $j = 0$. Equivalently the action induced in $\mathbb{K}[[x]]$ is:

$$T(f|g)(s) = \frac{f}{g}s\left(\frac{x}{g}\right) \quad \text{which represents the power series} \quad \frac{f(x)}{g(x)}s\left(\frac{x}{g(x)}\right)$$

while, in the usual notation, see [17]:

$$(d(x), h(x))(s(x)) = d(x)s(h(x));$$

equivalently the generating function of the j column is the series $d(x)h^j(x)$.

The basic formula relating them is

$$(d(x), h(x)) = T\left(\frac{xd}{h}\middle|\frac{x}{h}\right) = \left(\frac{f(x)}{g(x)}, \frac{x}{g}\right) = T(f|g) = (d_{i,j})_{i,j\geqslant 0}.$$

We are going to recall here some basic relations using both terminologies.

The representation of the product and the inverse (for a proper Riordan array) are, in both notations:

$$(d_1(x), h_1(x))(d_2(x), h_2(x)) = (d_1(x)d_2(h_1(x)), h_2(h_1(x))),$$

$$T(f_1|g_1)T(f_2|g_2) = T\left(f_1f_2\left(\frac{x}{g_1}\right)\middle| g_1g_2\left(\frac{x}{g_1}\right)\right).$$

The expression $f_1f_2\left(\frac{x}{g_1}\right)$ represents the power series $f_1(x)\cdot f_2\left(\frac{x}{g_1(x)}\right)$ and analogously for $g_1g_2\left(\frac{x}{g_1}\right)$

$$(d(x), h(x))^{-1} = \left(\frac{1}{d(\bar{h}(x))}, \bar{h}(x)\right) \quad (\bar{h}\circ h)(x) = (h\circ\bar{h})(x) = x,$$

$$T^{-1}(f|g) \equiv (T(f|g))^{-1} = T\left(\frac{1}{f(\bar{k})}\middle|\frac{1}{g(\bar{k})}\right), \quad k = \frac{x}{g}, \quad k\circ\bar{k} = \bar{k}\circ k = x$$

or mixing both notations:

$$T^{-1}(f|g) = T\left(\frac{1}{f(\bar{h})}\middle|\frac{1}{g(\bar{h})}\right)$$

if $T(f|g) = (d(x), h(x))$.

One of the main equalities is

$$T(f|g) = T(f|1)T(1|g)$$

or

$$\left(\frac{f(x)}{g(x)}, \frac{x}{g(x)}\right) = (f(x), x)\left(\frac{1}{g(x)}, \frac{x}{g(x)}\right)$$

or

$$(d(x), h(x)) = \left(\frac{xd(x)}{h(x)}, x\right)\left(\frac{h(x)}{x}, h(x)\right)$$

3.1. The main theorem

Definition 4. Consider an infinite lower triangular matrix $A = (a_{n,j})_{n,j\in\mathbb{N}}$. *The family of polynomials associated to A* is the sequence of polynomials $(p_n(x))_{n\in\mathbb{N}}$, given by

$$p_n(x) = \sum_{j=0}^{n} a_{n,j}x^j, \quad \text{with } n \in \mathbb{N}$$

Note that the coefficients of the polynomials are given by the entries in the rows of A in increasing order of the columns till the main diagonal. Note also that the degree of $p_n(x)$ is less than or equal to n. The family $p_n(x)$ becomes a *polynomial sequences*, in the usual sense, when the matrix A is invertible, that is, when all the elements in the main diagonal are non-null.

Our main result can be given in the following terms:

Theorem 5. *Let $D = (d_{n,j})_{n,j\in\mathbb{N}}$ be an infinite lower triangular matrix. D is a Riordan matrix, or an arithmetical triangle in the sense of [9], if and only if there exist two sequences (f_n) and (g_n) in $\mathbb{K}$ with $g_0 \neq 0$ such that the family of polynomials associated to D satisfies the recurrence relation*:

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0} \quad \forall n \geqslant 0.$$

Moreover, in this case, $D = T(f|g)$ where $f = \sum_{n\geqslant 0} f_n x^n$ and $g = \sum_{n\geqslant 0} g_n x^n$.

Proof. If D is a Riordan array we can identify this with an arithmetical triangle $D = T(f|g)$ such that $g_0 \neq 0$. Following Algorithm 1 we obtain that the family of polynomials associated to $T(f|g)$ satisfies:

$$\begin{aligned}
p_n(x) &= \sum_{j=0}^{n} d_{n,j}x^j = d_{n,0} + \sum_{j=1}^{n} d_{n,j}x^j \\
&= \frac{1}{g_0}\left(f_n - \sum_{k=1}^{n} g_k d_{n-k,0}\right) + \sum_{j=1}^{n}\left(\frac{1}{g_0}\left(d_{n-1,j-1} - \sum_{k=1}^{n} g_k d_{n-k,j}\right)\right)x^j \\
&= \frac{1}{g_0}\left(f_n - \sum_{j=1}^{n} d_{n-1,j-1}x^j - \sum_{k=1}^{n} g_k d_{n-k,0} - \sum_{j=1}^{n}\sum_{k=1}^{n} g_k d_{n-k,j}x^j\right) \\
&= \frac{1}{g_0}\left(f_n - xp_{n-1}(x) - \sum_{j=0}^{n}\sum_{k=1}^{n} g_k d_{n-k,j}x^j\right) = \frac{1}{g_0}\left(f_n - xp_{n-1}(x) - \sum_{k=1}^{n} g_k \sum_{j=0}^{n-k} d_{n-k,j}x^j\right) \\
&= \frac{1}{g_0}\left(f_n - xp_{n-1}(x) - \sum_{k=1}^{n} g_k p_{n-k}(x)\right) = \frac{1}{g_0}\left(f_n + (g_1 - x)p_{n-1}(x) - \sum_{k=2}^{n} g_k p_{n-k}(x)\right) \\
&= \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}
\end{aligned}$$

On the other hand, we suppose that

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}$$

for two sequences (f_n) and (g_n). We consider $D = (d_{n,k})$ such that $p_n(x) = \sum_{j=0}^{n} d_{n,j}x^j$. So $p_0(x) = \frac{f_0}{g_0}$ then $d_{0,0} = \frac{f_0}{g_0}$.

$$p_1(x) = \left(\frac{x-g_1}{g_0}\right)p_0(x) + \frac{f_1}{g_0} = -\frac{g_1}{g_0}d_{0,0} + \frac{f_1}{g_0} + \frac{d_{0,0}}{g_0}x$$

then

$$\begin{aligned} d_{1,0} &= -\frac{g_1}{g_0}d_{0,0} + \frac{f_1}{g_0}, \quad d_{1,1} = \frac{d_{0,0}}{g_0} \\ p_2(x) &= \left(\frac{x-g_1}{g_0}\right)p_1(x) - \frac{g_2}{g_0}p_0(x) + \frac{f_2}{g_0} \\ &= -\frac{g_1}{g_0}d_{1,0} - \frac{g_2}{g_0}d_{0,0} + \frac{f_2}{g_0} + \left(-\frac{g_1}{g_0}d_{1,1} + \frac{d_{1,0}}{g_0}\right)x + \frac{d_{1,1}}{g_0}x^2 \end{aligned}$$

so

$$d_{2,0} = -\frac{g_1}{g_0}d_{1,0} - \frac{g_2}{g_0}d_{0,0}, \quad d_{2,1} = -\frac{g_1}{g_0}d_{1,1} + \frac{d_{1,0}}{g_0}, \quad d_{2,2} = \frac{d_{1,1}}{g_0}$$

in general

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}$$

then

$$\begin{aligned} d_{n,0} &= -\frac{g_1}{g_0}d_{n-1,0} - \frac{g_2}{g_0}d_{n-2,0}\cdots - \frac{g_n}{g_0}d_{0,0} + \frac{f_n}{g_0} \\ d_{n,1} &= -\frac{g_1}{g_0}d_{n-1,1} - \frac{g_2}{g_0}d_{n-2,1}\cdots - \frac{g_n}{g_0}d_{0,1} + \frac{d_{n-1,0}}{g_0} \\ d_{n,j} &= -\frac{g_1}{g_0}d_{n-1,j} - \frac{g_2}{g_0}d_{n-2,j}\cdots - \frac{g_n}{g_0}d_{0,j} + \frac{d_{n-1,j-1}}{g_0} \end{aligned}$$

and

$$d_{n,n-1} = -\frac{g_1}{g_0}d_{n-1,n-1} + \frac{d_{n-1,n-2}}{g_0}, \quad d_{n,n} = \frac{d_{n-1,n-1}}{g_0}$$

then using our algorithm the matrix D is just $D = T(f|g)$ where $f(x) = \sum_{n\geqslant 0} f_n x^n$ and $g(x) = \sum_{n\geqslant 0} g_n x^n$. □

Corollary 6. *If $g(x) = g_0 + g_1x + g_2x^2 + \cdots + g_mx^m$ with $g_m \neq 0$ be a polynomial of degree m, the recurrence relation of Theorem 5 is eventually finite. It is,*

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_m}{g_0}p_{n-m}(x) + \frac{f_n}{g_0} \quad n \geqslant m$$

and

$$p_k(x) = \left(\frac{x-g_1}{g_0}\right)p_{k-1}(x) - \sum_{i=2}^{k}\frac{g_i}{g_0}p_{k-i}(x) + \frac{f_k}{g_0} \quad 0 \leqslant k \leqslant m-1.$$

Remark 7. Following [9] the arithmetical triangle $T(f|g)$ above is an element of the Riordan group when it is invertible for the product of matrices. It is obviously equivalent to the fact that $f_0 \neq 0$ in the sequence (f_n) above.

