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A Matrix Approach for General Higher Order Linear Recurrences

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Abstract. We consider k sequences of generalized order-k linear recurrences with arbitrary initial conditions and coefficients, and we give their generalized Binet formulas and generating functions. We also obtain a new matrix method to derive explicit formulas for the sums of terms of the k sequences. Further, some relationships between determinants of certain Hessenberg matrices and the terms of these sequences are obtained.

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1. Introduction

Linear recurrences have played (and will most certainly play) an important role in many areas of mathematics. Various authors have studied various properties of linear recurrences (such as the well-known Fibonacci and Pell sequences).

In [2], Er defined k linear recurring sequences of order at most k as shown: For $n > 0$ and $1 \le i \le k$,

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if } n = 1-i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0,$$

where g_n^i is the nth term of the ith generalized order-k Fibonacci sequence.

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More generally, in [6], the author gave the generalized order-k Fibonacci and Pell (F-P) sequence as follows: For $m \geq 0,\ n > 0$ and $1 \leq i \leq k$

$$u_n^i = 2^m u_{n-1}^i + u_{n-2}^i + \cdots + u_{n-k}^i$$

with initial conditions

$$u_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where u_n^i is the nth term of the ith generalized order-k F-P sequence.

When $m = 0$, the generalized order-k F-P sequence $\{u_n^i\}$ is reduced to the generalized order-k Fibonacci sequence $\{g_n^i\}$. Also when $m = 1$, the generalized order-k F-P sequence is reduced to the generalized order-k Pell sequence $\{P_n^i\}$ (for more details see [5]).

Define k sequences of kth order linear recurrence relation $\{f_n^i\}$ as shown, for $n > 0$ and $1 \leq i \leq k$

$$f_n^i = c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i \tag{1.1}$$

with initial conditions

$$f_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0$$

where $c_j,\ 1 \leq j \leq k$, are real constant coefficients, and f_n^i is the nth term of the ith sequence. When $k = 2, c_1 = c_2 = 1$, respectively, $k = c_1 = 2,\ c_2 = 1$ the sequence $\{f_n^2\}$ is reduced to the Fibonacci sequence $\{F_n\}$, respectively, the Pell sequence $\{P_n\}$.

Define the $k \times k$ companion matrix A and the matrix G_n as follows:

$$A = \begin{bmatrix} c_1 & c_2 & \dots & c_{k-1} & c_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad G_n = \begin{bmatrix} f_n^1 & f_n^2 & \cdots & f_n^k \\ f_{n-1}^1 & f_{n-1}^2 & \cdots & f_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-k+1}^1 & f_{n-k+1}^2 & \cdots & f_{n-k+1}^k \end{bmatrix} \tag{1.2}$$

Using the approach of Kalman [3], Er [2] showed that

$$G_n = A^n \tag{1.3}$$

and

$$f_{n+1}^i = c_i f_n^1 + f_n^{i+1}, \text{ for } 1 \leq i \leq k - 1 \tag{1.4}$$

$$f_{n+1}^k = c_k f_n^1. \tag{1.5}$$

Matrix methods are helpful and convenient in solving certain problems stemming from linear recursion relations, such as that of finding an explicit expression for the nth term of the Fibonacci sequence (see [9]), or of analyzing the vibration of a weighted string [10, pp. 152–154]. Here we will consider a more general case using matrix methods to obtain some explicit formulas for the nth term of a general recurrence relation and the sums of terms of the recurrence. The general linear recurrence relations have been considered by many mathematicians (for references, see [1, 2, 4, 5]). The authors of [4, 6, 7] give the generalized Binet formula for the generalized order-k Fibonacci, Lucas and Pell numbers by matrix methods.

In this paper, we consider k sequences of general order-k linear recurrences with k arbitrary initial conditions and coefficients. Then we study the properties of k linear recursive sequences and derive many applications to matrices.

2. General linear recurrence with k initial conditions

Define a set of k sequences satisfying the generalized order-k linear recurrence $\{t_n^i(r_1, r_2, \ldots, r_k)\}$ as shown: For $n > 0$ and $1 \le i \le k$,

$$t_n^i = c_1 t_{n-1}^i + c_2 t_{n-1}^i + \cdots + c_k t_{n-k}^i$$

with k initial conditions

$$t_n^i = \begin{cases} r_1 & \text{if } n = 1 - i, \\ r_2 & \text{if } n = 2 - i, \\ \vdots & \quad \vdots \\ r_k & \text{if } n = k - i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 - k \le n \le 0$$

where the coefficients c_i and the initial conditions r_i are arbitrary, for $1 \le i \le k$, and t_n^i is the nth term of ith sequence. Clearly, $\left\{t_n^i\,(1, 0, \ldots, 0)\right\} = \left\{f_n^i\right\}$, where f_n^i are given by (1.1).

Next, we define a $k \times k$ matrix $H_n = [h_{ij}]$ by

$$H_n = \begin{bmatrix} t_n^1 & t_n^2 & \cdots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \cdots & t_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \cdots & t_{n-k+1}^k \end{bmatrix}. \tag{2.1}$$

By Kalman's [3] approach, we find that

$$H_n = A H_{n-1} \text{ and so, } \; H_n = A^{n-1} H_1, \tag{2.2}$$

where the matrix A is given by (1.2).

