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SOME EXTENSIONS OF PROPERTIES OF THE SEQUENCE OF RECIPROCAL FIBONACCI POLYNOMIALS

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This paper is, in a sense, dual to the Fibonacci Association paper by J. R. Howell [4]. On the other hand, interest in the *reciprocal* Fibonacci-like polynomials is caused by the very effective propositions 7 and 4 of [3].

It is also the intention of this paper to draw the attention of the Fibonacci Association audience to the vast area of applications of its activities in the domain of computational techniques allowing one to perform quantitative comparisons among various data organizations in the framework defined by the authors of [3].

Let $W_n(x)$ be a polynomial in the variable x; $x \in (c,d) \subset \mathbf{R}$ and $\deg(W_n(x)) = N$. We define the reciprocal polynomial of $W_n(x)$ as follows.

Definition 1:
$$\overline{W}_n(x) = x^N W_n\left(\frac{1}{x}\right). \tag{1}$$

The purpose of this paper is to describe the reciprocal polynomials of Fibonacci-like polynomials that are defined by the recursion formula [4]

$$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x), \tag{2}$$

where a and b are real constants.

It is easy to verify that the reciprocal Fibonacci-like polynomials satisfy

$$\overline{g}_{n+2}(x) = a\overline{g}_{n+1}(x) + bx^2\overline{g}_n(x), \quad n > 2. \tag{3}$$

Indeed, if $\deg g_3(x) = m$, then

$$\deg g_n(x) = n-3+m, \quad \text{for } n > 2. \tag{4}$$

From (2),

$$x^{n+m-1}g_{n+2}\left(\frac{1}{x}\right) = ax^{n+m-2}g_{n+1}\left(\frac{1}{x}\right) + bx^{n+m-1}g_n\left(\frac{1}{x}\right).$$

Hence, (3) follows by (1) and (4). If $\deg g_2(x) \geq \deg g_1(x)$, then the recursion formula (3) is true for $n \geq 2$, and if $\deg g_2(x) = \deg g_1(x) + 1$, then (3) holds for each natural number n.

Theorem 1: Suppose that the sequence $\{\overline{g}_n(x)\}$ satisfies (3) for every natural number n. Then the following summation formula holds:

$$\sum_{j=1}^{p} \overline{g}_j(x) = \frac{\overline{g}_{p+1}(x) + bx^2\overline{g}_p(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a + bx^2 - 1}, \tag{5}$$

for each natural number p.

Proof: For $p = 1$, formula (5) is trivial. Let (5) hold for $p = k$, then

$$\begin{aligned}\sum_{j=1}^{k+1} \overline{g}_j(x) &= \frac{\overline{g}_{k+1}(x) + bx^2\overline{g}_k(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a + bx^2 - 1} + \overline{g}_{k+1}(x)\\ &= \frac{\overline{g}_{k+1}(x) + bx^2\overline{g}_k(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x) + a\overline{g}_{k+1}(x) + bx^2\overline{g}_{k+1}(x) - \overline{g}_{k+1}(x)}{a + bx^2 - 1}\\ &= \frac{\overline{g}_{k+2}(x) + bx^2\overline{g}_{k+1}(x) + (a-1)\overline{g}_1(x) - \overline{g}_2(x)}{a + bx^2 - 1}\end{aligned}$$

and the result follows by induction on p. □

The inverse of Theorem 1 is also true.

Proof: If the summation formula (5) holds for some sequence of polynomials (for each natural number p), then the identity

$$\overline{g}_{p+1}(x) = \sum_{j=1}^{p+1} \overline{g}_j(x) - \sum_{j=1}^{p} \overline{g}_j(x)$$

may be transformed easily into equation (3). □

For the remainder of this paper, we consider the sequence of Fibonacci-like polynomials $\{w_n(x)\}_{n=1}^{\infty}$ defined by recurrence (2), with initial values

$$w_1(x) = 1, \quad w_2(x) = ax. \tag{6}$$

If $a \neq 0$ and $b \neq 0$, then $w_p(x)$ can be written in the explicit form [4]:

$$w_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} (ax)^{p-1-2j} b^j. \tag{7}$$

The reciprocal polynomials of $w_p(x)$ are defined by recurrence (3) with the following initial conditions:

$$\overline{w}_1(x) = 1, \quad \overline{w}_2(x) = a. \tag{8}$$

We take $\overline{w}_n(x)$ and $\overline{w}_n$ to mean the same thing.