Suppose that we have two Riordan matrices $T(f|g)$, $T(l|m)$ with $f = \sum_{n\geqslant 0} f_n x^n$, $g = \sum_{n\geqslant 0} g_n x^n$ $l = \sum_{n\geqslant 0} l_n x^n$ and $m = \sum_{n\geqslant 0} m_n x^n$ with $g_0, m_0 \neq 0$. Consider the corresponding polynomial families $(p_n(x))_{n\in\mathbb{N}}$ and $(q_n(x))_{n\in\mathbb{N}}$ associated to $T(f|g)$ and $T(l|m)$ respectively, as in Theorem 5. Using the matrix representation of $T(f|g)$ and $T(l|m)$, [9], and the product of matrices, we can define an operation $\sharp$ on these sequences of polynomials as follows:

We say that

$$(p_n(x))_{n\in\mathbb{N}}\sharp(q_n(x))_{n\in\mathbb{N}} = (r_n(x))_{n\in\mathbb{N}}$$

if $(r_n(x))_{n\in\mathbb{N}}$ is the family of polynomials associated to the Riordan matrix

$$T(f|g)T(l|m) = T\left(fl\left(\frac{x}{g}\right)\middle| gm\left(\frac{x}{g}\right)\right)$$

see [9].

Suppose $T(f|g) = (p_{n,k})_{n,k\in\mathbb{N}}$, $T(l|m) = (q_{n,k})_{n,k\in\mathbb{N}}$ and $T\left(fl\left(\frac{x}{g}\right)\middle| gm\left(\frac{x}{g}\right)\right) = (r_{n,k})_{n,k\in\mathbb{N}}$. Consequently $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$, $q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k$ and $r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$.

So, by the product of matrices, the entries in the n-row of $(r_{n,k})$, which are just the coefficients of $r_n(x)$ in increasing order of the power of x, are given by:

$$\left(\sum_{k=0}^{n} p_{n,k}q_{k,0}, \sum_{k=1}^{n} p_{n,k}q_{k,1}, \ldots \sum_{k=j}^{n} p_{n,k}q_{k,j} \ldots p_{n,n}q_{n,n}, 0, \ldots\right)$$
$$= p_{n,0}(q_{0,0}, 0, \ldots, 0, \ldots) + p_{n,1}(q_{1,0}, q_{1,1}, 0, \ldots, 0, \ldots) + \cdots + p_{n,n}(q_{n,0}, q_{n,1}, \ldots, q_{n,n}, 0, \ldots)$$

Consequently

$$r_n(x) = \sum_{k=0}^{n} p_{n,k}q_k(x)$$

which corresponds to replace in the expression of $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$ the power x^k by the element $q_k(x)$ in the sequence of polynomials $(q_n(x))_{n\in\mathbb{N}}$. This is in the spirit of the Blissard symbolic's method, see [1] for an exposition on this topic. The product $(p_n(x))_{n\in\mathbb{N}}\sharp(q_n(x))_{n\in\mathbb{N}} = (r_n(x))_{n\in\mathbb{N}}$ is usually called the umbral composition of the sequences of polynomials $(p_n(x))$ and $(q_n(x))$. The formula for the umbral composition is given by

$$(p_n(x))_{n\in\mathbb{N}}\sharp(q_n(x))_{n\in\mathbb{N}} = (r_n(x))_{n\in\mathbb{N}}, \quad \text{where } r_{n,j} = \sum_{k=j}^{n} p_{n,k}q_{k,j}$$

As a summary of the above construction we have:

Theorem 8. *Suppose four sequences of elements of* $\mathbb{K}$, $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, $(l_n)_{n\in\mathbb{N}}$, $(m_n)_{n\in\mathbb{N}}$, *with* $g_0, m_0 \neq 0$. *Consider the sequences of polynomials* $(p_n(x))_{n\in\mathbb{N}}$ $(q_n(x))_{n\in\mathbb{N}}$ *satisfying the following recurrences relations*

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x)\cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}$$

with $p_0(x) = \frac{f_0}{g_0}$,

$$q_n(x) = \left(\frac{x-m_1}{m_0}\right)q_{n-1}(x) - \frac{m_2}{m_0}q_{n-2}(x)\cdots - \frac{m_n}{m_0}q_0(x) + \frac{l_n}{m_0}$$

with $q_0(x) = \frac{l_0}{m_0}$. *Then the umbral composition* $(p_n(x))_{n\in\mathbb{N}}\sharp(q_n(x))_{n\in\mathbb{N}} = (r_n(x))_{n\in\mathbb{N}}$ *satisfies the following recurrence relation*

$$r_n(x) = \left(\frac{x-\alpha_1}{\alpha_0}\right)r_{n-1}(x) - \frac{\alpha_2}{\alpha_0}r_{n-2}(x)\cdots - \frac{\alpha_n}{\alpha_0}r_0(x) + \frac{\beta_n}{\alpha_0},$$

where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ *are sequences such that* $fl\left(\frac{x}{g}\right)=\sum_{n\geqslant 0}\beta_n x^n$, $gm\left(\frac{x}{g}\right)=\sum_{n\geqslant 0}\alpha_n x^n$, *with* $f=\sum_{n\geqslant 0} f_n x^n$, $g=\sum_{n\geqslant 0} g_n x^n$ $l=\sum_{n\geqslant 0} l_n x^n$ *and* $m=\sum_{n\geqslant 0} m_n x^n$.

Of special interest is when we restrict ourselves to the so called proper Riordan arrays, see [17]. As noted in Remark 7 this is the case when $f_0\neq 0$ or, equivalently, $T(f|g)$ is in the Riordan group. Moreover, in this case, the assignment $T(f|g)\to(p_n(x))_{n\in\mathbb{N}}$ is injective, obviously, and since the product of matrices converts to the umbral composition of the corresponding associated polynomial sequences, we have the following alternative description of the Riordan group.

Theorem 9. *Let* $\mathbb{K}$ *be a field of characteristic zero. Consider* $\mathcal{R}=\{(p_n(x))_{n\in\mathbb{N}}\}$ *where* $(p_n(x))_{n\in\mathbb{N}}$ *is a polynomial sequence with coefficients in* $\mathbb{K}$ *satisfying that there are two sequences* $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ *of elements of* $\mathbb{K}$, *depending on* $(p_n(x))_{n\in\mathbb{N}}$, *with* $f_0, g_0\neq 0$ *and such that*

$$p_n(x)=\left(\frac{x-g_1}{g_0}\right)p_{n-1}(x)-\frac{g_2}{g_0}p_{n-2}(x)\cdots-\frac{g_n}{g_0}p_0(x)+\frac{f_n}{g_0}$$

with $p_0(x)=\frac{f_0}{g_0}$.

Given $(p_n(x))_{n\in\mathbb{N}}$, $(q_n(x))_{n\in\mathbb{N}}\in\mathcal{R}$ *Define* $(p_n(x))_{n\in\mathbb{N}}\sharp(q_n(x))_{n\in\mathbb{N}}=(r_n(x))_{n\in\mathbb{N}}$ *where* $r_n(x)=\sum_{k=0}^n p_{n,k}q_k(x)$ *with* $p_n(x)=\sum_{k=0}^n p_{n,k}x^k$. *Then* $(\mathcal{R},\sharp)$ *is a group isomorphic to the Riordan group. Moreover*

$$\sum_{n\geqslant 0} p_n(t)x^n=\frac{f(x)}{g(x)-xt}$$

if $f=\sum_{n\geqslant 0} f_n x^n$ *and* $g=\sum_{n\geqslant 0} g_n x^n$ *and* (f_n) *and* (g_n) *are the sequences generating the polynomial sequence* $(p_n(x))$ *in* $\mathcal{R}$.

Proof. Only a proof of the final part is needed. As we know, from Theorem 5, $T(f|g)=(p_{n,k})_{n,k\in\mathbb{N}}$ is a proper Riordan array where $p_n(x)=\sum_{k=0}^n p_{n,k}x^k$, $\frac{1}{1-xt}=\sum_{n\geqslant 0} t^n x^n$. We consider, symbolically, $\frac{1}{1-xt}$ as a power series on x with *parametric* coefficients $a_n=t^n$. From this point of view, [9],

$$T(f|g)\left(\frac{1}{1-xt}\right)=\begin{pmatrix} p_{0,0} & & & & & \\ p_{1,0} & p_{1,1} & & & & \\ p_{2,0} & p_{2,1} & p_{2,2} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix}1\\ t\\ t^2\\ \vdots\\ t^n\\ \vdots\end{pmatrix}=\sum_{k=0}^{n} p_n(t)x^k$$

$$T(f|g)\left(\frac{1}{1-xt}\right)=\frac{f(x)}{g(x)}\frac{1}{1-t\frac{x}{g}}=\frac{f(x)}{g(x)-xt}\qquad\square$$

Remark 10. Note that $\sum_{k=0}^n p_n(t)x^k$ is just the bivariate generating function of the Riordan array $T(f|g)=(p_{n,k})_{n,k\in\mathbb{N}}$ in the sense of [17].

3.2. Some relationships between polynomials sequences of Riordan type. Some classical examples

Now we are going to describe some relations between polynomial sequences associated to different Riordan arrays. From now on we are going to use the following definition:

Definition 11. Let $(p_n(x))_{n\in\mathbb{N}}$ be a sequence of polynomials in $\mathbb{K}[[x]]$, $p_n(x)=\sum_{k=0}^n p_{n,k}x^k$. We say that $(p_n(x))_{n\in\mathbb{N}}$ is a *polynomial sequence of Riordan type* if the matrix $(p_{n,k})$ is an element of the Riordan group.

Using the basic equality $T(f|g) = T(f|1)T(1|g)$ we can get some formulas.

