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Determinants and Recurrence Sequences

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Abstract

We examine relationships between two minors of order n of some matrices of n rows and $n+r$ columns. This is done through a class of determinants, here called n-determinants, the investigation of which is our objective.

As a consequence of our main result we obtain a generalization of theorem of the product of two determinants.

We show the upper Hessenberg determinants, with -1 on the subdiagonal, belong to our class. Using such determinants allow us to represent terms of various recurrence sequences in the form of determinants. We illustrate this with several examples. In particular, we state a few determinants, each of which equals a Fibonacci number.

Also, several relationships among terms of sequences defined by the same recurrence equation are derived.

1 Introduction

In the second section of this paper we define n-determinants and derive a relationship between n-determinants and recurrence sequences. We apply the result on a particular kind of n-determinants to obtain an extension of the theorem of the product of two determinants.

In Section 3 we consider 1-determinants, which appear to be the upper Hessenberg determinants. Some important mathematical objects may be represented as 1-determinants. This is found to be the case for the Catalan numbers, the Bell numbers, the Fibonacci numbers, the Fibonacci polynomials, the generalized Fibonacci numbers, the Tchebychev polynomials of both kinds, the continuants, the derangements, the factorials and the terms of any homogenous linear recurrence equation. We also find several 1-determinants, each of which equals a Fibonacci number.

The case $n = 2$ is examined in Section 4. We show that, in a particular case, 2-determinants produce relationships between two sequences given by the same recurrence equation, with possibly different initial conditions. In this sense, we prove a formula for the Fibonacci polynomials from which several well-known formulas follow. For example, this is the case with the Ocagne's formula and the index reduction formula. Analogous formulas for the Tchebychev polynomials are then stated. Also, we derive a result for the continuants, generalizing the fundamental theorem of convergents. Another result generalizes the standard recurrence equation for the derangements.

In Section 5, we consider 3-determinants which connect terms of three sequences given by the same recurrence equation.

The obtained results are concerned with several sequences from [5].

2 n-determinants

In this section we define n-determinants and give its relationships with recurrence sequences. We also consider a particular class of n-determinants which leads to a generalization of the theorem of the product of determinants. Note that entries of the considered matrices may belong to arbitrary commutative ring.

Let n and r be positive integers. We consider the following $n+r-1$ by r matrix

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,r-1} & p_{1,r} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,r-1} & p_{2,r} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,r-1} & p_{n,r} \\ -1 & p_{n+1,2} & \cdots & p_{n+1,r-1} & p_{n+1,r} \\ 0 & -1 & \cdots & p_{n+2,r-1} & p_{n+2,r} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n+r-2,r-1} & p_{n+r-2,r} \\ 0 & 0 & \cdots & -1 & p_{n+r-1,r} \end{pmatrix}. \tag{1}$$

Note that entries $(n+i,i),\ (i = 1,\ldots,r-1)$ in P equal -1.

Definition 1. If Q is a submatrix of order r of P, we say that $\det Q$ is an n-determinant.

We connect matrix (1) with a recursively given sequence of vector-columns in the following way: Let A be a square matrix of order n. We define a block matrix $A_r = [A|A_{n+1}|\cdots|A_{n+r}]$ of n rows and $n+r$ columns (where $A_{n+j},\ (j = 1,2,\ldots,r)$ are vector -columns) as follows:

$$A_{n+j} = \sum_{i=1}^{n+j-1} p_{i,j}A_i,\ (j = 1,2,\ldots,r). \tag{2}$$

We shall establish a relationship of n-determinants and minors of order n of A_r.

For a sequence $1 \le j_1 < j_2 < \cdots < j_r < n+r$ of positive integers, we let $M = M(\widehat{j_1},\widehat{j_2},\ldots,\widehat{j_r})$ denote the minor of A_r of order n, obtained by deleting columns $j_1, j_2, \ldots, j_r$

of A_r. We shall also write $M(\widehat{j_1},\dots,\widehat{j_r},A_{i_1},A_{i_2},\dots,A_{i_k})$ if we want to stress that M contains $i_1,i_2,\dots,i_k$ columns of A_r.