By a simple transformation of formula (7), we obtain the explicit form of $\overline{w}_p(x)$. From (7),

$$x^{p-1} w_p\left(\frac{1}{x}\right) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} x^{p-1} \left(\frac{a}{x}\right)^{p-1-2j} b^j.$$

Since $\deg w_n(x) = n-1$, we have

$$\overline{w}_p(x) = \sum_{j=0}^{[(p-1)/2]} \binom{p-1-j}{j} x^{2j} a^{p-1-2j} b^j. \tag{9}$$

Thus, $\overline{w}_p(x)$ is a polynomial of degree $2[(p-1)/2]$ with only even powers of x.

Theorem 2:
$$\overline{w}_p(x) = \frac{A^p(x) - B^p(x)}{A(x) - B(x)}, \tag{10}$$
where
$$A(x) = \frac{a + \sqrt{a^2 + 4bx^2}}{2} \quad \text{and} \quad B(x) = \frac{a - \sqrt{a^2 + 4bx^2}}{2}.$$

Proof: It is easy to verify that
$$A^2(x) = aA(x) + bx^2 \quad \text{and} \quad B^2(x) = aB(x) + bx^2.$$
Multiplying both sides of the above identities by A^{p-2} and B^{p-2}, respectively, we see that the sequences $, A, A^2, A^3, \ldots$ and $, B, B^2, B^3, \ldots$ satisfy (3). From these two facts, it follows that the recursion formula (3) holds also for the sequence $A - B, A^2 - B^2, A^3 - B^3, \ldots$ Since
$$\overline{w}_{n+2}(x)(A-B) = a\overline{w}_{n+1}(x)(A-B) + bx^2\overline{w}_n(x)(A-B),$$
the result follows from the identities
$$\overline{w}_1(x)(A-B) = A - B \quad \text{and} \quad \overline{w}_2(x)(A-B) = A^2 - B^2. \ \square$$

Theorem 3: Let $Q = \begin{pmatrix} a & 1 \\ bx^2 & 0 \end{pmatrix}$ and let the sequence $\{\overline{g}_n\}$ be defined by the recursion formula (3). Then, for every natural number p,
$$\begin{pmatrix} \overline{g}_{p+2} & \overline{g}_{p+1} \\ \overline{g}_{p+1} & \overline{g}_p \end{pmatrix} = \begin{pmatrix} \overline{g}_3 & \overline{g}_2 \\ \overline{g}_2 & \overline{g}_1 \end{pmatrix} Q^{p-1}. \tag{11}$$

The proof of this theorem may be realized by a simple induction argument [5]. Theorem 3 provides standard means of obtaining identities for the sequence of reciprocal Fibonacci-like polynomials [5].

For example, computing the determinants in identity (11) leads to
$$\overline{g}_{p+2}\overline{g}_p - \overline{g}_{p+1}^2 = (-b)^{p-1}x^{2p-2}(\overline{g}_3\overline{g}_1 - \overline{g}_2^2). \tag{12}$$
Now if we consider $\{\overline{w}_n\}$ with initial conditions (8), then from identities (11) and (12) we get:
$$\begin{pmatrix} \overline{w}_{p+2} & \overline{w}_{p+1} \\ \overline{w}_{p+1} & \overline{w}_p \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} Q^p; \tag{13}$$
$$\overline{w}_{p+2}\overline{w}_p - \overline{w}_{p+1}^2 = -(-b)^p x^{2p}. \tag{14}$$
Multiplying both sides of (13) on the left by $\begin{pmatrix} 1 & 0 \\ 0 & bx^2 \end{pmatrix}$ yields
$$\begin{pmatrix} \overline{w}_{p+2} & \overline{w}_{p+1} \\ bx^2\overline{w}_{p+1} & bx^2\overline{w}_p \end{pmatrix} = Q^{p+1}. \tag{15}$$