Proposition 12. *Let $T(f|g)$ an element of the Riordan group and suppose $(p_n(x))$ the corresponding associated family of polynomials. Let $h(x) = h_0 + h_1x + h_2x^2 + \cdots + h_mx^m$ be a m degree polynomial, $h_m \neq 0$. Let $(q_n(x))$ be the associated family of polynomials of $T(h|1)T(f|g)$ then*

$$\begin{aligned}
&q_0(x) = h_0p_0(x)\\
&q_1(x) = h_1p_0(x) + h_0p_1(x)\\
&\vdots\\
&q_m(x) = h_mp_{n-m}(x) + \cdots + h_0p_m(x)\\
&q_n(x) = h_mp_{n-m}(x) + \cdots + h_0p_n(x) \quad n \geqslant m
\end{aligned}$$

Remark 13. Note that to multiply by the left by the Toepliz matrix $T(h|1)$ above corresponds eventually to make some fixed elementary operations by rows on the matrix $T(f|g)$. These operations are completely determined by the coefficients of the polynomial h. For example if $h(x) = a + bx$ then $q_0(x) = ap_0(x)$ and $q_n(x) = bp_{n-1}(x) + ap_n(x)$.

As a direct application of Proposition 12 we will obtain the known relationships between Chebysev polynomials of the first and second kind.

Example 14 (*The Chebyshev polynomials of the first and the second kind*)**.** Consider the Chebyshev polynomials of the second kind:

$$\begin{aligned}
&U_0(x) = 1\\
&U_1(x) = 2x\\
&U_2(x) = 4x^2 - 1\\
&U_3(x) = 8x^3 - 4x\\
&U_4(x) = 16x^4 - 12x^2 + 1\\
&U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad \text{for } n \geqslant 2
\end{aligned} \tag{4}$$

Let the sequences $(l_n)_{n\in\mathbb{N}}$, $(m_n)_{n\in\mathbb{N}}$ given by $l_0 = \frac{1}{2}$ and $l_n = 0$ for $n \geqslant 1$ and $m_0 = \frac{1}{2}$, $m_2 = \frac{1}{2}$ and $m_n = 0$ otherwise. In this case (4) can be converted to

$$\begin{aligned}
&U_0(x) = \tfrac{l_0}{m_0}\\
&U_n(x) = \left(\tfrac{x-m_1}{m_0}\right)U_{n-1}(x) - \tfrac{m_2}{m_0}U_{n-2}(x)\cdots - \tfrac{m_n}{m_0}U_0(x) + \tfrac{l_n}{m_0}, \quad \text{for } n \geqslant 1
\end{aligned} \tag{5}$$

If $U = (u_{n,k})_{n,k\in\mathbb{N}}$ where $U_n(x) = \sum_{k=0}^{n} u_{n,k}x^k$ then using our algorithm, or equivalently Theorem 5, we obtain that $U = T\left(\frac{1}{2}\middle|\,\frac{1}{2} + \frac{1}{2}x^2\right)$ is a Riordan matrix:

$$\left(\begin{array}{c|cccccc}
\frac{1}{2} & & & & & & \\
0 & 1 & & & & & \\
0 & 0 & 2 & & & & \\
0 & -1 & 0 & 4 & & & \\
0 & 0 & -4 & 0 & 8 & & \\
0 & 1 & 0 & -12 & 0 & 16 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

So the associated polynomials of this arithmetical triangle are the Chebyshev polynomials of the second kind. Consequently

$$\sum_{n\geqslant 0} U_n(t)x^n = T\left(\frac{1}{2}\middle|\,\frac{1}{2} + \frac{1}{2}x^2\right)\left(\frac{1}{1-xt}\right) = \frac{1}{1+x^2-2xt}.$$

The first few Chebyshev polynomials of the first kind are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1 \cdots$ In general

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{for } n \geqslant 2.$$

We first produce a small perturbation in this classical sequence. Consider a new sequence $(\widetilde{T}(x))_{n\in\mathbb{N}}$ where $\widetilde{T}_0(x) = \frac{1}{2}$ and $\widetilde{T}_n(x) = T_n(x)$ for every $n \geqslant 1$. For this new sequence we have the following recurrence relation

$$\begin{aligned}
\widetilde{T}_0(x) &= \frac{1}{2} \\
\widetilde{T}_1(x) &= 2x\widetilde{T}_0(x) \\
\widetilde{T}_2(x) &= 2x\widetilde{T}_1(x) - \widetilde{T}_0(x) - \frac{1}{2} \\
\widetilde{T}_n(x) &= 2x\widetilde{T}_{n-1}(x) - \widetilde{T}_{n-2}(x) \quad \text{for } n \geqslant 3
\end{aligned} \tag{6}$$

to unify the above equalities we consider the sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ given by $f_0 = \frac{1}{4}, f_2 = -\frac{1}{4}$ and $f_n = 0$ otherwise, $g_0 = \frac{1}{2}, g_2 = \frac{1}{2}$ and $g_n = 0$ otherwise. We note that the equalities in (6) can be converted to

$$\begin{aligned}
&\widetilde{T}_0(x) = \tfrac{f_0}{g_0} \\
&\widetilde{T}_n(x) = \left(\tfrac{x-g_1}{g_0}\right)\widetilde{T}_{n-1}(x) - \tfrac{g_2}{g_0}\widetilde{T}_{n-2}(x) \cdots - \tfrac{g_n}{g_0}\widetilde{T}_0(x) + \tfrac{f_n}{g_0}, \quad \text{for } n \geqslant 1
\end{aligned} \tag{7}$$

Let $\widetilde{T} = (\tilde{t}_{n,k})$ be the matrix given by $\widetilde{T}_n(x) = \sum_{k=0}^{n} \tilde{t}_{n,k}x^k$. One can verifies that (7) converts to $\tilde{t}_{n,k} = 0$ if $k > n$ and the following rules for $n \geqslant k$:

$$\tilde{t}_{n,j} = -\frac{g_1}{g_0}\tilde{t}_{n-1,j} - \frac{g_2}{g_0}\tilde{t}_{n-2,j} \cdots - \frac{g_n}{g_0}\tilde{t}_{0,j} + \frac{\tilde{t}_{n-1,j-1}}{g_0} \quad \text{if } j \geqslant 1$$

and if $j = 0$

$$\tilde{t}_{n,0} = -\frac{g_1}{g_0}\tilde{t}_{n-1,0} - \frac{g_2}{g_0}\tilde{t}_{n-2,0} \cdots - \frac{g_n}{g_0}\tilde{t}_{0,0} + \frac{f_n}{g_0}$$

Note that $\tilde{t}_{0,0} = \frac{f_0}{g_0}$ because the empty sum evaluates to zero.

Using our algorithm in [9], we obtain that $\widetilde{T}$ is a Riordan matrix. In fact we get $\widetilde{T} = T\left(\frac{1}{4} - \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}x^2\right)$ in our notation, because $f(x) = \frac{1}{4} - \frac{1}{4}x^2$ is the generating function of the sequence (f_n) and $g(x) = \frac{1}{2} + \frac{1}{2}x^2$ is the generating function of the sequence (g_n). So

$$\left(\begin{array}{c|cccccc}
\frac{1}{4} & & & & & & \\
0 & \frac{1}{2} & & & & & \\
\frac{1}{4} & 0 & 1 & & & & \\
0 & -1 & 0 & 2 & & & \\
0 & 0 & -3 & 0 & 4 & & \\
0 & 1 & 0 & -8 & 0 & 8 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

But now more can be said because

$$\sum_{n\geqslant 0} \widetilde{T}_n(t)x^n = T\left(\frac{1}{4} - \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}x^2\right)\left(\frac{1}{1-tx}\right) = \frac{1}{2}\frac{1-x^2}{1+x^2-2tx}.$$

Since

$$\sum_{n\geqslant 0} T_n(t)x^n = \frac{1}{2} + \sum_{n\geqslant 0} \widetilde{T}_n(t)x^n$$

we get the generating function

$$\sum_{n\geqslant 0} T_n(t)x^n = \frac{1-tx}{1+x^2-2tx}$$

of the classical Chebyshev polynomials of the first kind.

Using the involved Riordan matrices we can find the known relation between $T_n(x)$ and $U_n(x)$. Since

$$T\left(\frac{1}{4}-\frac{1}{4}x^2\middle|\frac{1}{2}+\frac{1}{2}x^2\right) = T\left(\frac{1}{2}-\frac{1}{2}x^2\middle|1\right)T\left(\frac{1}{2}\middle|\frac{1}{2}+\frac{1}{2}x^2\right)$$

Consequently

$$\widetilde{T}_n(x) = -\frac{1}{2}U_{n-2}(x)+\frac{1}{2}U_n(x) \quad \text{or} \quad 2\widetilde{T}_n(x) = U_n(x)-U_{n-2}(x)$$

and then

$$2T_n(x) = U_n(x)-U_{n-2}(x), \quad n\geqslant 3$$

As we noted in Section 4 of [8], if we delete the first row and the first column in the Riordan matrix $T(f|g)$ we obtain the new Riordan matrix $T\left(\frac{f}{g}\middle|g\right)$. On the other hand to add suitably a new column to the left of $T(f|g)$, one place shifted up, and complete the new first row only with zeros we have the Riordan matrix $T(fg|g)$. So deleting or adding, in the above sense, any amount of rows and columns to $T(f|g)$ we obtain the intrinsically related family of Riordan matrices

$$\ldots, T(g^3f|g), T(g^2f|g), T(gf|g), \mathbf{T(f|g)}, T\left(\frac{f}{g}\middle|g\right), T\left(\frac{f}{g^2}\middle|g\right), T\left(\frac{f}{g^3}\middle|g\right), \ldots$$

We can easily obtain a recurrence to get the associated polynomials to $T\left(\frac{f}{g^n}\middle|g\right)$ in terms of that of $T(f|g)$. We have an analogous conclusion on $T(fg^n|g)$ $n\geqslant 0$. Anyway, once we know the polynomial associated to $T(f|g)$ we can calculate that of $T(fg^n|g)$ for $n\in\mathbb{Z}$.