Note that the last column of A_r cannot be deleted.

The sign $\sigma(M)$ of M is defined as

$$\sigma(M)=(-1)^{nr+j_1+j_2+\cdots+j_r+\frac{(r-1)r}{2}}.$$

We let $Q=Q(j_1,\dots,j_r)$ denote the submatrix of order r, laying in $j_1,j_2,\dots,j_r$ rows of P. Hence, $\det Q$ is an n-determinant.

We now prove the following result.

Theorem 2. *Let $1\le j_1<\cdots<j_r<r+n$ be a sequence of positive integers. Then,*

$$M(\widehat{j_1},\dots,\widehat{j_r})=\sigma(M)\cdot\det Q\cdot\det A. \tag{3}$$

Proof. The proof is by induction on r. For $r=1$, we have $1\le j_1\le n$, since the case $j_1>r$ makes no sense. It follows that $M=M(\widehat{j_1})$. Taking $i=1$ in (2), we obtain

$$A_{n+1}=\sum_{m=1}^{n}p_{m,1}A_m.$$

Hence, we obtain $M(\widehat{j_1})$ as a sum of n terms. It is clear that this sum reduces to a single term, in which $m=j_1$. We conclude that

$$M(\widehat{j_1})=p_{j_1,1}M(\widehat{j_1},A_{j_1}),$$

where $M(\widehat{j_1},A_{j_1})$ denotes the minor of order n of A_r, in which the j_1th column is shifted at the nth place. In $M(\widehat{j_1},A_{j_1})$, we interchange the last column with the preceding one and repeat this until the j_1th column takes the j_1th place. For this, we need $n-j_1$ interchanges. It follows that

$$M(\widehat{j_1})=(-1)^{n+j_1}p_{j_1,1}\cdot\det A.$$

On the other hand, we obviously have $\det Q=p_{j_1,1}$, and $\sigma(M)=(-1)^{n+j_1}$, which proves the theorem, for $r=1$.

Assume that the theorem is true for $1\le k<r$. The last column of the minor $M(\widehat{j_1},\dots,\widehat{j_r})$ is column $n+r$ of A_r. The condition (2) implies

$$M(\widehat{j_1},\dots,\widehat{j_r})=\sum_{m=1}^{n+r-1}p_{m,r}M(\widehat{j_1},\dots,\widehat{j_r},A_m).$$

Note that the column A_m is the last column in each minor on the right-hand side of the preceding equation. Hence, in the sum on the right-hand side only terms obtained for $j_m\in\{j_1,\dots,j_r\}$ remain. They are of the form:

$$S(t)=p_{j_t,r}M(\widehat{j_1},\dots,\widehat{j_t},\dots\widehat{j_r},A_{j_t}),\ (t=1,2,\dots,r),$$

that is,

$$S(t)=(-1)^{n+r-1-j_t}p_{j_t,r}M(\widehat{j_1},\dots,A_{j_t},\dots\widehat{j_r}).$$

Applying the induction hypothesis yields

$$S(t) = (-1)^{n+t-1-j_t}\sigma(M(\widehat{j_1}, \dots, A_{j_t}, \dots \widehat{j_r}))p_{j_t,r}Q(j_1, \dots, \widehat{j_t}, \dots, j_r) \cdot \det A.$$

By a simple calculation, we obtain

$$(-1)^{n+t-1-j_t}\sigma(M(\widehat{j_1}, \dots, A_{j_t}, \dots \widehat{j_r})) = (-1)^{r+t}\sigma(M),$$

hence,

$$S(t) = \sigma(M)(-1)^{r+t}Q(j_1, \dots, \widehat{j_t}, \dots, j_r) \cdot \det A.$$

Summing over all t gives

$$\sum_{t=1}^{r} S(t) = \sigma(M)\left[\sum_{t=1}^{r}(-1)^{r+t}p_{j_t,r}Q(j_1, \dots, \widehat{j_t}, \dots, j_r)\right] \cdot \det A.$$

The expression in the square brackets is the expansion of $\det Q$ by elements of the last column, and the theorem is proved. $\square$