Theorem 2.1. *For $n > 0$,*

$$t_n^i = \sum_{j=1}^{i} r_{i+1-j} f_n^j,$$

where f_n^i is defined as before.

Proof. From (2.2), we have $H_n = A^{n-1} H_1$. From (2.1) we get

$$H_1 = \begin{bmatrix} t_1^1 & t_1^2 & \cdots & t_1^k \\ t_0^1 & t_0^2 & \cdots & t_0^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{2-k}^1 & t_{2-k}^2 & \cdots & t_{2-k}^k \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{1} c_j r_{2-j} & \sum_{j=1}^{2} c_j r_{3-j} & \cdots & \sum_{j=1}^{k} c_j r_{k+1-j} \\ r_1 & r_2 & \cdots & r_k \\ 0 & r_1 & \cdots & r_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_1 \end{bmatrix},$$

which implies that

$$H_1 = AE, \tag{2.3}$$

where the matrix E is the $k \times k$ upper tridiagonal matrix of the form

$$E = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_k \\ & r_1 & r_2 & \dots & r_{k-1} \\ & & r_1 & \dots & r_{k-2} \\ & & & \ddots & \vdots \\ 0 & & & & r_1 \end{bmatrix}.$$

Using Er's approach [2] and (1.3), we obtain $A^n = G_n$. Since $H_n = A^{n-1}H_1$ and $H_1 = AE$, we get

$$H_n = A^n E, \tag{2.4}$$

which can be re-written as

$$t_n^i = \sum_{j=1}^{i} r_{i+1-j} f_n^j, \tag{2.5}$$

and the proof is complete. ∎

Therefore we see that the general recurrence with arbitrary initial conditions can be written as a linear combination of terms of the recurrence $\{f_n^i\}$. By this result, we can easily derive some properties of the recurrence $\{t_n^i\}$.

Corollary 2.1. *For $n \in \mathbb{Z}$,*

$$\det \begin{pmatrix} t_n^1 & t_n^2 & \dots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \dots & t_{n-1}^k \\ \vdots & \vdots & & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \dots & t_{n-k+1}^k \end{pmatrix} = (-1)^{k+1} c_k r_1^k.$$

Proof. Let H_n, G_n and E be the matrices defined in the proof of Theorem 2.1. It is clear that $\det G_n = (-1)^{k+1} c_k$ and $\det E = r_1^k$. Taking the determinant in $H_n = G_n E$ shows our claim. ∎

Corollary 2.1 is a vast generalization of the well-known Cassini's identity for the Fibonacci numbers, that is, $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$.

Corollary 2.2. *Let $x^k - c_1x^{k-1} - c_2x^{k-2} - \dots - c_k = (x - \lambda_1)\cdots(x - \lambda_k)$ and $e_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_k^n$. Then*

$$e_n = \sum_{i=1}^{k} \left(\sum_{m=1}^{i} r_{i+1-m} f_{n+1-t}^m \right).$$

Proof. A is the companion matrix from (1.2) and $x^k - c_1x^{k-1} - c_2x^{k-2} - \dots - c_k$ is its characteristic polynomial, whose roots (also, eigenvalues of A) are $\lambda_1, \dots, \lambda_k$. Thus the eigenvalues of A^n are $\lambda_1^n, \dots, \lambda_k^n$. Denote the trace of the matrix W by $\operatorname{tr}(W)$. By Theorem 2.1,

$$e_n = \lambda_1^n + \lambda_2^n + \dots + \lambda_k^n = \operatorname{tr}(H_n) = \operatorname{tr}(G_n E)$$

$$= \sum_{i=1}^{k} \left(\sum_{m=1}^{i} r_{i+1-m} f_{n+1-t}^{m} \right).$$

Thus the proof is complete. ∎

3. Sums of the terms of recurrence $\{t_n^k\}$

In this section we deal with the sums of the terms of recurrence $\{t_n^k\}$ subscripted from 1 to n. By the result of Theorem 2.1, clearly

$$t_n^k = \sum_{j=1}^{k} r_{k-j+1} f_n^j. \tag{3.1}$$

The characteristic polynomial of both the matrix A and the sequence $\{f_n^k\}$ is $E(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_{k-1} x - c_k$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the characteristic roots of the equation.

Hypothesis 1. *Throughout this paper, we suppose that the roots $\lambda_1, \ldots, \lambda_k$ are distinct (which happens if $\gcd(E, E') = 1$) and not equal to 1.*

As special cases, we note that when $c_i = 1$ for $1 \le i \le k$, the equation $x^k - x^{k-1} - \cdots - x - 1 = 0$ does not have multiple roots (see [7]). Also, when $c_1 = 2$ and $c_i = 1$ for $2 \le i \le k$, the equation $x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$ does not have multiple roots (see [5]). For the case $c_1 = 2^m$, $c_i = 1$ for $2 \le i \le k$ and $m \ge 0$, we refer to [6].

Let $V = \Lambda^T$ be a $k \times k$ Vandermonde matrix, where

$$\Lambda = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \ldots & \lambda_1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \ldots & \lambda_k & 1 \end{bmatrix}. \tag{3.2}$$

Let w_k^i be the column matrix

$$w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $\Lambda_j^{(i)}$ be the $k \times k$ matrix obtained from Λ by replacing the jth column of Λ by w_k^i.