Proposition 15. *Let $f=\sum_{n\geqslant 0} f_nx^n$, $g=\sum_{n\geqslant 0} g_nx^n$ be two power series such that $f_0\neq 0$, $g_0\neq 0$. Suppose that $(p_n(x))_{n\in\mathbb{N}}$ is the associated polynomial sequence of the Riordan array $T(f|g)$, then*

(a) If $(q_n(x))_{n\in\mathbb{N}}$ is the associated sequence to $T(fg|g)$ we obtain

$$q_n(x) = xp_{n-1}(x)+f_n \quad \text{if } n\geqslant 1$$

and $q_0(x)=f_0$.

(b) If $(r_n(x))_{n\in\mathbb{N}}$ is the associated polynomial sequence to $T\left(\frac{f}{g}\middle|g\right)$ then

$$r_{n-1}(x) = \frac{p_n(x)-p_n(0)}{x} \quad \text{for } n\geqslant 1.$$

Proof. (a) $T(fg|g) = T(g|1)T(f|g)$. Using the umbral composition we have

$$q_n(x) = g_np_0(x)+g_{n-1}p_1(x)+\cdots+g_0p_n(x).$$

Using now our Theorem 5 we obtain

$$q_n(x) = g_np_0(x)+g_{n-1}p_1(x)+\cdots + g_0\left(\left(\frac{x-g_1}{g_0}\right)p_{n-1}(x)-\frac{g_2}{g_0}p_{n-2}(x)\cdots-\frac{g_n}{g_0}p_0(x)+\frac{f_n}{g_0}\right).$$

Consequently

$$q_n(x) = xp_{n-1}(x)+f_n.$$

(b) Now $T(g|1)T\left(\frac{f}{g}\middle| g\right) = T(f|g)$. So

$$p_n(x) = g_n r_0(x) + g_{n-1} r_1(x) + \cdots + g_0 r_n(x)$$

using again Theorem 5 for the sequences $r_n(x)$ we obtain

$$p_n(x) = g_n r_0(x) + g_{n-1} r_1(x) + \cdots + g_0\left(\left(\frac{x-g_1}{g_0}\right) r_{n-1}(x) - \frac{g_2}{g_0} r_{n-2}(x) \cdots - \frac{g_n}{g_0} r_0(x) + \frac{d_n}{g_0}\right),$$

where the d_n is the n-coefficient of the series $\frac{f}{g}$. Consequently $p_n(x) = x r_{n-1}(x) + d_n$. Note that $p_n(0) = d_n$, so

$$r_{n-1}(x) = \frac{p_n(x) - p_n(0)}{x} \quad \text{if } n \geqslant 1 \qquad \square$$

Corollary 16. *Suppose $g = \sum_{n \geqslant 0} g_n x^n$ with $g_0 \neq 0$. Let $(p_n(x))_{n \in \mathbb{N}}$ be the polynomial sequence associated to $T(1|g)$ and $(q_n(x))_{n \in \mathbb{N}}$ that associated to $T(g|g)$. Then:*

$$q_n(x) = x p_{n-1}(x) \quad \text{for } n \geqslant 1 \quad \text{and} \quad q_0(x) = 1.$$

Example 17. As an application of Proposition 15 and as we noted in Section 2, the relationships between both kind of Morgan-Voyce polynomial families are

$$B_n(x) - B_{n-1}(x) = b_n(x) \quad \text{and} \quad b_n(x) - b_{n-1}(x) = x B_{n-1}(x).$$

That in terms of Riordan arrays means

$$T(1-x|1)T(1|(1-x)^2) = T(1-x|(1-x)^2),$$
$$T(1-x|1)T(1-x|(1-x)^2) = T((1-x)^2|(1-x)^2)$$

because $(T(1|(1-x)^2))$ gives rise to $(B_n(x))$ and $T(1-x|(1-x)^2)$ gives rise to $(b_n(x))$.

In the following expressions we consider $(p_n(x))$ as the family of polynomials associated to $T(f|g)$, and we denote by $(q_n(x))$ the family of polynomials associated to each of the matrix products. Moreover a, b are constant series with $b \neq 0$:

$$T(a|1)T(f|g) = T(af|g), \quad \text{then } q_n(x) = a p_n(x),$$
$$T(1|b)T(f|g) = T\left(f\left(\frac{x}{b}\right)\middle| bg\left(\frac{x}{b}\right)\right), \quad \text{then } q_n(x) = \frac{1}{b^{n+1}} p_n(x),$$
$$T(f|g)T(a|1) = T(af|g), \quad \text{then } q_n(x) = a p_n(x),$$
$$T(f|g)T(1|b) = T(f|bg), \quad \text{then } q_n(x) = \frac{1}{b} p_n\left(\frac{x}{b}\right).$$

The above results can be summarized and extended in the following way:

Proposition 18. *Let $T(f|g)$ and $T(l|m)$ be two elements of the Riordan group. Suppose that $(p_n(x))$ and $(q_n(x))$ are the corresponding associated families of polynomials. If*

$$T(l|m) = T(\gamma|\alpha + \beta x)T(f|g)T(c|a + bx)$$

where $\alpha, \gamma, a, c \neq 0$. Then

$$q_n(x) = \frac{\gamma c}{\alpha a}\left(\sum_{k=0}^{n} \binom{n}{k}\left(-\frac{\beta}{\alpha}\right)^{n-k} \frac{1}{\alpha^k} p_k\left(\frac{x-b}{a}\right)\right)$$

Proof. Using Theorem 5 we have that if $(s_n(x))$ is the family of polynomials associated to $T(\gamma|\alpha+\beta x)$ then

$$s_0(x)=\frac{\gamma}{\alpha}\quad\text{and}\quad s_n(x)=\left(\frac{x-\beta}{\alpha}\right)s_{n-1(x)}\quad\forall n\geqslant 1$$

consequently

$$s_n(x)=\frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^n\quad n\in\mathbb{N}$$

Proposition 14 in [10] says that if $(r_n(x))$ is the family of polynomials associated to $T(f|g)T(c|a+bx)$ then

$$r_n(x)=\frac{c}{a}p_n\left(\frac{x-b}{a}\right)$$

Since $(q_n(x))=(s_n(x))\sharp(r_n(x))$ we obtain that

$$(q_n(x))=\left(\frac{\gamma}{\alpha}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{\beta}{\alpha}\right)^{n-k}\frac{1}{\alpha^k}x^k\right)\sharp\left(\frac{c}{a}p_n\left(\frac{x-b}{a}\right)\right).$$

Hence

$$q_n(x)=\frac{\gamma c}{\alpha a}\left(\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{\beta}{\alpha}\right)^{n-k}\frac{1}{\alpha^k}p_k\left(\frac{x-b}{a}\right)\right)\qquad\square$$

Example 19. As we noted in Section 2 the relation between the Pell and the Fibonacci polynomials is $P_n(x)=F_n(2x)$. Recall that $T\left(\frac{1}{2}\middle|1\right)T(1|1-x^2)\;T\left(1|\,\frac{1}{2}\right)=T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right)$ and $T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right)$ gives rise to the Pell polynomials and $T(1|1-x^2)$ gives rise to the Fibonacci polynomials.

Example 20. The Fermat polynomials are the polynomials given by $\mathcal{F}_0(x)=1$, $\mathcal{F}_1(x)=3x$ and $\mathcal{F}_n(x)=3x\mathcal{F}_{n-1}(x)-2\mathcal{F}_{n-2}(x)$ for $n\geqslant 2$. Using our Theorem 5 this means that Fermat polynomials are the polynomials associated to the Riordan matrix $T\left(\frac{1}{3}\middle|\frac{1}{3}+\frac{2}{3}x^2\right)$. For this case, $g_0=\frac{1}{3}$, $g_1=0$, $g_2=\frac{2}{3}$, $g_n=0$, $\forall n\geqslant 3$ and $f_0=\frac{1}{3}$, $f_n=0$ $\forall n\geqslant 1$. The construction rule of this triangle is: $d_{n,k}=-2d_{n-2,k}+3d_{n-1,k-1}$ for $k>0$. The few first Fermat polynomials are $\mathcal{F}_0(x)=1$, $\mathcal{F}_1(x)=3x$, $\mathcal{F}_2(x)=-2+9x^2$, $\mathcal{F}_3(x)=-12x+27x^3$, $\mathcal{F}_4(x)=4-54x^2+81x^4,\ldots$ Since

$$T\left(\frac{1}{3}\middle|\frac{1}{3}(1+2x^2)\right)=T\left(1\middle|\frac{1}{\sqrt{2}}\right)T\left(\frac{1}{2}\middle|\frac{1}{2}(1+x^2)\right)T\left(\frac{2}{3}\middle|\frac{2\sqrt{2}}{3}\right)$$

and using Proposition 18 we obtain the following relation to the Chebysev polynomials of the second kind:

$$\mathcal{F}_n(x)=\left(\sqrt{2}\right)^nU_n\left(\frac{3x}{2\sqrt{2}}\right)$$

Recently, it has been introduced a special family of polynomials in [3,7] related to the so called spray pyrolysis techniques. Now we are going to find a relation of these polynomials with the Chebyshev polynomials of the second kind and then also with the Fermat polynomials. This new sequences of polynomials is given by $\mathcal{B}_0(x)=1$, $\mathcal{B}_1(x)=x$, $\mathcal{B}_2(x)=2+x^2$ and $\mathcal{B}_n(x)=x\mathcal{B}_{n-1}(x)-\mathcal{B}_{n-2}(x)$ for $n\geqslant 3$. Using our Theorem 5, we find that $\mathcal{B}_n(x)$ polynomials are the polynomials associated to the Riordan matrix $T(1+3x^2|1+x^2)$. For this case, $g_0=1$, $g_1=0$, $g_2=1$, $g_n=0$, $\forall n\geqslant 3$ and $f_0=1, f_1=0, f_2=3$ $f_n=0$ $\forall n\geqslant 3$. And the rule of construction of this triangle is: $d_{n,k}=-d_{n-2,k}+d_{n-1,k-1}$, with generating function

$$\sum_{n\geqslant 0}\mathcal{B}_n(t)x^n=T\left(1+3x^2|1+x^2\right)\left(\frac{1}{1-xt}\right)=\frac{1+3x^2}{1-xt+x^2}$$

Table 1
Polynomial families, their associated Riordan arrays in both notation, and their generating functions.