We now investigate a particular class of n-determinants, arising from an $n+r-1$ by r block-matrix P of the form

$$P = \left[\frac{S}{T}\right], \tag{4}$$

where

$$S = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,r-1} & p_{1,r} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,r-1} & p_{2,r} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,r-1} & p_{n,r} \end{pmatrix}, \; T = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

Proposition 3. *Each n-determinant of the matrix (4) equals, up to the sign, a minor of S.*

Proof. We obtain an n-determinant by deleting $n-1$ rows of P. It follows that each n-determinant must contain at least one row of S. Let $1 \le j_1 < j_2 < \cdots < j_k \le n < j_{k+1} < \cdots < j_r < n+r$, $(k \ge 1)$ be arbitrary, and consider the submatrix Q laying in $j_1, j_2, \dots, j_r$ rows of P. It has the following form

$$Q = \left(\frac{Q_1}{Q_2}\right),$$

where Q_1 lies in rows $j_1, j_2, \dots, j_k$ of S, and Q_2 lies in rows $j_{k+1}-n, \dots, j_r-n$ of T. If $k = r$, then $Q = Q_1$, therefore $\det Q$ is a minor of S.

Thus, we may assume that $k < r$. We calculate $\det Q$ by expansion across the rows of Q_2. The matrix Q_2 has a unique nonzero minor D of order $r-k$, which lies in $k+1, \dots, r$ rows of Q. Its value obviously equals $(-1)^{r-k}$. The sum of the indices of rows of Q, in which D lies, is

$$(k+1) + (k+2) + \cdots + r = \frac{(r-k)(r+k+1)}{2}.$$

The sum of indices of columns in which D lies equals

$$(j_{k+1} - n) + \cdots + (j_r - n) = j_{k+1} + \cdots + j_r - (r - k)n.$$

We thus obtain that

$$\det Q = (-1)^\tau \det Q_4,$$

where $\tau = \frac{(r-k)(k+r+3-2n)}{2} + j_{k+1} + \cdots + j_r$, and $\det Q_4$ is the complement minor of D, laying in columns different from $j_{k+1}, \ldots, j_r$. Clearly, $\det Q_4$ is a minor of S, and the proposition is true. □

As a consequence of Theorem (3) we obtain

Proposition 4. *Let A be arbitrary matrix of order n, let B be arbitrary n by r matrix. Consider the block matrix $A_r = [A|AB]$. Let $1 \leq j_1 < j_2 < \cdots < j_k \leq n < j_{k+1} < \cdots < j_r < n + r$, $(k \geq 1)$ be arbitrary. If $M(\widehat{j_1}, \ldots, \widehat{j_r})$ has the same meaning as before, then*

$$(-1)^\sigma M = \det A \cdot \det Q_4, \tag{5}$$

where $\sigma = \frac{k^2-2kn+k}{2} + j_1 + j_2 + \cdots + j_k$, and the submatrix Q_4 lies in $j_1, j_2, \ldots, j_k$ rows, and in columns different from $j_{k+1}, j_{k+2}, \ldots, j_r$ of B.

Proof. It is easy to see that the matrix A_r, corresponding to the matrix (4), has the form

$$A_r = [A|AB].$$

After a simple calculation we obtain $\sigma = \frac{k^2-2kn+k}{2} + j_1 + j_2 + \cdots + j_k$, and the assertion follows from the preceding proposition. □

Remark 5. Note that equation (5) is a generalization of the theorem of the product of two determinants.

Namely, in the case $r = n = k$, and $j_i = i,\ (i = 1, 2, \ldots, n)$, we have $\det Q = \det B,\ M = \det AB$, and $\sigma(M) = 1$, so that equation (5) becomes

$$\det(AB) = \det A \cdot \det B.$$

3 1-determinants

In this case, A is a matrix of order 1, that is, a single element. We also have $j_1 = 1, j_2 = 2, \ldots, j_r = r$, hence, the minor $M(\widehat{j_1}, \widehat{j_2}, \ldots, \widehat{j_r})$ must be of the form: $M(\hat{1}, \hat{2}, \ldots, \hat{r})$. We easily obtain that $\sigma(M) = 1$. The matrix P is as follows:

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,r-1} & p_{1,r} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,r-1} & p_{2,r} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,r-1} & p_{3,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{r-1,r-1} & p_{r-1,r} \\ 0 & 0 & 0 & \cdots & -1 & p_{r,r} \end{pmatrix}. \tag{6}$$

We see that $Q = P$. Therefore, each 1-determinant is an upper Hessenberg determinant. Applying Theorem 2, we obtain the following result.