Let p and q denote natural numbers. Using (15) with Q^{p+q}, Q^p, and Q^q, one has
$$\begin{pmatrix} \overline{w}_{p+q+1} & \overline{w}_{p+q} \\ bx^2\overline{w}_{p+q} & bx^2\overline{w}_{p+q-1} \end{pmatrix} = \begin{pmatrix} \overline{w}_{p+1} & \overline{w}_p \\ bx^2\overline{w}_p & bx^2\overline{w}_{p-1} \end{pmatrix} \begin{pmatrix} \overline{w}_{q+1} & \overline{w}_q \\ bx^2\overline{w}_q & bx^2\overline{w}_{q-1} \end{pmatrix}.$$

If we compare the entries on both sides of the above identity, we obtain

$$\overline{w}_{p+q+1} = \overline{w}_{p+1}\overline{w}_{q+1} + bx^2\overline{w}_p\overline{w}_q. \tag{16}$$

Identity (16) is a special case of identity (7) in [5].

We shall now describe some of the divisibility properties of $\{\overline{w}_n\}$. If $a = 0$, then

$$\overline{w}_n(x) = \frac{1-(-1)^n}{2} b^{(n-1)/2} x^{n-1};$$

if $b = 0$, then

$$\overline{w}_n(x) = a^{n-1}.$$

In these cases, the investigation of divisibility properties of $\{\overline{w}_n(x)\}$ is easy. Suppose that a and b are nonzero numbers.

Theorem 4: Let $W(x)$ be a polynomial that divides both $\overline{w}_p$ and $\overline{w}_{p+1}$ for a fixed $p > 1$. Then $W(x)$ divides $\overline{w}_{p-1}$.

Proof: Suppose that $\overline{w}_p = W(x)S(x)$ and $\overline{w}_{p+1} = W(x)T(x)$, where $S(x)$ and $T(x)$ are certain polynomials. From (3), we have

$$x^2\overline{w}_{p-1}(x) = \frac{1}{b}W(x)(T(x) - aS(x)).$$

Thus, $W(x) | x^2\overline{w}_{p-1}$. From the fact that x^n does not divide $\overline{w}_n$ for any natural number n [see (9)], we conclude that x^2 and $\overline{w}_{p-1}$ are relatively prime. Finally, since $W(x)$ does not divide x^2, then $W(x) | \overline{w}_{p-1}$. □

Theorem 5: For natural numbers p and q, $\overline{w}_p | \overline{w}_{pq}$.

Proof: Let p be an arbitrary natural number. The fact that $\overline{w}_p | \overline{w}_p$ is trivial. If $\overline{w}_p | \overline{w}_{pk}$ for a certain k, then using formula (16) and the fact that $p(k+1) = (pk-1) + p + 1$, we obtain the following identity:

$$\overline{w}_{p(k+1)} = \overline{w}_{pk}\overline{w}_{p+1} + bx^2\overline{w}_{pk-1}\overline{w}_p.$$

Since $\overline{w}_p$ divides the right-hand side of the above identity, we have $\overline{w}_p | \overline{w}_{p(k+1)}$. This completes the proof of Theorem 5. □

We now consider some natural corollaries of Theorems 4 and 5.

Corollary 1: Let $W(x)$ be a polynomial that divides both $\overline{w}_p$ and $\overline{w}_{p+1}$ for a fixed $p > 1$. Then $W(x)$ is a constant.

Proof: Corollary 1 follows from Theorem 4 by induction. □

Corollary 2: If n, p, q, and r are natural numbers ($p > 1$) such that $q = np + r$ and if $\overline{w}_p | \overline{w}_q$, then $\overline{w}_p | \overline{w}_r$.