The generalized Binet formula for the recurrence $\{f_n^i\}$ can be expressed using $V = \Lambda^T$ and $V_j^{(i)} = \Lambda_j^{(i)}$.

Theorem 3.1. *For $n > 0$ and $1 \le i \le k$,*

$$f_{n-i+1}^j = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

Proof. Since the eigenvalues of A are distinct (by our Hypothesis 1), we infer that A is diagonalizable. It is readily seen that $AV = VD$, where $D = diag\,(\lambda_1, \lambda_2, \dots, \lambda_k)$. Since V is invertible, $V^{-1}AV = D$. Hence, A is similar to D. So we obtain $A^nV = VD^n$. Since $A^n = G_n = [g_{ij}]$, we obtain the following linear system of equations:

$$\begin{array}{ll} g_{i1}\lambda_1^{k-1} + g_{i2}\lambda_1^{k-2} + \cdots + g_{ik} & = \lambda_1^{n+k-i} \\ g_{i1}\lambda_2^{k-1} + g_{i2}\lambda_2^{k-2} + \cdots + g_{ik} & = \lambda_2^{n+k-i} \\ \qquad\qquad \vdots & \qquad \vdots \\ g_{i1}\lambda_k^{k-1} + g_{i2}\lambda_k^{k-2} + \cdots + g_{ik} & = \lambda_k^{n+k-i}. \end{array}$$

Thus, for $j = 1, 2, \dots, k$, we get $g_{ij} = \det\left(\Lambda_j^{(i)}\right)/\det(\Lambda)$, where $G_n = [g_{ij}]$ and $g_{ij} = f_{n-i+1}^{j}$. The proof is complete. ∎

Corollary 3.1. *For $n > 0$, we have*

$$t_n^i = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} r_{k+1-j} \det\left(\Lambda_j^{(1)}\right).$$

For example, when $c_1 = 2$ and $c_i = 1$ for all $2 \le j \le k$, the sequence $\left\{f_n^i\right\}$ is reduced to the generalized order-k Pell sequence $\left\{P_n^i\right\}$ and so the sums of the generalized order-k Pell numbers is given by

$$\sum_{i=1}^{n} P_i^k = \frac{\left(P_n^1 + P_n^2 + \cdots + P_n^k - 1\right)}{k}.$$

When $k = 3$, $c_i = 1$ for $1 \le i \le 3$, the sequence $\left\{f_n^i\right\}$ is reduced to the generalized Tribonacci sequence $\left\{T_n^i\right\}$ and so

$$\sum_{i=1}^{n} T_i^3 = \frac{\left(T_n^1 + T_n^2 + T_n^3 - 1\right)}{2}$$

and by the definition of the $\left\{T_n^i\right\}$, we have $T_n^1 = T_{n+1}^3$ and $T_n^2 = T_n^3 + T_{n-1}^3$. For easy writing, we denote T_n^3 by T_n. Thus we can write

$$\sum_{i=1}^{n} T_i = \frac{(T_{n+1} + 2T_n + T_{n-1} - 1)}{2} = \frac{(T_{n+2} + T_n - 1)}{2}.$$

We expand our matrix method to find all sums of terms of k sequences of generalized order-k recurrences $\left\{f_n^i\right\}$ subscripted 1 to n for all $1 \le i \le k$.

Define the following two sums: For $1 \le i \le k$, let $S_n^{(i)} = \sum_{m=1}^{n-1} f_m^i$ and $T_n^{(i)} = \sum_{m=1-i}^{n-i} f_m^i$. Then $T_n^{(i)} = S_{n-i+1}^{(i)} + 1$, since

$$f_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0.$$

Further,

$$S_{n+1}^{(i)} = f_n^i + S_n^{(i)} \tag{3.3}$$

$$T_{n+1}^{(i)} = f_{n-i+1}^i + T_n^{(i)} \tag{3.4}$$

We next define two $(k+1)\times(k+1)$ matrices as follows:

$$B_i = \begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & A & \\ 1 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow (i+1)\,\text{th row} \\ \\ \\ \\ \end{array}$$

and

$$Y_{n,i} = \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ S_n^{(i)} & & & \\ S_{n-1}^{(i)} & & & \\ \vdots & & G_n & \\ S_{n-i+2}^{(i)} & & & \\ T_n^{(i)} & & & \\ T_{n-1}^{(i)} & & & \\ \vdots & & & \\ T_{n-k+i}^{(i)} & & & \end{array}\right] \begin{array}{ll} & \\ \leftarrow & \text{1st row} \\ \leftarrow & \text{2nd row} \\ \vdots & \quad\vdots \\ \leftarrow & (i-1)\,\text{th row} \\ \leftarrow & i\text{th row} \\ \leftarrow & (i+1)\,\text{th row} \\ \vdots & \quad\vdots \\ \leftarrow & k\text{th row} \end{array}$$

where the matrices A and G_n were defined before. We have the following result.

Theorem 3.2. *For $n > 0$,*

$$Y_{n,i} = B_i^n.$$

Proof. Combining the identities (3.3) and (3.4), we obtain

$$Y_{n+1,i} = Y_{n,i}B_i = \cdots = Y_{1,i}B_i^n.$$

From the definitions of $\left\{T_n^{(i)}\right\}$ and $\left\{S_n^{(i)}\right\}$, we can easily check that $Y_{1,i} = B_i$, and the theorem is proven. ∎

Now we are going to derive an explicit expression for every sum $S_n^{(i)}$ for $1 \le i \le k$ by matrix methods.