Polynomial family	$(d(x), h(x))$	$T(f\|g)$	$\sum p_n(t)x^n$
Pascal	$\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$	$T(1\|1-x)$	$\frac{1}{1-x-t}$
Fibonacci	$\left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right)$	$T(1\|1-x^2)$	$\frac{1}{1-x^2-tx}$
Pell	$\left(\frac{1}{1-x^2}, \frac{2x}{1-x^2}\right)$	$T\left(\frac{1}{2}\middle\|\frac{1}{2}-\frac{x^2}{2}\right)$	$\frac{1}{1-x^2-2tx}$
Morgan-Voyce (1)	$\left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$	$T(1\|(1-x)^2)$	$\frac{1}{1+x^2-(2+t)x}$
Morgan-Voyce (2)	$\left(\frac{1}{(1-x)}, \frac{x}{(1-x)^2}\right)$	$T(1-x\|(1-x)^2)$	$\frac{1-x}{1+x^2-(2+t)x}$
Chebyshev 2° kind	$\left(\frac{1}{1+x^2}, \frac{2x}{1+x^2}\right)$	$T\left(\frac{1}{2}\middle\|\frac{1}{2}+\frac{x^2}{2}\right)$	$\frac{1}{1+x^2-2tx}$
* Chebyshev 1° kind	$\left(\frac{1-x^2}{2(1+x^2)}, \frac{2x}{1+x^2}\right)$	$T\left(\frac{1}{4}-\frac{x^2}{4}\middle\|\frac{1}{2}+\frac{x^2}{2}\right)$	$\frac{1-x^2}{2(1+x^2-2tx)}+\frac{1}{2}$ *
Fermat	$\left(\frac{1}{1+2x^2}, \frac{3x}{1+2x^2}\right)$	$T\left(\frac{1}{3}\middle\|\frac{1}{3}+\frac{2x^2}{3}\right)$	$\frac{1}{1+2x^2-3tx}$

* See Example 14.

Since

$$T\left(1+3x^2|1+x^2\right) = T\left(1+3x^2|1\right)T\left(\frac{1}{2}\middle|\frac{1}{2}(1+x^2)\right)T(2|2)$$

and using Proposition 12 and Proposition 18 we obtain the following relation to the Chebysev polynomials of the second kind:

$$\mathcal{B}_n(x) = U_n\left(\frac{x}{2}\right) + 3U_{n-2}\left(\frac{x}{2}\right) \quad \text{for } n \geqslant 2.$$

On the other hand we can relate these polynomials and Fermat polynomials:

$$T\left(1+3x^2|1+x^2\right) = T\left(1+3x^2|1\right)T\left(1|\sqrt{2}\right)T\left(\frac{1}{3}\middle|\frac{1}{3}(1+2x^2)\right)T\left(3\middle|\frac{3}{\sqrt{2}}\right)$$

and using again Proposition 12 and 18 we obtain

$$\mathcal{B}_n(x) = \frac{1}{\left(\sqrt{2}\right)^n}\mathcal{F}_n\left(\frac{\sqrt{2}x}{3}\right) + \frac{3}{\left(\sqrt{2}\right)^{n-2}}\mathcal{F}_{n-2}\left(\frac{\sqrt{2}x}{3}\right) \quad \text{for } n \geqslant 2.$$

4. Some applications to the generalized umbral calculus: the associated polynomials and its recurrence relations.

There are many other types of polynomial sequences in the literature that can be constructed by means of Riordan arrays. We are going to characterize by means of recurrences relations all the polynomial sequences called *generalized Appell polynomials* in Boas–Buck [2, pp. 17–18]. We will follow their definitions there.

We first introduce some concepts. Suppose we have any polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ with $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$ and let $h(x) = \sum_{n\geqslant 0} h_n x^n$ any power series, we call the *Hadamard h-weighted sequence* generated by $(p_n(x))$ to the sequence $p_n^h(x) = (p_n \star h)(x)$ where $\star$ means the Hadamard product of series. Recall that if $f = \sum_{n \geqslant 0} f_n x^n$ and $g = \sum_{n \geqslant 0} g_n x^n$, then the Hadamard product $f \star g$ is given by $f \star g = \sum_{n \geqslant 0} f_n g_n x^n$.

Note that p_n^h is a polynomial for every $n \in \mathbb{N}$ and $h \in \mathbb{K}[[x]]$. In fact $p_n^h(x) = \sum_{k=0}^n p_{n,k} h_k x^k$.

Note also that the original definition of generalized Appell polynomials defined by Boas–Buck in [2] can be rewritten in terms of Riordan matrices in the following way:

Proposition 21. *A sequence of polynomials $(s_n(x))$ is a family of generalized Appell polynomials if and only if there are three series $f, g, h \in \mathbb{K}[[x]]$, $f = \sum_{n \geqslant 0} f_n x^n$, $g = \sum_{n \geqslant 0} g_n x^n$ and $h(x) = \sum_{n \geqslant 0} h_n x^n$ with $f_0, g_0 \neq 0$, and $h_n \neq 0$ for all n such that*

$$T(f|g)h(tx) = \sum_{n \geqslant 0} s_n(t) x^n.$$

Moreover in this case, $s_n(x) = p_n^h(x)$ in the above sense where $(p_n(x))$ is the associated polynomial sequence of $T(f|g)$. Consequently

$$\sum_{n \geqslant 0} s_n(t) x^n = \sum_{n \geqslant 0} (p_n \star h)(t) x^n = \frac{f(x)}{g(x)} h\left(t \frac{x}{g(x)}\right).$$

Proof. If $T(f|g)(h(tx)) = \sum_{n \geqslant 0} s_n(t) x^n$ then obviously $(s_n(x))$ is a generalized Appell sequence because $\sum_{n \geqslant 0} s_n(t) x^n = \frac{f(x)}{g(x)} h\left(t \frac{x}{g(x)}\right)$. Suppose now that $(s_n(x))$ is a generalized Appell sequence, then there are three series A, B, Φ where $A = \sum_{n \geqslant 0} A_n x^n$, $A_0 \neq 0$, $B = \sum_{n \geqslant 1} B_n x^n$, $B_1 \neq 0$ and $\Phi = \sum_{n \geqslant 0} \Phi_n x^n$ with $\Phi_n \neq 0$, $\forall n \in \mathbb{N}$ such that $\sum_{n \geqslant 0} s_n(t) x^n = A(x)\Phi(tB(x))$. If we take $\Phi = h, g(x) = \frac{x}{B(x)}$ and $f(x) = \frac{xA(x)}{B(x)}$ we are done. $\square$

Remark 22. Note that if $h(x) = \frac{1}{1-x}$ the family of $\left(p_n^{\frac{1}{1-x}}(x)\right)$ is exactly the associated polynomials $(p_n(x))$ of $T(f|g)$, because $\frac{1}{1-x}$ is the neutral element in the Hadamard product.

Example 23 (*The Sheffer polynomials*)**.** Following the previous proposition we have that $(S_n(x))$ is a Sheffer sequence if and only if there is a Riordan matrix $T(f|g)$ such that

$$T(f|g)(e^{tx}) = \sum_{n \geqslant 0} S_n(t) x^n.$$

The usual way to introduce Sheffer sequences is by means of the corresponding generating function

$$\sum_{n \geqslant 0} S_n(t) x^n = A(x) e^{tH(x)},$$

where $A = \sum_{n \geqslant 0} A_n x^n, H = \sum_{n \geqslant 1} H_n x^n$ with $A_0 \neq 0, H_1 \neq 0$. Note that for this case the corresponding Riordan matrix is

$$T\left(\frac{xA(x)}{H(x)} \middle| \frac{x}{H(x)}\right).$$

The general term of a Sheffer sequence, $S_n(x)$ is given by

$$S_n(x) = p_n(x) \star e^x,$$

where $(p_n(x))$ are the associated polynomials to $T(f|g)$. Consequently

$$S_n(x) = \sum_{k=0}^{n} \frac{p_{n,k}}{k!} x^k$$

if $p_n(x) = \sum_{k=0}^n p_{n,k} x^k$.

WARNING. Note that in many places [13–15] they call a Sheffer sequence to the sequence $(n!S_n(x))_{n\in\mathbb{N}}$ where $(S_n(x))_{n\in\mathbb{N}}$ is our Sheffer sequence.

In the following example we can note that applying a fixed $T(f|1)$ to different series h gives rise to some classical families of polynomials.

Example 24 (*The Brenke polynomials*)**.** Following [2], $(B_n(x))$ is in the class of Brenke polynomials if

$$T(f|1)(h(tx)) = \sum_{n\geqslant 0} B_n(t)x^n.$$

Some particular cases are:

$$T(f|1)\left(\frac{1}{1-tx}\right) = \sum_{n\geqslant 0} T_n^*(t)x^n,$$

where (T_n^*) are the reversed Taylor polynomial of f.

$$T(f|1)(e^{tx}) = \sum_{n\geqslant 0} A_n(t)x^n,$$

where $(A_n(x))$ are the Appell polynomials of f.

