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More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences

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ABSTRACT

Recently [Z. Wenpeng, W. Tingting, Applied Mathematics and Computation 218 (10) (2012) 6164–6167; T. Komatsu, V. Laohakosol, Journal of Integer Sequences 13 (5) (2010) Article 10.5.8.] computed partial infinite sums including reciprocal usual Fibonacci, Pell and generalized order-k Fibonacci numbers. In this paper we will present generalizations of earlier results by considering more generalized higher order recursive sequences with additional one coefficient parameter.

1. Introduction

Let p and q be real numbers such that $p^2 + 4q \neq 0$. Define the generalized Fibonacci sequence $\{U_n(p,q)\}$, briefly $\{U_n\}$, and Lucas sequence $\{V_n(p,q)\}$, briefly $\{V_n\}$, as shown: for $n > 1$

$$U_n(p,q) = pU_{n-1}(p,q) + qU_{n-2}(p,q),$$
$$V_n(p,q) = pV_{n-1}(p,q) + qV_{n-2}(p,q),$$

where $U_0 = 0,\ U_1 = 1$, and, $V_0 = 2,\ V_1 = p$, respectively. The Binet formulae for $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = (p \pm \sqrt{p^2 + 4q})/2$. Here note that $U_n(1,1) = F_n$ (nth Fibonacci Number), $V_n(1,1) = L_n$ (nth Lucas number) and $U_n(2,1) = P_n$ (nth Pell number).

Ohtsuka and Nakamura [5] introduced and computed the following partial infinite sums including reciprocal Fibonacci numbers:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geqslant 2, \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geqslant 1 \end{cases} \tag{1}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geqslant 2, \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases}$$

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where $\lfloor\cdot\rfloor$ is the floor function.

Wenpeng and Tingting [6] gave analogue of the result (1) for the Pell numbers:

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{P_k}\right)^{-1}\right\rfloor = \begin{cases} P_{n-1}+P_{n-2} & \text{if } n \text{ is even and } n \geqslant 2; \\ P_{n-1}+P_{n-2}-1 & \text{if } n \text{ is odd and } n \geqslant 1. \end{cases}$$

Also the same authors [7] gave the similar results for partial infinite sums including reciprocal squared-Pell numbers.

Holliday and Komatsu [1] obtained similar results for the terms of generalized Fibonacci sequence $\{U_n(p,1)\}$:

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k}\right)^{-1}\right\rfloor = \begin{cases} U_n-U_{n-1} & \text{if } n \text{ is even and } n \geqslant 2, \\ U_n-U_{n-1}-1 & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases} \tag{2}$$

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k^2}\right)^{-1}\right\rfloor = \begin{cases} pU_nU_{n-1}-1 & \text{if } n \text{ is even and } n \geqslant 2, \\ pU_nU_{n-1} & \text{if } n \text{ is odd and } n \geqslant 1. \end{cases}$$

In this paper we will consider on the following type higher order recurrence sequences and then give general results similar to the above partial sums. For any positive reals p and q, we define a kth order linear recursive sequence $\{u_n(p,q,k)\}$, briefly $\{u_n\}$, for $n > k$ as follows

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k}, \tag{3}$$

with nonnegative initials $u_t \geqslant 0$ for $0 \leqslant t < k$ and assumed that at least one of them is different from zero.

The author [2] generalized the results given in (2) for the terms of generalized order-k Fibonacci sequence $\{u_n(p,q,2)\}$ as shown: then there exist a positive integer n_0 such that

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_n(p,q,2)}\right)^{-1}\right\| = u_n(p,q,2) - u_{n-1}(p,q,2), \quad (n \geqslant n_0),$$

where $p \geqslant q$ and $\|\cdot\|$ denotes the nearest integer (clearly $\|x\| = \lfloor x + \frac{1}{2}\rfloor$).

Recently the authors [3] presented the following results for the order- k recursion $\{u_n(p,1,k)\}$ (with an arbitrary coefficient p and arbitrary k initials but not all of them are zero):

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_k(p,1,k)}\right)^{-1}\right\| = u_n(p,1,k) - u_{n-1}(p,1,k), \quad (n \geqslant n_0),$$

$$\left\|\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{u_k(p,1,k)}\right)^{-1}\right\| = (-1)^n(u_n(p,1,k) - u_{n-1}(p,1,k)), \quad (n \geqslant n_1)$$

and

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_{2k}(p,1,k)}\right)^{-1}\right\| = u_{2n}(p,1,k) - u_{2n-2}(p,1,k), \quad (n \geqslant n_2),$$

where n_0, n_1, n_2 are natural numbers depending on p.