We first make some observations. If we expand $\det B_i$ with respect to the first row, we get

$$\det B_i = \det A$$

and the characteristic polynomials of A, B_i satisfy

$$C_{B_i}(\lambda) = (1-\lambda)\, C_A(\lambda).$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $C_A(\lambda)$ (distinct and nonequal to 1), the eigenvalues of matrix B_i are $\lambda_1, \lambda_2, \ldots, \lambda_k, 1$. Therefore the eigenvalues of the matrix B_i are distinct, and so B_i is diagonalizable.

For easy writing, let

$$\mu_i = \frac{\sum_{t=i}^{k} c_t}{1 - \sum_{t=1}^{k} c_t} \text{ for } 1 < i \leq k \text{ and } \mu_1 = \frac{1}{1 - \sum_{t=1}^{k} c_t}.$$

The following $(k+1) \times (k+1)$ matrix for $1 < i \leq k$

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \mu_i & \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \mu_i & \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_i & \lambda_1^{k-i+1} & \lambda_2^{k-i+1} & & \lambda_k^{k-i+1} \\ \mu_i + 1 & \lambda_1^{k-i} & \lambda_2^{k-i} & & \lambda_k^{k-i} \\ \mu_i + 1 & \lambda_1^{k-i-1} & \lambda_2^{k-i-1} & & \lambda_k^{k-i-1} \\ \vdots & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \mu_i + 1 & 1 & 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \quad 0 & \dots & 0 \\ \mu_i & & & \\ \mu_i & & & \\ \vdots & & V & \\ \mu_i & & & \\ \mu_i + 1 & & & \\ \vdots & & & \\ \mu_i + 1 & & & \end{bmatrix}$$

satisfies $B_i P = P D_1$, where D_1 is the $(k+1) \times (k+1)$ diagonal matrix defined previously, $D_1 = diag\,(1, \lambda_1, \lambda_2, \dots, \lambda_k)$. Here we note that if we expand $\det P$ with respect to the first row, then we get $\det P = \det \Lambda$. Since Λ is the Vandermonde matrix, the matrix P is invertible.

Theorem 3.3. *For $n > 0$ and $1 < i < k$,*

$$S_n^{(i)} = \mu_i \left(1 - \sum_{j=1}^{k} f_n^j \right) - \sum_{m=i}^{k} f_n^m$$

and

$$S_n^{(1)} = \mu_1 \left(1 - \sum_{j=1}^{k} f_n^j \right).$$

Proof. Since $B_i P = P D_1$ for $1 < i \leq k$ and the matrix P is invertible, we write $B_i^n P = P D_1^n$ and so $Y_{n,i} P = P D_1^n$. By equating the $(2,1)$ entries of the equality $Y_{n,i} P = P D_1^n$, we have the conclusion.

For the case $i = 1$, one can see that $B P_1 = P_1 D_1$ where the $(k+1) \times (k+1)$ matrices B and P_1 are as follows

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & A & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \mu_1 & & & \\ \vdots & & V & \\ \mu_1 & & & \end{bmatrix}.$$

By induction on n, we see that

$$Y = B^n = \begin{bmatrix} 1 & 0 \ \dots \ 0 \\ S_n^{(i)} & \\ S_{n-1}^{(i)} & G_n \\ \vdots & \\ S_{n-k+1}^{(i)} & \end{bmatrix}.$$

Similar to the cases $1 < i \le k$, the proof is easily seen for the case $i = 1$. ∎

As a consequence of Theorem 3.3, we get

$$S_n = \sum_{i=1}^{n} f_i^k = \frac{c_k \left(\sum_{j=1}^{k} f_n^j - 1 \right)}{c_1 + c_2 + \dots + c_k - 1}.$$

Let $V_{i,j}$ be a $k \times k$ matrix obtained from the Vandermonde matrix V by replacing the jth column of V by e_i where $V = \Lambda^T$ is defined as in (3.2) and e_i is the ith element of the natural basis for $\mathbb{R}^n$, that is,

$$e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{th}}}{1}, 0, \dots 0)^T$$

and

$$V_{i,j} = \begin{bmatrix} \lambda_1^{k-1} & \dots & \lambda_{j-1}^{k-1} & 0 & \lambda_{j+1}^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \dots & \lambda_{j-1}^{k-2} & 0 & \lambda_{j+1}^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \dots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \dots & \lambda_k^{k-i+1} \\ \lambda_1^{k-i} & \dots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \dots & \lambda_k^{k-i} \\ \lambda_1^{k-i-1} & \dots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \dots & \lambda_k^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \dots & \lambda_{j-1} & 0 & \lambda_{j+1} & \dots & \lambda_k \\ 1 & \dots & 1 & 0 & 1 & \dots & 1 \end{bmatrix}$$

Let $q_j^{(i)} = \frac{|V_{i,j}|}{|V|}$.