Proof: From $p > 1$, $np - 1 > 0$, $q = (np-1) + r + 1$ and formula (16), we have

$$\overline{w}_q = \overline{w}_{np}\overline{w}_{r+1} + bx^2\overline{w}_{np-1}\overline{w}_r.$$

Since $\overline{w}_p | \overline{w}_q$ and $\overline{w}_p | \overline{w}_{np}$, we have

$$\overline{w}_p \mid x^2 \overline{w}_{np-1} \overline{w}_r .$$

The greatest common divisor of $\overline{w}_{np}$ and $\overline{w}_{np-1}$ is a constant (Corollary 1), so the greatest common divisor of $\overline{w}_p$ and $\overline{w}_{np-1}$ is a constant. Thus, $\overline{w}_p \mid x^2 \overline{w}_r$. Now, reasoning as in the proof of Theorem 4 completes the proof of Corollary 2. □

Corollary 2 implies our final theorem which is analogous to Theorem 10 in [4].

Theorem 6: If p and q are natural numbers and $\overline{w}_p \mid \overline{w}_q$, then $p \mid q$.

THE MAIN REMARK

If we put $a = 1$ and $b = -1$ in (2) and (6), we obtain

$$w_1(x) = 1, \quad w_2(x) = x, \quad \text{and} \quad w_{n+2}(x) = x w_{n+1}(x) - w_n(x). \tag{17}$$

These are the well-known *Tchebycheff polynomials.* Then the reciprocal Tchebycheff polynomials do satisfy

$$\overline{w}_1(x) = 1, \overline{w}_2(x) = 1, \quad \text{and} \quad \overline{w}_{n+2}(x) = x^2 \overline{w}_n(x) - w_n(x).$$

From [3], it is known that these polynomials are associated with stacks.

Specifically, orthogonal Tchebycheff polynomials are used to calculate the numbers $H_{k,l,n}$ of histories of length n starting at level k and ending at level l, while the reciprocal Tchebycheff polynomials of degree h are used to derive generating functions for histories of height $\leq h$. In this context, the Tchebycheff polynomials are distinguished among the family of Fibonacci-like polynomials defined by (2) and (6), as only for that case (i.e., for $a = 1$ and $b = -1$) the Fibonacci-like polynomials associate with **standard** organizations [3]. This can be seen easily after consulting Theorem 4.1 of [3]. For other admissible values of a and b, the Fibonacci-like polynomials also provide us with an orthogonal polynomial system with respect to the corresponding positive-definite moment functional [3].

The resulting dynamical data organizations are then **nonstandard** ones. (A paper on nonstandard data organizations and Fibonacci-like polynomials is now in preparation.)

FINAL REMARKS

The following two attempts now seem to be natural. First, one may use the number-theoretic properties of Tchebycheff and reciprocal Tchebycheff polynomials developed in [4] and this paper to investigate further stacks in the framework created in [3]. Second, one may look for other data structure organizations relaxing the positive-definiteness of the moment functional. This might be valuable if we knew how to convey contiguous quantum-mechanics-like descriptions of dynamic data structures, which is one of several considerations in [2].

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REFERENCES

1. T. S. Chihara. *An Introduction to Orthogonal Polynomials*. Paris: Gordon and Breach Science Publishers, 1978.
2. Philip Feinsilver & René Schott. *Algebraic Structures and Operator Calculus*. Vol. II: *Special Functions and Computer Science*. London: Kluwer Academic Publishers, 1993.
3. P. Flajolet, J. Françon, & J. Vuillemin. "Sequence of Operations for Dynamic Data Structures." *Journal Algorithms* **1** (1980):111-14.
4. John R. Howell. "Some Extensions of Properties of the Sequence of Fibonacci Polynomials." *A Collection of Manuscripts Related to the Fibonacci Sequence–18th Anniversary Volume*. Ed. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. Santa Clara, CA: The Fibonacci Association, 1980.
5. M. E. Waddill. "Matrices and Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **12.4** (1974):381-86.

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