In the rest of this paper, we will obtain generalizations of the results of [3] on the reciprocal sums of order-k recurrence sequence $\{u_n(p,1,k)\}$ mentioned just above. To obtain such generalizations, we will consider the order-k recurrence sequence $\{u_n(p,q,k)\}$ (with two arbitrary coefficients p,q and arbitrary k initials) instead of the sequence $\{u_n(p,1,k)\}$.

2. Main results

While considering the order-k sequences defined by (3), we assume that the restriction $p \geqslant q \geqslant 1$ throughout this paper.

Our first main result is

Theorem 1. *Let $\{u_n(p,q,k)\}$, briefly $\{u_n\}$, be an order-k sequence defined by (3) with the restriction $p \geqslant q \geqslant 1$. Then there exists a positive integer n_0 such that*

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_k}\right)^{-1}\right\| = u_n - u_{n-1}, \quad (n \geqslant n_0).$$

Before the proof, we need the following lemmas:

Lemma 2. *Let p and q be positive reals with $p \geqslant q \geqslant 1$ and $k \in \mathbb{N}$ with $k \geqslant 2$. Then for the polynomial*

$$f(x) = x^k - px^{k-1} - qx^{k-2} - x^{k-3} - \cdots - x - 1, \tag{4}$$

we have

(i) $f(x)$ *has exactly one positive real root* α *with* $p < \alpha < p + 1$.
(ii) *Other* $k - 1$ *roots of* $f(x)$ *are within the unit circle in the complex plane.*

Proof. Let

$$g(x) = (x-1)f(x) = x^{k+1} - (p+1)x^k + (p-q)x^{k-1} + (q-1)x^{k-2} + 1.$$

The case $q = 1$ was given in [3]. We will consider two cases $p = q$ and $p > q$.

Case 1: If $p = q$, then

$$g(x) = x^{k+1} - (p+1)x^k + (p-1)x^{k-2} + 1.$$

This case is very similar to the case $q = 1$ so we omit it here.

Case 2: For $p > q > 1$, we have five nonnegative coefficients in the polynomial $g(x)$ given by

$$g(x) = x^{k+1} - (p+1)x^k + (p-q)x^{k-1} + (q-1)x^{k-2} + 1.$$

According to Descarte's rule, $f(x)$ has at most one positive real root and so $g(x)$ has at most two positive real roots (clearly one of them is 1).

Now we examine that there exists an another positive real root. Since $p > 1$ and $k \geqslant 2$ then

$$g(p) = \frac{1}{p^2}\left(p^k q + p^2 - p^k - p^{k+1}q\right) = \frac{1}{p^2}\left(p^k q(1-p) + (p^2 - p^k)\right) < 0$$

and also since $p^2 > pq$ and $p > 1$ we have

$$g(p+1) = \frac{1}{(p+1)^2}\left((p+1)^k(p^2 - 1 + p - pq) + 2p + p^2 + 1\right) > 0.$$

Thus there exist an another positive real root α of $g(x)$ with $p < \alpha < p + 1$. As a result of this $f(x)$ has exactly one positive real root ($\alpha \in \mathbb{R}$) with $p < \alpha < p + 1$. So the proof of Lemma 2 (i) is complete.

By considering the Lemma (i), we have

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \quad \text{then } f(x) > 0, \tag{5}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 0 < x < a, \quad \text{then } f(x) < 0$$

and

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \quad \text{then } g(x) > 0,$$

$$\text{if } x \in \mathbb{R} \text{ such that } 1 < x < a, \quad \text{then } g(x) < 0. \tag{6}$$

To complete the proof of Lemma 2 (ii), it is sufficient to show that there is no root on and outside of the unit circle.

Claim 1: $f(x)$ has no complex root z_1 with $|z_1| > \alpha$.
Assume that there exists such a root. So we have

$$f(z_1) = z_1^k - pz_1^{k-1} - qz_1^{k-2} - z_1^{k-3} - \cdots - z_1 - 1 = 0$$

and then we obtain

$$\left|z_1^k\right| \leqslant p\left|z_1^{k-1}\right| + q\left|z_1^{k-2}\right| + \left|z_1^{k-3}\right| + \cdots + |z_1| + 1,$$
$$f(|z_1|) = |z_1|^k - p|z_1|^{k-1} - q|z_1|^{k-2} - |z_1|^{k-3} - \cdots - |z_1| - 1 \leqslant 0.$$

This contradicts with (5).