Theorem 3.4. *For any integer n and $1 \le i \le k$,*

$$f_n^i = \sum_{j=1}^{k} q_j^{(i)} \lambda_j^{n+k-1}.$$

Proof. We consider the following system of k linear equations in k unknowns $x_1, x_2, \dots, x_k$:

$$\begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-i} & \lambda_2^{k-i} & \dots & \lambda_k^{k-i} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_i}.$$

Using Vandermonde's determinants and Cramer rule, we get

$$q_j^{(i)} = \frac{|V_{i,j}|}{|V|} \ (i = 1, 2, \ldots, k),$$

and so, for $n, k > 0$ and $1 \le i \le k$, $f_n^i = \sum_{j=1}^{k} q_j^{(i)} \lambda_j^{n+k-1}$, which completes the proof. ∎

Consequently, we extend the result of Theorem 3.4 to the general order linear recurrences $\{t_n^i\}$ by the result given by (2.5).

Corollary 3.2. *For any integer n and $1 \le i \le k$,*

$$t_n^i = \sum_{j=1}^{i} \sum_{s=1}^{k} r_{i+1-j} q_s^{(j)} \lambda_s^{n+k-1}.$$

As an example, we consider the sequence $\{T_n^i\}$,

$$T_n^i = T_{n-1}^i + 3T_{n-2}^i + T_{n-2}^i, n \ge 2, 1 \le i \le 3$$

with

$$T_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \le n \le 0,$$

displayed in the following table:

$i\backslash n$	1	2	3	4	5	6	7	8		
1	1	4	8	21	49	120	288	697	...	$\{T_n^1\}$
2	3	4	13	28	71	168	409	984	...	$\{T_n^2\}$
3	1	1	4	8	21	49	120	288	...	$\{T_n^3\}$

Here we note that $\gamma_1 = -1,\ \gamma_2 = 1 + \sqrt{2}, \gamma_3 = 1 - \sqrt{2}$ and

$$q_1^{(1)} = \tfrac{1}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(1)} = \tfrac{1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ q_3^{(1)} = \tfrac{1}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$
$$q_1^{(2)} = -\tfrac{\gamma_2+\gamma_3}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ q_2^{(2)} = \tfrac{\gamma_1+\gamma_3}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)},\ q_3^{(2)} = -\tfrac{\gamma_1+\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$
$$q_1^{(3)} = \tfrac{\gamma_2\gamma_3}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(3)} = -\tfrac{\gamma_1\gamma_3}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)},\ q_3^{(3)} = \tfrac{\gamma_1\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}.$$

Therefore, by Theorem 3.4, we get

$$T_n^1 = \frac{\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)} + \frac{\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$
$$T_n^2 = -\frac{(\gamma_2+\gamma_3)\gamma_1^{n+2}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{(\gamma_1+\gamma_3)\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)} - \frac{(\gamma_1+\gamma_2)\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}$$

and since $\gamma_1\gamma_2\gamma_3 = 1$,

$$\begin{aligned} T_n^3 &= \frac{\gamma_2\gamma_3\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} - \frac{\gamma_1\gamma_3\gamma_2^{n+2}}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)} + \frac{\gamma_1\gamma_2\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)} \\ &= \frac{\gamma_1^{n+1}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+1}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)} \\ &= T_{n-1}^1. \end{aligned}$$

Observe (from table above) that $T_n^3 = T_{n-1}^1$.

4. Generating Functions

In this section we derive the family of generating functions $G(i,x) = \sum_{n=0}^{\infty} f_n^i x^n$ for the generalized order-k recurrences $\{f_n^i\}$ for all i, $1 \le i \le k$.

Theorem 4.1. *For* $1 \le i \le k$,

$$G(i,x) = \frac{f_0^i + \sum_{m=1}^{k-1} \left(\sum_{v=m+1}^{k} c_v f_{m-v}^i \right) x^m}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}.$$

Proof. Let $G(i,x) = f_0^i x^0 + f_1^i x^1 + f_2^i x^2 + \cdots + f_n^i x^n + \cdots$. Consider

$$\begin{aligned}
&\left(1 - c_1 x - c_2 x^2 - \cdots - c_k x^k\right) G(i,x) \\
&= f_0^i + f_1^i x + f_2^i x^2 + \cdots + f_k^i x^k + \cdots + f_n^i x^n + \cdots \\
&\quad - c_1 f_0^i x - c_1 f_1^i x^2 - c_1 f_2^i x^3 - \cdots - c_1 f_{k-1}^i x^k - \cdots - c_1 f_{n-1}^i x^n - \cdots \\
&\quad - c_k f_0^i x^k - c_k f_1^i x^{k+1} - c_k f_2^i x^{k+2} - \cdots - c_k f_{n-k}^i x^n - \cdots \\
&= f_0^i + \left(f_1^i - c_1 f_0^i\right) x + \left(f_2^i - c_1 f_1^i - c_2 f_0^i\right) x^2 + \cdots \\
&\quad + \left(f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i\right) x^{k-1} \\
&\quad + \left(f_k^i - c_1 f_{k-1}^i - c_2 f_{k-2}^i - \cdots - c_{k-1} f_0^i - c_k f_1^i\right) x^k + \cdots \\
&\quad + \left(f_n^i - c_1 f_{n-1}^i - c_2 f_{n-2}^i - \cdots - c_k f_{n-k}^i\right) x^n + \cdots .
\end{aligned}$$