Using analogous arguments as in the previous section for polynomials of Riordan type, we can get some relationships between some classical Sheffer sequences once we know, easily, some relation between their corresponding Riordan matrices.

Using our main theorem in Section 3 we can obtain the following recurrence relations for the generalized Appell polynomials, which is the main result in this section.

Theorem 25. *Let $(s_n(x))_{n\in\mathbb{N}}$ be a sequence of polynomials with $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$. Then $(s_n(x))_{n\in\mathbb{N}}$ is a family of generalized Appell polynomials if and only if there are three sequences (f_n), (g_n), $(h_n) \in \mathbb{K}$ with $f_0, g_0 \neq 0$ and $h_n \neq 0\ \forall n \in \mathbb{N}$ such that*

$$s_n(x) = \frac{1}{g_0}(xs_{n-1}(x)\star\hat{h}(x)) - \frac{g_1}{g_0}s_{n-1}(x) - \cdots - \frac{g_n}{g_0}s_0(x) + \frac{h_0f_n}{g_0} \quad \forall n\in\mathbb{N} \text{ with } s_0(x) = \frac{h_0f_0}{g_0}$$

where $\hat{h}(x) = \sum_{k=1}^{\infty} \frac{h_k}{h_{k-1}}x^k$. Moreover the coefficients of this family of polynomials satisfy the following recurrence:

If $k \geqslant 1$

$$s_{n,k} = -\frac{g_1}{g_0}s_{n-1,k} - \cdots - \frac{g_n}{g_0}s_{0,k} + \frac{h_k}{h_{k-1}}s_{n-1,k-1}.$$

If $k = 0$

$$s_{n,0} = -\frac{g_1}{g_0}s_{n-1,0} - \cdots - \frac{g_n}{g_0}s_{0,0} + \frac{h_0f_n}{g_0}, \quad s_{0,0} = \frac{h_0f_0}{g_0}.$$

Proof. If $(s_n(x))$ is a family of generalized Appell polynomials then there are three sequence (f_n), (g_n), (h_n) of elements in $\mathbb{K}$ with $f_0, g_0 \neq 0$ and $h_n \neq 0\ \forall n \in \mathbb{N}$, such that if $f = \sum_{n\geqslant 0} f_nx^n$, $g = \sum_{n\geqslant 0} g_nx^n$ and $h = \sum_{n\geqslant 0} h_nx^n$ then

$$T(f|g)h(tx) = \sum_{n\geqslant 0} s_n(t)x^n$$

since $s_n(x) = p_n^h(x) = p_n(x)\star h(x)$, the family of polynomials $(p_n(x))$ associated to $T(f|g)$ obeys the recurrence relation of Theorem 5: Using the distributivity of Hadamard product we get

$$p_n(x)\star h(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x)\star h(x) - \frac{g_2}{g_0}p_{n-2}(x)\star h(x) \cdots - \frac{g_n}{g_0}p_0(x)\star h(x) + \frac{f_n}{g_0}\star h(x)$$

$$= p_n^h(x) = \frac{x}{g_0}p_{n-1}(x)\star h(x) - \frac{g_1}{g_0}p_{n-1}^h(x) - \frac{g_2}{g_0}p_{n-2}^h(x)\cdots - \frac{g_n}{g_0}p_0^h(x) + \frac{f_n h_0}{g_0}$$

since

$$xp_{n-1}(x)\star h(x) = p_{n-1,0}h_1x + p_{n-1,1}h_2x^2 + \cdots + p_{n-1,n-1}h_nx^n$$

then

$$\begin{aligned} xp_{n-1}(x)\star h(x) &= p_{n-1,0}h_0\frac{h_1}{h_0}x + p_{n-1,1}h_1\frac{h_2}{h_1}x^2 + \cdots + p_{n-1,n-1}h_{n-1}\frac{h_n}{h_{n-1}}x^n \\ &= xp_{n-1}^h(x)\star\hat{h}(x) \end{aligned}$$

so we get the result.

On the other hand if there are three sequences $(f_n), (g_n), (h_n) \in \mathbb{K}$ with $f_0, g_0 \neq 0$ and $h_n \neq 0\ \forall n \in \mathbb{N}$ such that

$$\begin{aligned} s_n(x) &= \frac{1}{g_0}(xs_{n-1}(x)\star\hat{h}(x)) - \frac{g_1}{g_0}s_{n-1}(x) - \cdots - \frac{g_n}{g_0}s_0(x) + \frac{h_0f_n}{g_0} \quad \forall n \in \mathbb{N} \\ &\text{with } s_0(x) = \frac{h_0f_0}{g_0}, \end{aligned}$$

where $\hat{h}(x) = \sum_{k=1}^{\infty}\frac{h_k}{h_{k-1}}x^k$. Let

$$p_n(x) = s_n(x)\star h^{(-1)^\bigstar}(x)$$

where $h^{(-1)^\bigstar}(x) = \sum_{n\geqslant 0}\frac{1}{h_n}x^n$. Then

$$\begin{aligned} s_n(x)\star h^{(-1)^\bigstar}(x) &= \frac{1}{g_0}(xs_{n-1}(x)\star\hat{h}(x))\star h^{(-1)^\bigstar}(x) - \frac{g_1}{g_0}s_{n-1}(x)\star h^{(-1)^\bigstar}(x) \\ &\quad - \cdots - \frac{g_n}{g_0}s_0(x)\star h^{(-1)^\bigstar}(x) + \frac{h_0f_n}{g_0}\star h^{(-1)^\bigstar}(x), \end{aligned}$$

$$p_n(x) = \frac{1}{g_0}(xs_{n-1}(x)\star\hat{h}(x))\star h^{(-1)^\bigstar} - \frac{g_1}{g_0}p_{n-1}(x) - \cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}$$

since

$$xs_{n-1}(x)\star\hat{h}(x) = s_{n-1,0}\frac{h_1}{h_0}x + s_{n-1,1}\frac{h_2}{h_1}x^2 + \cdots + s_{n-1,n-1}\frac{h_n}{h_{n-1}}x^n$$

then

$$xs_{n-1}(x)\star\hat{h}(x)\star h^{(-1)^\bigstar}(x) = xs_{n-1}(x)\star h^{(-1)^\bigstar}(x) = xp_{n-1}(x)$$

consequently

$$p_n(x) = \frac{1}{g_0}(xp_{n-1}(x)) - \frac{g_1}{g_0}p_{n-1}(x) - \cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0}$$

so $(p_n(x))$ obeys Theorem 5 and then $(p_n(x))$ is the associated polynomials to $T(f|g)$. Hence $(s_n(x))$ is a family of generalized Appell polynomials.

The second part of the result is an easy consequence of our Algorithm 1 in the Introduction. □

Remark 26. Note that if $k \geqslant 1$, some terms in the recurrence are null, in fact $s_{l,k} = 0$ if $l < k$. Consequently:

$$s_{n,k} = -\frac{g_1}{g_0}s_{n-1,k} - \cdots - \frac{g_{n-k}}{g_0}s_{k,k} + \frac{h_k}{h_{k-1}}s_{n-1,k-1}.$$

A consequence that we can obtain from the recurrence relation for the generalized Appell sequences is the following relation between the Hadamard h-weighted and h'-weighted sequences for a polynomials sequence of Riordan type. For notational convenience we represent now by $\mathcal{D}(\alpha)$ to the derivative of any series α. The result obtained below when we consider the classical Appell sequences, is just what Appell took as the definition for these classical sequences.

Corollary 27. *Let $T(f|g)$ be any element of the Riordan group with $f = \sum_{n\geqslant 0} f_n x^n$, $g = \sum_{n\geqslant 0} g_n x^n$, and with associated sequence $(p_n(x))$. Suppose that $h \in \mathbb{K}[[x]]$ is Hadamard invertible. Then the $\mathcal{D}(h)$ is Hadamard invertible and*

$$p_{n-1}^{\mathcal{D}(h)}(x) = \sum_{k=0}^{n} g_k \mathcal{D}(p_{n-k}^h)(x).$$

Proof. We know that

$$p_n^h(x) = \frac{1}{g_0}(xp_{n-1}^h(x)\star\hat{h}(x)) - \frac{g_1}{g_0}p_{n-1}^h(x) - \cdots - \frac{g_n}{g_0}p_0^h(x) + \frac{h_0 f_n}{g_0}.$$

Applying the derivative in both sides we obtain

$$\mathcal{D}(p_n^h)(x) = \frac{1}{g_0}\mathcal{D}(xp_{n-1}^h(x)\star\hat{h}(x)) - \sum_{k=1}^{n}\frac{g_k}{g_0}\mathcal{D}(p_{n-k}^h)(x).$$

Consequently

$$\mathcal{D}(xp_{n-1}^h(x)\star\hat{h}(x)) = \sum_{k=0}^{n} g_k \mathcal{D}(p_{n-k}^h)(x).$$

It is easy to prove that

$$\mathcal{D}(m(x)\star l(x)) = \frac{m(x) - m(0)}{x}\star\mathcal{D}(l(x)) = \mathcal{D}(m(x))\star\frac{(l(x) - l(0)}{x}$$

for any series $l, m \in \mathbb{K}[[x]]$. Using the first equality above we get

$$p_{n-1}^h\star\mathcal{D}(\hat{h})(x) = \sum_{k=0}^{n} g_k \mathcal{D}(p_{n-k}^h)(x)$$

but

$$(p_{n-1}(x)\star h(x))\star\mathcal{D}(\hat{h})(x) = p_{n-1}(x)\star(h(x)\star(\mathcal{D}(\hat{h})(x))$$

and since $\hat{h}(x) = \sum_{k\geqslant 1}\frac{h_k}{h_{k-1}}x^k$ we obtain that

$$h(x)\star\mathcal{D}(\hat{h})(x) = \mathcal{D}(h)(x)$$

and so we have the announced equality. $\square$

In some cases the above formulas allow us to compute easily some generalized Appell sequences in terms of the associated sequences of Riordan type.