Proposition 6. *Let $a_1, a_2, \ldots$ be a sequence such that*

$$a_{1+r} = \sum_{i=1}^{r} p_{i,r} a_i. \tag{7}$$

Then,

$$a_{r+1} = a_1 \det P.$$

This result is known. For instance, it follows from Theorem 4.20,[1].

Remark 7. In the paper [3], we found a combinatorial interpretation for the coefficients of the characteristic polynomials of some 1-determinants.

We give a number of examples for sequences given by the formula (7). Some of them are well-known. In all examples, we take $a_1 = 1$.

1° **Catalan numbers** (A000108). We let C_n denote the nth Catalan number. If we take $p_{i,j} = C_{j-i}$, then equation (7) becomes

$$a_{1+r} = \sum_{i=1}^{r} C_{r-i} a_i.$$

The Segner's recurrence formula for Catalan numbers implies that $a_{r+1} = C_r$. Hence, a way to write the Segner's formula in terms of determinants is

$$C_r = \begin{vmatrix} C_0 & C_1 & C_2 & \cdots & C_{r-2} & C_{r-1} \\ -1 & C_0 & C_1 & \cdots & C_{r-3} & C_{r-1} \\ 0 & -1 & C_0 & \cdots & C_{r-4} & C_{r-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_0 & C_1 \\ 0 & 0 & 0 & \cdots & -1 & C_0 \end{vmatrix}.$$

2° **Bell numbers** (A000110). If one takes $p_{i,j} = \binom{j-1}{i-1}$ in (6), then (7) becomes the recursion for the Bell numbers. Thus, a determinantal expression for the Bell number B_r is

$$B_r = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{r-2}{0} & \binom{r-1}{0} \\ -1 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{r-2}{1} & \binom{r-1}{1} \\ 0 & -1 & \binom{2}{2} & \cdots & \binom{r-2}{2} & \binom{r-1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{r-2}{r-2} & \binom{r-1}{r-2} \\ 0 & 0 & 0 & \cdots & -1 & \binom{r-1}{r-1} \end{vmatrix}.$$

The order of the determinant equals r.

3° **Eigensequences for Stirling numbers** (A003659). If $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind, and $p_{i,j} = \left\{ {j-1 \atop i-1} \right\}$ in (6), then (7) becomes the recursion for the so-called eigensequence $(E_1, E_2, \ldots)$ of the Stirling number of the second kind. Therefore,

$$E_r = \begin{vmatrix} \left\{ {0 \atop 0} \right\} & \left\{ {1 \atop 0} \right\} & \left\{ {2 \atop 0} \right\} & \cdots & \left\{ {r-2 \atop 0} \right\} & \left\{ {r-1 \atop 0} \right\} \\ -1 & \left\{ {1 \atop 1} \right\} & \left\{ {2 \atop 1} \right\} & \cdots & \left\{ {r-2 \atop 1} \right\} & \left\{ {r-1 \atop 1} \right\} \\ 0 & -1 & \left\{ {2 \atop 2} \right\} & \cdots & \left\{ {r-2 \atop 2} \right\} & \left\{ {r-1 \atop 2} \right\} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \left\{ {r-2 \atop r-2} \right\} & \left\{ {r-1 \atop r-2} \right\} \\ 0 & 0 & 0 & \cdots & -1 & \left\{ {r-1 \atop r-1} \right\} \end{vmatrix}.$$

Note that analogous identity holds for the unsigned Stirling numbers of the first kind (A143805).