Now we compute the coefficients of x^n of the equation above. From the definition of $\{f_n^i\}$, we get

$$\begin{aligned}
f_1^i &= c_1 f_0^i + c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\
&\vdots \\
f_{k-1}^i &= c_1 f_{k-2}^i + c_2 f_{k-3}^i + \cdots + c_{k-1} f_0^i + c_k f_{-1}^i \\
&\vdots \\
f_n^i &= c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i.
\end{aligned}$$

and so

$$\begin{aligned}
f_1^i - c_1 f_0^i &= c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\
f_2^i - c_1 f_1^i - c_2 f_0^i &= c_3 f_{-1}^i + \cdots + c_k f_{2-k}^i \\
&\vdots \\
f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i &= c_k f_{-1}^i.
\end{aligned}$$

Then for $n \ge k$, by the definition of $\{f_n^i\}$, the coefficients of x^n are all 0. ∎

For example, for fixed k and $1 \le i \le k$, we take $i = 1$. Thus

$$G(1,x) = f_0^1 x^0 + f_1^1 x^1 + f_2^1 x^2 + \cdots + f_n^1 x^n + \cdots .$$

From the definition of $\{f_n^i\}$, the initial conditions of the recurrence $\{f_n^1\}$ are given by

$$f_n^1 = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0,$$

which implies

$$G(1,x) = \frac{1}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}. \tag{4.1}$$

More generally, we derive the generating function of recurrence $\{t_n^i\}$, namely $g(i,x) = \sum_{k\ge 0} t_k^i x^k$.

Corollary 4.1. *For $1 \le i \le k$,*

$$g(i,x) = \frac{t_0^i + \sum_{m=1}^{k-1} \left(\sum_{v=m+1}^{k} c_v t_{m-v}^i \right) x^m}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}.$$

As an example, if we take $k = i = 2$, $c_1 = c_2 = 1$ and $r_1 = -1, r_2 = 0$, then the sequence $\{t_n^2\}$ is

$$1, 3, 4, 7, 11, 18, 29, \ldots$$

which is the well-known Lucas sequence $\{L_n\}$. Then by Corollary 4.1, we obtain

$$g(2,x) = \sum_{n=0}^{\infty} t_n^2 x^n = \sum_{n=0}^{\infty} L_n x^n = \frac{t_0^i - \left(t_{-1}^i\right) x^1}{1 - x - x^2}$$

where $t_0^2 = r_2 = 2$ and $t_{-1}^2 = r_1 = 1$. Thus we have the well known result for the Lucas numbers:

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}.$$

5. nth powers of a companion and k-superdiagonal determinants

In [8], the author gave a relationship between determinants of certain $n \times n$ k-superdiagonal matrices and the terms of the nth power of matrix A given by (1.2). In this section, we derive some new relationships between some Hessenberg determinants and the terms of generalized recurrences $\{f_n^i\}$ for all $1 \le i \le k$.

Here, we recall a result of [8]. Define an $n \times n$ k-superdiagonal matrix M_n in the following form:

$$M_n = \begin{bmatrix} c_1 & c_2 & \ldots & c_k & & & 0 \\ -1 & c_1 & c_2 & \ldots & c_k & & \\ & -1 & c_1 & c_2 & \ldots & \ddots & \\ & & & & \ddots & \ldots & \vdots \\ 0 & & & & & -1 & c_1 \end{bmatrix}.$$

Lemma 5.1. *For $n > 0$,*

$$\det M_n = f_n^1.$$

Indeed, expanding $\det M_n$ by the elements of the first row gives us

$$\det M_n = c_1 \det M_{n-1} + c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k}, \tag{5.1}$$

$$= f_n^1 = c_1 f_{n-1}^1 + c_2 f_{n-2}^1 + \cdots + c_k f_{n-k}^1. \tag{5.2}$$

Now we extend the above result for the generalized sequences $\{f_n^i\}$ for $1 \leq i \leq k$. For this purpose we introduce some new notations: For $1 \leq t \leq k$, let $M_n(t, t+1, \ldots, k; r) = [\hat{m}_{ij}]$ denote the matrix obtained from $M_n = [m_{ij}]$ with $\hat{m}_{ij} = 0$ for $i \leq j \leq r$, $i \in \{t, t+1, \ldots, k\}$ and otherwise $\hat{m}_{ij} = m_{ij}$. Clearly $M_n(1, 2, \ldots, k; 0) = M_n$.

Recalling that $G_n = [g_{ij}] = A^n$, we give the following theorem for the diagonal elements $g_{jj} = f_{n-j}^{(j+1)}$.

Theorem 5.1. *For $n > j$ and $1 \leq j \leq k-1$,*

$$\det M_n(1; j) = f_{n-j}^{j+1}$$

where $\det M_n(1; 0) = f_n^1$.

Proof. First consider the case $j = 1$. If we expand the $\det M_n(1; 1)$ by the elements of the first row, then

$$\begin{aligned} \det M_n(1; 1) &= 0\,(\det M_{n-1}) + c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k} \\ &= c_2 \det M_{n-2} + \cdots + c_k \det M_{n-k}. \end{aligned}$$

By (5.1) and (5.2),

$$\begin{aligned} \det M_n(1; 1) &= c_2 f_{n-2}^1 + c_3 f_{n-3}^1 + \cdots + c_k f_{n-k}^1 \\ &= f_n^1 - c_1 f_{n-1}^1 = f_{n-1}^2. \end{aligned}$$

Thus the proof is complete for the case $j = 1$.