Claim 2: $f(x)$ has no complex root z_2 with $1 < |z_2| < \alpha$.
Suppose that there exists such a root. Since $f(z_2) = 0$,

$$g(z_2) = z_2^{k+1} - (p+1)z_2^k + (p-q)z_2^{k-1} + (q-1)z_2^{k-2} + 1 = 0,$$

which implies

$$(p+1)|z_2|^k \leqslant |z_2|^{k+1} + (p-q)|z_2|^{k-1} + (q-1)|z_2|^{k-2} + 1.$$

So we have $g(|z_2|) \geqslant 0$. But this is a contradiction with (6).

Claim 3: On the circle $|z_3| = \alpha$ and $|z_3| = 1$, $f(x)$ has the unique root α.
Let $z_3 \neq \alpha$ and either $|z_3| = \alpha$ or $|z_3| = 1$ and also $f(z_3) = 0$, then

$$g(z_3) = z_3^{k+1} - (p+1)z_3^k + (p-q)z_3^{k-1} + (q-1)z_3^{k-2} + 1 = 0.$$

So we get

$$(p+1)|z_3|^k \leqslant |z_3|^{k+1} + (p-q)|z_3|^{k-1} + (q-1)|z_3|^{k-2} + 1.$$

Since α and 1 are also the roots of $g(z)$,

$$\left|z_3^{k+1} + (p-q)z_3^{k-1} + (q-1)z_3^{k-2} + 1\right| = |z_3|^{k+1} + (p-q)|z_3|^{k-1} + (q-1)|z_3|^{k-2} + 1.$$

The equality holds if and only if all parts lie on the same ray issuing from the origin. One of the parts is 1 (see [4]). So the other parts, $z_3^{k+1},\ (p-q)z_3^{k-1}, (q-1)z_3^{k-2}$, must be element of $\mathbb{R}^+$. Since $(p-q),\ (q-1) \in \mathbb{R}^+,\ z_3^{k+1},\ z_3^{k-1}$ and z_3^{k-2} must be elements of $\mathbb{R}^+$. Therefore we obtain $z_3 \in \mathbb{R}^+$. There are two possibilities $z_3 = 1$ or $z_3 = \alpha$. Since $f(1) \neq 0$ the case $z_3 = 1$ is ruled out. From Lemma (i) we know that $f(x)$ has exact one positive real root α. So the case $z_3 = \alpha$ has already known. Since multiple roots are counted separately by Descarte's rule, there is not an another positive real root. From these tree claims, Lemma (ii) is proven. Consequently, $f(x)$ has exactly one positive real root α with $p < \alpha < p+1$ and the other roots are within the unit circle. □

Lemma 3. *Let $k \geqslant 2$, then the closed formula of $\{u_n\}$ is given by*

$$u_n = a\alpha^n + O(c^{-n}), \quad (n \to \infty),$$

where $a > 0,\ c > 1$ and α is the positive real root of (4).

Proof. Let $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_t$ with $|\alpha_i| < 1$ for $1 \leqslant i \leqslant t$ be distinct roots of $f(x)$ and r_j for $j = 1, 2, \ldots, t$ denotes the multiplicity of the root α_j. Then u_n can be written as follows

$$u_n = a\alpha^n + \sum_{i=1}^{t} P_i(n)\alpha_i^n,$$

where $P_i(n) \in \mathbb{R}[x]$ with $\deg P_i = r_i - 1, r_1 + r_2 + \cdots + r_t = k-1$ and $a \in \mathbb{R}^+$. Since $|\alpha_i| < 1$ for $1 \leqslant i \leqslant t$, each term of tail goes to 0 as $n \to \infty$. So we can find constant $K \in \mathbb{R}$ and $c \in \mathbb{R}$ with $c > 1$ for $n > n_0$ such that

$$\sum_{i=1}^{t} P_i(n)\alpha_i^n \leqslant Kc^{-n},$$

which completes the proof (note that if all roots of $f(x)$ are distinct we can choose $c^{-1} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{k-1}|\}$ and $K = k-1$). □

Proof (*Proof of Theorem 1*). From the geometric series as $\epsilon \to 0$ we have

$$\frac{1}{1 \pm \epsilon} = 1 \pm \epsilon + O(\epsilon^2) = 1 + O(\epsilon).$$

Using Lemma 2, we have

$$\frac{1}{u_k} = \frac{1}{a\alpha^k + O(c^{-k})} = \frac{1}{a\alpha^k\left(1 + O((\alpha c)^{-k})\right)} = \frac{1}{a\alpha^k}\left(1 + O((\alpha c)^{-k})\right) = \frac{1}{a\alpha^k} + O\left((\alpha^2 c)^{-k}\right).$$

Thus

$$\sum_{k=n}^{\infty} \frac{1}{u_k} = \frac{1}{a}\sum_{k=n}^{\infty} \frac{1}{\alpha^k} + O\left(\sum_{k=n}^{\infty} (\alpha^2 c)^{-k}\right) = \frac{\alpha}{a(\alpha-1)}\alpha^{-n} + O\left((\alpha^2 c)^{-n}\right).$$