Example 28 (*Some easy computations related to the geometric series*)**.** As an easy application of the above result we have: let $(p_n(x))$ be a polynomial sequence of Riordan type. Then

(i) $p_n^{\frac{1}{(1-x)^2}}(x) = xp_n'(x) + p_n(x) = (xp_n(x))' \quad \forall n \geqslant 0.$

(ii) If $a \neq 0$ then

$$p_n^{a-\log(1-x)}(x) = ap_n(0) + \int_0^x \frac{p_n(t) - p_n(0)}{t} \quad \forall n \geqslant 0.$$

The previous results convert to the following formulas in the important class of Sheffer sequences.

Example 29 (*The recurrence relation for the Sheffer polynomials. Some classical examples*)**.** Since for Sheffer polynomials $h(x) = e^x = \sum_{n \geqslant 0} \frac{x^n}{n!}$ and $\hat{h}(x) = \sum_{n \geqslant 1} \frac{x^n}{n} = -\log(1-x)$, the recurrence relation is:

$$S_n(x) = \frac{1}{g_0}(xS_{n-1}(x)\star(-\log(1-x))) - \frac{g_1}{g_0}S_{n-1}(x) - \cdots - \frac{g_n}{g_0}S_0(x) + \frac{f_n}{g_0} \quad \forall n \in \mathbb{N}$$
$$\text{with } S_0(x) = \frac{f_0}{g_0}$$

and the recurrence relations for the coefficients are

If $k \geqslant 1$

$$S_{n,k} = -\frac{g_1}{g_0}S_{n-1,k} - \cdots - \frac{g_n}{g_0}S_{0,k} + \frac{1}{k}S_{n-1,k-1}.$$

If $k = 0$

$$S_{n,0} = -\frac{g_1}{g_0}S_{n-1,0} - \cdots - \frac{g_n}{g_0}S_{0,0} + \frac{f_n}{g_0}, \quad S_{0,0} = \frac{f_0}{g_0}.$$

For its derivatives. Since

$$(xS_{n-1}(x)\star(-\log(1-x)))' = S_{n-1}(x)\star\frac{1}{1-x} = S_{n-1}(x).$$

Then

$$S_n'(x) = \frac{1}{g_0}S_{n-1}(x) - \frac{g_1}{g_0}S_{n-1}'(x) - \cdots - \frac{g_n}{g_0}S_0'(x) \quad \text{So} \quad S_{n-1}(x) = \sum_{k=0}^{n} g_k S_{n-k}'(x).$$

Pidduck and Mittag-Leffler polynomials. Consider the sequence $(\mathcal{P}_n(x))$ satisfying

$$\sum_{n \geqslant 0} \mathcal{P}_n(t)x^n = T\left(\frac{x}{(1-x)\log\left(\frac{1+x}{1-x}\right)} \middle| \frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)(e^{tx})$$

in matricial form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 4 & 0 & 0 & \cdots \\ 1 & \frac{8}{3} & 4 & 8 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2} \\ \frac{t^3}{6} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 2t+1 \\ 2t^2+2t+1 \\ \frac{4}{3}t^3 + 2t^2 + \frac{8}{3}t + 1 \\ \vdots \end{pmatrix}.$$

If we take $\widetilde{P}_n(x) = n!\mathcal{P}_n(x)$, then $\widetilde{P}_n(x)$ are the usual Pidduck polynomials: $\widetilde{P}_0(x) = 1, \widetilde{P}_1(x) = 2x + 1$, $\widetilde{P}_2(x) = 4x^2 + 4x + 2$, $\widetilde{P}_3(x) = 8x^3 + 12x^2 + 16x + 6, \ldots$ On the other hand we get the Mittag-Leffler polynomials, in the following way. If $(M_n(x))$ is given by the formula:

$$\sum_{n \geqslant 0} M_n(t)x^n = T\left(\frac{x}{\log\left(\frac{1+x}{1-x}\right)} \middle| \frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)(e^{tx})$$

then, if we take now $\widetilde{M}_n(x) = n!M_n(x)$, then $\widetilde{M}_n(x)$ are the usual Mittag-Leffler polynomials: $\widetilde{M}_0(x) = 1, \widetilde{M}_1(x) = 2x, \widetilde{M}_2(x) = 4x^2, \widetilde{M}_3(x) = 8x^3 + 4x, \ldots$. Both families of polynomials are related because:

$$T\left(\frac{x}{(1-x)\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right) = T\left(\frac{1}{1-x}\middle|1\right)T\left(\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right).$$

So

$$\mathcal{P}_n(x) = \sum_{k=0}^{n} M_k(x) \quad \text{or equivalently} \quad \widetilde{P}_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-k)!\widetilde{M}_k(x).$$

The following two particular examples are Sheffer polynomials which can be easily described with a different representation as generalized Appell polynomial. We choose, in particular, Laguerre sequence because it is very close to the Pascal triangle.

The Laguerre polynomials. We consider

$$T(-1|x-1)(e^{tx}) = T(1|1-x)T(-1|-1)(e^{tx}) = T(1|1-x)(e^{-tx}) = \sum_{k=0}^{n} L_n(t)x^n,$$

where $L_n(x)$ are the Laguerre polynomials. Note that $T(1|1-x)$ is the Pascal triangle. From the definition of the polynomials we obtain easily the well-known general term:

$$L_n(x) = p_n(x)\star e^{-x} = \sum_{k=0}^{n}\binom{n}{k}x^k\star\sum_{k\geqslant 0}\frac{(-1)^k}{k!}x^k = \sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n}{k}x^k$$

Our recurrence relation for Laguerre polynomials is:

$$L_n(x) = xL_{n-1}(x)\star(-\log(1-x)) + L_{n-1}(x)$$

and the recurrence relations for the coefficients are

If $k \geqslant 1$, $L_{n,k} = L_{n-1,k} - \frac{1}{k}L_{n-1,k-1}$ and $L_{n,0} = L_{n-1,0}$, $L_{0,0} = 1$.

Using Corollary 27 we have:

$$L'_n(x) = L'_{n-1}(x) - L_{n-1}(x) \quad \text{consequently} \quad L'_n(x) = -\sum_{k=0}^{n-1} L_k(x).$$

The Hermite polynomials. We consider

$$\sum_{n\geqslant 0} H_n(t)x^n = T\left(\frac{1}{2e^{x^2}}\middle|\frac{1}{2}\right)(e^{tx}) = T\left(\frac{1}{e^{x^2}}\middle|1\right)T\left(\frac{1}{2}\middle|\frac{1}{2}\right)(e^{tx}) = T\left(\frac{1}{e^{x^2}}\middle|1\right)(e^{2tx}) = e^{2tx-x^2}.$$

If $\widetilde{H}_n(x) = n!H_n(x)$, we obtain $\widetilde{H}_n(x)$ are the usual Hermite polynomials: $\widetilde{H}_0(x) = 1$, $\widetilde{H}_1(x) = 2x$, $\widetilde{H}_2(x) = 4x^2 - 2$, $\widetilde{H}_3(x) = 8x^3 - 12x$, $\widetilde{H}_4(x) = 16x^4 - 48x^2 + 12, \cdots$. Since $\sum_{n\geqslant 0} H_n(t)x^n = T\left(\frac{1}{e^{x^2}}\middle|1\right)(e^{2tx})$, the recurrence for the $(H_n(x))$ is: $H_n(x) = xH_{n-1}(x)\star\hat{h}(x) + f_n$ where

$$\hat{h}(x) = \sum_{n\geqslant 1}\frac{2}{n}x^n = -2\log(1-x) \quad \text{and} \quad f_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!}, & \text{if } n \text{ is even} \end{cases}$$

and the recurrence relations for the coefficients are: If $k \geqslant 1$, $H_{n,k} = \frac{2}{k}H_{n-1,k-1}$ and $H_{n,0} = f_n$, $H_{0,0} = 1$. Using Corollary 27 we obtain $H'_n(x) = 2H_{n-1}(x)$ or equivalently, the known relation $\widetilde{H}'_n(x) = 2n\widetilde{H}_{n-1}(x)$. We can also obtain the general term for the Hermite polynomials:

$$H_{2m}(x) = \sum_{j=0}^{m}\frac{(-1)^{m-j}2^{2j}}{(m-j)!(2j)!}x^{2j}, \qquad H_{2m+1}(x) = \sum_{j=0}^{m}\frac{(-1)^{m-j}2^{2j+1}}{(m-j)!(2j+1)!}x^{2j+1}$$

From here the known equality $\widetilde{H}_n(-x) = (-1)^n \widetilde{H}_n(x)$ is obvious.

Now we are going to translate the operations in the Riordan group to the set of Hadamard h-weighted families of polynomials. Suppose that $(p_n(x))$ is the associated sequences of polynomials to the element of the Riordan group $T(f|g)$. If $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$, $T(f|g) = (p_{n,k})_{n,k\in\mathbb{N}}$. Let $h(x) = \sum_{n\geqslant 0} h_n x^n$ be such that $h_n \neq 0\ \forall n \in \mathbb{N}$. So, h admits a reciprocal for the Hadamard product, we represent it by $h^{(-1)\star}$. In fact $h^{(-1)\star}(x) = \sum_{n\geqslant 0} \frac{1}{h_n} x^n$.