Now, we take the general case for $1 \leq j \leq k-1$. By expanding $\det M_n(1; j)$ with respect to the first row, we get

$$\det M_n(1; j) = \det \begin{bmatrix} 0 & \ldots & 0 & c_{j+1} & c_{j+2} & \ldots & c_k & 0 & \ldots & 0 \end{bmatrix},$$

which, by (5.1) and (5.2), becomes

$$\begin{aligned} \det M_n(1; j) &= c_{j+1} \det M_{n-j-1} + c_{j+2} \det M_{n-j-2} + \cdots + c_k \det M_{n-k} \\ &= c_{j+1} f_{n-j-1}^1 + c_{j+2} f_{n-j-2}^1 + \cdots + c_k f_{n-k}^1. \end{aligned}$$

From (5.2) and after repeating j times the identity (1.4), we get

$$\begin{aligned} \det M_n(1; j) &= c_{j+1} f_{n-j-1}^1 + c_{j+2} f_{n-j-2}^1 + \cdots + c_k f_{n-k}^1 \\ &= f_n^1 - c_1 f_{n-1}^1 - c_2 f_{n-2}^1 - \cdots - c_j f_{n-j}^1 \\ &= f_{n-1}^2 - c_2 f_{n-2}^1 - c_3 f_{n-3}^1 - \cdots - c_j f_{n-j}^1 \\ &\cdots \\ &= f_{n-j+1}^j - c_j f_{n-j}^1 = f_{n-j}^{j+1}, \end{aligned}$$

and the proof is complete. ∎

According to the definition of $M_n(t, t+1, \ldots, k; r)$, the matrix $M_n(2, 3; n)$ can be expressed in the compact form

$$M_n(2,3;n) = \begin{bmatrix} c_1 & c_2 & \ldots & c_k & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ & -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ & & -1 & c_1 & c_2 & \ldots & c_k & 0 & \ldots & 0 \\ & & & \ddots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 & \ldots & c_k & 0 \\ & & & & & -1 & c_1 & c_2 & \ldots & c_k \\ & & & & & & \ddots & \ddots & \ldots & \vdots \\ & & & & & & & -1 & c_1 & c_2 \\ 0 & & & & & & & & -1 & c_1 \end{bmatrix}.$$

Theorem 5.2. *For $n > k+2$,*

$$\det M_{n+1}(2, 3, \ldots, k; n) = f^k_{n-k+2}.$$

Proof. First we consider the case of $k = 2$, and $\det M_{n+1}(2; n)$. The matrix $M_n(2; n)$ has the following form:

$$M_n(2;n) = \begin{bmatrix} c_1 & c_2 & \ldots & c_k & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ & -1 & c_1 & c_2 & \ldots & c_k & 0 & \ldots & 0 \\ & & \ddots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ & & & -1 & c_1 & c_2 & \cdots & c_k & 0 \\ & & & & -1 & c_1 & c_2 & \ldots & c_k \\ & & & & & \ddots & \ddots & \ldots & \vdots \\ & & & & & & -1 & c_1 & c_2 \\ & & & & & & & -1 & c_1 \end{bmatrix}.$$

Expanding $\det M_{n+1}(2; n)$ with respect to the first row, we obtain

$$\det M_{n+1}(2; n) = c_2 \det M_{n-1} + c_3 \det M_{n-2} + \cdots + c_k \det M_{n-k+1}.$$

Since the first principal subdeterminant include a zero row, by Lemma 5.1, we write

$$\begin{aligned} \det M_{n+1}(2; n) &= c_2 f^1_{n-1} + c_3 f^1_{n-3} + \cdots + c_k f^1_{n-k+1} \\ &= -c_1 f^1_n + c_1 f^1_n + c_2 f^1_{n-1} + c_3 f^1_{n-3} + \cdots + c_k f^1_{n-k+1} \\ &= -c_1 f^1_n + f^1_{n+1}. \end{aligned}$$

By (1.4), we obtain $\det M_{n+1}(2; n) = -c_1 f^1_n + f^1_{n+1} = f^2_n$. Thus, the proof is complete for $k = 2$.

Continuing this expanding process with respect to the first row for the $\det M_{n+1}$ $(2, 3, \ldots, k; n)$, for $j \geq 2$, we get

$$\det M_{n+1}(2, 3, \ldots, j; n) = c_j \det M_{n-j+1} + c_{j+1} \det M_{n-j} + \cdots + c_k \det M_{n-k+1}$$

which, by Lemma 5.1, gives

$$\begin{aligned}
\det M_{n+1}(2,3,\ldots,j;n) &= c_j f^1_{n-j+1} + c_3 f^1_{n-j} + \cdots + c_k f^1_{n-k+1} \\
&= \left(c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_{j-1} f^1_{n-j+2}\right) \\
&\quad - \left(c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_{j-1} f^1_{n-j+2}\right) \\
&\quad + c_j f^1_{n-j+1} + c_3 f^1_{n-j} + \cdots + c_k f^1_{n-k+1} \\
&= f^1_{n+1} - \left(c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_{j-1} f^1_{n-j+2}\right).
\end{aligned}$$

By (1.4), we obtain

$$\begin{aligned}
\det M_{n+1}(2,3,\ldots,j;n) &= f^1_{n+1} - c_1 f^1_n - c_2 f^1_{n-1} - \cdots - c_{j-1} f^1_{n-j+2} \\
&= f^2_n - c_2 f^1_{n-1} - \cdots - c_{j-1} f^1_{n-j+2} \\
&\vdots \\
&= f^{j-1}_{n-j+3} - c_{j-1} f^1_{n-j+2} = f^j_{n-j+2},
\end{aligned}$$

and the proof is complete. ∎

Now we present further relations including other entries of G_n and the determinant of certain matrices.