By taking reciprocal we get

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_k}\right)^{-1} = \frac{1}{\frac{\alpha}{a(\alpha-1)}\alpha^{-n} + O\left((\alpha^2 c)^{-n}\right)} = \frac{\alpha-1}{\alpha}a\alpha^n\left(1+O\left((\alpha c)^{-n}\right)\right) = \frac{\alpha-1}{\alpha}a\alpha^n + O(c^{-n}) = u_n - u_{n-1} + O(c^{-n}).$$

So there exists n_0 such that the last error term becomes less than 1/2 which completes the proof. □

Theorem 4. *Let $\{u_n(p,q,k)\}$, briefly $\{u_n\}$, be an order-k sequence defined by (3) with a restriction $p \geqslant q \geqslant 1$. Then there exists a positive integer n_1, such that*

$$\left\|\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{u_k}\right)^{-1}\right\| = (-1)^n(u_n + u_{n-1}), \quad (n \geqslant n_1).$$

Proof. Here we start to the proof with computing the summand term

$$\frac{(-1)^k}{u_k} = \frac{(-1)^k}{a\alpha^k + O(c^{-k})} = \frac{(-1)^k}{a\alpha^k}\left(1+O\left((\alpha c)^{-k}\right)\right).$$

Then we have

$$\sum_{k=n}^{\infty}\frac{(-1)^k}{a\alpha^k}\left(1+O\left((\alpha c)^{-k}\right)\right) = \frac{\alpha}{a(-\alpha)^n(\alpha+1)} + O\left((\alpha^2 c)^{-n}\right).$$

By taking reciprocal,

$$\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{u_k}\right)^{-1} = \frac{a(-\alpha)^n(\alpha+1)}{\alpha}\left(1+O\left((\alpha c)^{-n}\right)\right) = (-1)^n\left(a\alpha^n + a\alpha^{n-1}\right) + O(c^{-n}) = (-1)^n(u_n + u_{n-1}) + O(c^{-n}).$$

Then we can find integer n_1 such that the error term is less than 1/2 for $n \geqslant n_1$. □

The following result could be proven similar to the previous results.

Theorem 5. *For the sequence which defined in (3) with a restriction $p \geqslant q \geqslant 1$. Then there exist positive integers n_2 and n_3 such that*

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_{tk+r}}\right)^{-1}\right\| = (u_{tn+r} - u_{tn-t+r}), \quad (n \geqslant n_2),$$

$$\left\|\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{u_{tk+r}}\right)^{-1}\right\| = (-1)^n(u_{tn+r} + u_{tn-t+r}), \quad (n \geqslant n_3),$$

where t and r positive integers with $0 \leqslant r < t$.

Now we present some examples of our results. When $q = 1,\ t = 2,\ r = 0$ and $r = 1$ in the previous theorem, respectively, we get same results given in [3].

When we take $p = 2,\ q = 1,\ k = 2$, with initials $u_0 = 0$ and $u_1 = 1$, respectively, we have same result in [6]. In addition we have more results such as

$$\left\|\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{P_k}\right)^{-1}\right\| = (-1)^n(P_n + P_{n-1}), \quad (n \geqslant 1), \tag{7}$$

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{P_{tk+r}}\right)^{-1}\right\| = (P_{tn+r} - P_{tn-t+r}), \quad (n \geqslant n_0).$$

Identity (7) can be also found in [3].

When $p = q = 1,\ k = 2,\ t = 5$ and $r = 3$ with initials $u_0 = 0$ and $u_1 = 1$, we obtain new result as follows,

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{F_{5k+3}}\right)^{-1}\right\| = (F_{5n+3} - F_{5n-2}), \quad (n \geqslant 1),$$

$$\left\|\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{F_{5k+3}}\right)^{-1}\right\| = (-1)^n(F_{5n+3} + F_{5n-2}), \quad (n \geqslant 1).$$

For example, we consider the sequence $\{u_n\}$ defined for $n > 3$ by

$$u_n = 7u_{n-1} + 4u_{n-2} + u_{n-3} + u_{n-4},$$

with initials $u_0 = 0,\ u_1 = 1,\ u_2 = 2$ and $u_3 = 3$. Then, by Theorems 1 and 5, we obtain

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \quad (n \geqslant n_0),$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_{tk+r}} \right)^{-1} \right\| = (-1)^n (u_{tn+r} + u_{tn-t+r}), \quad (n \geqslant n_1),$$

where n_0 and n_1 are determined according to the initial values and the roots of characteristic equation of sequence $\{u_n\}$.

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