Consider the set $\mathcal{R}_h = \{(p_n^h(x))_{n\in\mathbb{N}} \ /\ (p_n(x))_{n\in\mathbb{N}} \in \mathcal{R}\}$, we can prove:

Proposition 30. *The function*

$$\begin{array}{llll} H_h: & \mathcal{R} & \longrightarrow & \mathcal{R}_h \\ & (p_n(x))_{n\in\mathbb{N}} & \longmapsto & (p_n^h(x))_{n\in\mathbb{N}} \end{array}$$

is bijective if h is a Hadamard unit in $\mathbb{K}[[x]]$. *Consequently the umbral composition* $\sharp$ *defined in* $\mathcal{R}$ *is transformed into an operation* $\sharp_h$ *converting so* $(\mathcal{R}_h, \sharp_h)$ *into a group and* H_h *converts into a group isomorphism. Moreover if* $(s_n(x))_{n\in\mathbb{N}},\ (t_n(x))_{n\in\mathbb{N}} \in \mathcal{R}_h$ *with* $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$, $t_n(x) = \sum_{k=0}^{n} t_{n,k}x^k \in \mathcal{R}_h$, $(r_n(x))_{n\in\mathbb{N}} = (s_n(x))_{n\in\mathbb{N}} \sharp_h (t_n(x))_{n\in\mathbb{N}}$ *with* $r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$ *then*

$$r_{n,j} = \sum_{k=j}^{n} \frac{1}{h_k} s_{n,k} t_{k,j}$$

Proof. The first part is obvious, because if the function

$$\begin{array}{llll} G_{h^{(-1)\star}}: & \mathcal{R}_h & \longrightarrow & \mathcal{R} \\ & (s_n(x))_{n\in\mathbb{N}} & \longmapsto & (s_n(x)\star h^{(-1)\star})_{n\in\mathbb{N}} \end{array}$$

is the inverse, for the composition of H_h.

Now given $(s_n(x))_{n\in\mathbb{N}},\ (t_n(x))_{n\in\mathbb{N}} \in \mathcal{R}_h$ we define $(s_n(x))_{n\in\mathbb{N}} \sharp_h (t_n(x))_{n\in\mathbb{N}} = (r_n(x))_{n\in\mathbb{N}}$ where $r_n(x) = H_h(p_n(x) \sharp q_n(x))$ where $s_n(x) = p_n^h(x), t_n(x) = q_n^h(x)$ for every $n \in \mathbb{N}$. If $p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k$ and $q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k$ then if $(p_n(x))\sharp(q_n(x)) = (u_n(x))$ with $u_n(x) = \sum_{k=0}^{n} u_{n,k}x^k$ then $u_{n,j} = \sum_{k=j}^{n} p_{n,k}q_{k,j}$. Consequently $r_{n,j} = u_{n,j}h_j$ then

$$r_{n,j} = \sum_{k=j}^{n} \frac{p_{n,k}h_k q_{k,j} h_j}{h_k} = \sum_{k=j}^{n} \frac{s_{n,k}t_{k,j}}{h_k} \qquad \square$$

Another important kind of polynomial sequences in the literature are the sequences of binomial type [15] or the closely related sequences, of convolution polynomials, see [6]. In fact $(s_n(x))_{n\in\mathbb{N}}$ is a convolution polynomial family if and only if $(n!s_n(x))_{n\in\mathbb{N}}$ is a sequence of binomial type.

As one can deduce from [6] a polynomial sequence $(s_n(x))_{n\in\mathbb{N}}$ forms a convolution family if and only if there is a formal power series $b(x) = \sum_{n\geqslant 1} b_n x^n$ with $b_1 \neq 0$ such that $e^{tb(x)} = \sum_{n\geqslant 0} s_n(t)x^n$. So the convolution condition

$$s_n(t+r) = \sum_{k=0}^{n} s_{n-k}(t)s_k(r)$$

come directly from the fact that

$$e^{tb(x)}e^{rb(x)} = e^{(t+r)b(x)}.$$

So, symbolically, the Cauchy product

$$\left(\sum_{n\geqslant 0} s_n(t)x^n\right)\left(\sum_{n\geqslant 0} s_n(r)x^n\right) = \sum_{n\geqslant 0} s_n(t+r)x^n$$

is just the convolution condition.

Now suppose again a power series $g = \sum_{n \geqslant 0} g_n x^n$ with $g_0 \neq 0$. Then

$$T(g|g)(e^{tx}) = \sum_{n \geqslant 0} s_n(t) x^n = e^{\frac{tx}{g}}.$$

Consequently we have:

Theorem 31. *A polynomial sequence $(s_n(x))_{n\in\mathbb{N}}$ is a convolution sequence if and only if there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ with $g_0 \neq 0$ such that*

$$s_n(x) = \frac{1}{g_0}(xs_{n-1}(x)\star(-\log(1-x))) - \frac{g_1}{g_0}s_{n-1}(x) - \cdots - \frac{g_{n-1}}{g_0}s_1(x) \quad \text{for } n \geqslant 2$$

and $s_0(x) = 1$, $s_1(x) = \frac{x}{g_0}$.

Proof. $(s_n(x))_{n\in\mathbb{N}}$ is a convolution family if and only if there is a series $\sum_{n\geqslant 0} g_n x^n$ with $g_0 \neq 0$ such that $T(g|g)(e^{tx}) = \sum_{n\geqslant 0} s_n(t)x^n$. So $(s_n(x))_{n\in\mathbb{N}}$ is the e^x-Hadamard weighted sequence generated by the Riordan sequence $(q_n(x))_{n\in\mathbb{N}}$ associated, as in Theorem 5, to the $T(g|g)$. Consequently $q_0(x) = \frac{g_0}{g_0} = 1$

$$q_n(x) = \left(\frac{x-g_1}{g_0}\right)q_{n-1}(x) - \frac{g_2}{g_0}q_{n-2}(x)\cdots - \frac{g_{n-1}}{g_0}q_1(x) - \frac{g_n}{g_0}q_0(x) + \frac{g_n}{g_0}$$

$$\text{so} \quad q_1(x) = \frac{x}{g_0}, \quad \text{and} \quad q_n(x) = \left(\frac{x-g_1}{g_0}\right)q_{n-1}(x) - \frac{g_2}{g_0}q_{n-2}(x)\cdots - \frac{g_{n-1}}{g_0}q_1(x) \quad \text{for } n \geqslant 2.$$

The result follows directly multiplying Hadamard by e^x. □

The polynomial sequences of binomial types are closely related to the so called delta-operator, see [15]. In [12,17,11] it was introduced the so called A-sequence associated to a Riordan array. In our notation the A-sequence associated to the Riordan array $T(f|g)$ is just the unique power series $A = \sum_{n\geqslant 0} a_n x^n$ with $a_0 \neq 0$ such that $A\left(\frac{x}{g}\right) = \frac{1}{g}$. As a consequence of the results in [8] we get that A is the A-sequence of $T(g|g)$ if and only if $T(A|A) = T^{-1}(g|g)$ where the inverse is taken in the Riordan group. So A is the A-sequence of $T(g|g)$ if and only if g is the A-sequence of $T(A|A)$. Let us denote by $\mathcal{D}$ to the derivative operator on polynomials. Using Theorem 1 and Corollary 3 in [15] we have

Theorem 32. *Suppose that $(s_n(x))_{n\in\mathbb{N}}$ is the convolution sequences associated to the Riordan array $T(g|g)$. Consider the corresponding sequence $(r_n(x))_{n\in\mathbb{N}}$ of binomial type, i.e. $r_n(x) = n!s_n(x)$. Then the delta-operator Q having $(r_n(x))_{n\in\mathbb{N}}$ as its basic sequences is just $\frac{x}{A(x)}(\mathcal{D})$ where A is the A-sequence of $T(g|g)$. On the opposite, if we have the delta-operator $\frac{x}{g(x)}(\mathcal{D})$ and $(r_n(x))_{n\in\mathbb{N}}$ is the basis sequence then $\left(\frac{r_n(x)}{n!}\right)_{n\in\mathbb{N}}$ is the convolution sequence associated to the Riordan array $T(A|A)$ where A is the A-sequence of $T(g|g)$.*

We would like to say that in [8] it is described a recurrence process, related to Banach Fixed Point Theorem and to the Lagrange inversion formula, to get $\frac{x}{A}$ using only the series g.

Now we are going to give a characterization of a generalized Appell sequence using linear transformations in the $\mathbb{K}$-linear space $\mathbb{K}[[x]]$.

Usually a Riordan matrix is defined by means of the natural linear action on $\mathbb{K}[[x]]$, in fact, a matrix $A = (a_{n,k})$ is a Riordan matrix $T(f|g)$ if and only if the action of A on any power series α is given by $T(f|g)(\alpha) = \frac{f}{g}\alpha\left(\frac{x}{g}\right)$. In these terms we have

Proposition 33. *A matrix $s = (s_{n,k})$ has as associated sequence of polynomials a generalized Appell sequence if and only if there are three power series $f = \sum_{n\geqslant 0} f_n x^n$, $g = \sum_{n\geqslant 0} g_n x^n$, $h = \sum_{n\geqslant 0} h_n x^n$, with $f_0, g_0 \neq 0$ and $h_n \neq 0$, $\forall\, n \in \mathbb{N}$ such that the natural linear action induced by s is given by $s(\alpha) = \frac{f(x)}{g(x)}(h\star\alpha)\left(\frac{x}{g}\right)$ for any $\alpha \in \mathbb{K}[[x]]$.*

Remark 34. From the above proposition we could develop the exponential Riordan arrays or more generally the generalized Riordan matrices, see [18].

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