Define the $n \times n$ matrix $M_n(c_{i,k})$ in the compact form:

$$M_n(c_{i,k}) = \begin{bmatrix} c_i & c_{i+1} & \ldots & c_k & 0 & \cdots & 0 \\ -1 & & & & & & \\ 0 & & M_{n-1} & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}$$

where M_n is defined as before.

For $2 \le t \le r$, let $M_n(c_{i,k}, t, t+1, \ldots, r) = [\check{m}_{ij}]$ denote the $n \times n$ matrix obtained from $M_n(c_{i,k}) = [\tilde{m}_{ij}]$ with taking $\check{m}_{ij} = 0$ for $i \le j \le r,\ i \in \{t, t+1, \ldots, n\}$ and otherwise $\check{m}_{ij} = \tilde{m}_{ij}$.

For example, $M_7(c_{2,4}, 3, 4)$ takes the form:

$$M_7(c_{2,4}, 3, 4) = \begin{bmatrix} c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 \\ -1 & c_1 & c_2 & c_3 & c_3 & c_4 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & -1 & c_1 & c_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & c_1 \end{bmatrix}.$$

Theorem 5.3. *For $n > j-1,\ 2 \le r \le k-1$ and $2 \le j \le k$*

$$\det M_n(c_{r,k}, 2, 3, \ldots, j) = g_{j-1,j+r-1} = f^{j+r-1}_{n-j+1}$$

where $G_n = [g_{ij}]$.

Proof. First we prove the case $r = 2$ and $2 \leq j \leq k$. If we expand $\det M_n(c_{2,k}, 2, 3, \ldots, j)$ by the Laplace expansion of determinant, then we obtain the following equation by combining (5.1) and (5.2)

$$\begin{aligned}
&\det M_n(c_{2,k}, 2, 3, \ldots, j) \\
&= c_{j+1} \det M_{n-j} + c_{j+2} \det M_{n-j-1} + \cdots + c_k \det M_{n-k+1} \\
&= c_{j+1} f^1_{n-j} + c_{j+2} f^1_{n-j-1} + \cdots + c_k f^1_{n-k+1}.
\end{aligned}$$

By adding and subtracting $c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_j f^1_{n-j+1}$ to both sides of the above equation, we get

$$\begin{aligned}
&\det M_n(c_{2,k}, 2, 3, \ldots, j) \\
&= \left(c_1 f^1_n + \cdots + c_j f^1_{n-j+1}\right) + c_{j+1} f^1_{n-j} + \cdots + c_k f^1_{n-k+1} \\
&\quad - \left(c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_j f^1_{n-j+1}\right) \\
&= f^1_{n+1} - c_1 f^1_n - c_2 f^1_{n-1} - \cdots - c_j f^1_{n-j+1}.
\end{aligned}$$

By (1.4), we get

$$\begin{aligned}
\det M_n(c_{2,k}, 2, 3, \ldots, j) &= f^2_n - c_2 f^1_{n-1} - c_3 f^1_{n-2} - \cdots - c_j f^1_{n-j+1} \\
&= f^3_{n-1} - c_3 f^1_{n-2} - \cdots - c_j f^1_{n-j+1} \\
&\ \ \vdots \\
&= f^j_{n-j+2} - c_j f^1_{n-j+1} = f^{j+1}_{n-j+1}.
\end{aligned}$$

Thus the proof is complete for $r = 2$.

Now we consider the case $r > 2$. If j is greater than $k - 2$, then the matrix $M_n(c_{r,k}, 2, 3, \ldots, j)$ has a zero row and so we ignore this case. For $r > 2$ and $j \leq k - 2$, we obtain, by (1.4), (5.1) and (5.2)

$$\begin{aligned}
&\det M_n(c_{r,k}, 2, 3, \ldots, j) \\
&= c_{r+j-1} \det M_{n-j} + \cdots + c_k \det M_{n-k+1} \\
&= c_{r+j-1} f^1_{n-j} + c_{r+j} f^1_{n-j-1} + \cdots + c_k f^1_{n-k+1} \\
&= f^1_{n+1} - c_1 f^1_n - c_2 f^1_{n-1} - \cdots - c_{r+j-2} f^1_{n-j+1} \\
&= f^2_n - c_2 f^1_{n-1} - \cdots - c_{r+j-2} f^1_{n-j+1} \\
&\ \ \vdots \\
&= f^{r+j-2}_{n-j+2} - c_{r+j-2} f^1_{n-j+1} = f^{r+j-1}_{n-j+1},
\end{aligned}$$

which completes the proof for all cases. ∎

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