4° **Factorials** (A000142). Let k be a positive integer. Consider the following 1-determinant D of order $r > 1$:

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & k & k & k & \cdots & k & k \\ 0 & -1 & k+1 & k+1 & \cdots & k+1 & k+1 \\ 0 & 0 & -1 & k+2 & \cdots & k+2 & k+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & k+r-3 & k+r-3 \\ 0 & 0 & 0 & \cdots & 0 & -1 & k+r-2 \end{vmatrix}.$$

In this case, the formula (7) becomes

$$a_{r+1} = 1 + \sum_{i=2}^{r} (k+i-2)a_i.$$

Subtracting the equation

$$a_{r+2} = 1 + \sum_{i=2}^{r+1} (k+i-2)a_i$$

from the preceding easily yields

$$a_{r+2} = (k+r)a_{r+1},$$

which is the recursion for the falling factorials. Hence,

$$D = \frac{(k+r-1)!}{k!}.$$

5° **Derangements** (A000166). We let D_r denote the number of derangements of r. The recurrence equation for the derangements is

$$D_2 = 1, D_3 = 2,\ D_r = (r-1)(D_{r-2} + D_{r-1}),\ (r \geq 4).$$

Hence,

$$D_{r+1} = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 3 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & r-1 & r \\ 0 & 0 & 0 & \cdots & 0 & -1 & r \end{vmatrix}.$$

6° **Fibonacci polynomials**. In this case, we consider the recurrence equation

$$a_1 = 1, a_2 = x,\ a_{k+1} = a_{k-1} + xa_k,\ (k \geq 2)$$

for Fibonacci polynomials. Hence, for Fibonacci polynomial $F_{r+1}(x)$ we have

$$F_{r+1}(x) = \begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ -1 & x & 1 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}.$$

The order of the determinant equals r. Taking $x = 1,$, in particular, we obtain the well-known determinantal expression for Fibonacci numbers.

7° **Tchebychev polynomials of the first kind**. The recurrence relation for the Tchebychev polynomials of the first kind is

$$T_0(x) = 1,\ T_1(x) = x,\ T_k(x) = -T_{k-2}(x) + 2xT_{k-1}(x),\ (k > 2).$$

Theorem 6 now implies the following equation:

$$T_r(x) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}.$$

The order of the determinant is r. A similar formula holds for Tchebychev polynomials $U_r(x)$ of the second kind.

8° **Hermite polynomials.** For the Hermite polynomials $H_r(x)$, we have the following recurrence equation:

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_{r+1}(x) = -2rH_{r-1}(x) + 2xH_r(x),\ (r \geq 2).$$

Applying Theorem 6, we obtain the following expression:

$$H_r(x) = \begin{vmatrix} 2x & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -4 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -2(r-1) \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}.$$

9° **Continuants**. Take in (7) $p_{k,k} = p_k,\ p_{k-1,k} = 1$, and $p_{i,j} = 0$ otherwise. We obtain the recursion:

$$a_1 = 1, p_1,\ a_{1+k} = a_{k-1} + p_k a_k,\ (k = 2, \ldots).$$

The terms of this sequence are the continuants, and are denoted by $(p_1, p_2, \ldots, p_r)$. We thus obtain the following well-known formula:

$$(p_1, p_2, \ldots, p_r) = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & p_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{r-1} & 1 \\ 0 & 0 & 0 & \cdots & -1 & p_r \end{vmatrix}. \tag{8}$$

10° **Linear homogenous recurrence equation**. Let $b_1, b_2, \ldots, b_k$ be given elements. Consider the sequence $1, a_2, a_3, \ldots$ defined as follows:

$$a_2 = b_1, \ldots, a_{k+1} = b_k,\ a_{r+1} = \sum_{i=r-k+1}^{r} p_{i,r} a_i,\ (r > k).$$

We thus have a linear homogenous recurrence equation of order k. From Theorem 2 follows

$$a_{r+1} = \begin{vmatrix} b_1 & b_2 & \cdots & b_k & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & \cdots & 0 & p_{k+1,1} & 0 & \cdots & \cdots & 0 \\ 0 & -1 & \cdots & 0 & p_{k+1,2} & p_{k+2,2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & p_{k+1,k-1} & p_{k+2,k-1} & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & -1 & p_{k+1,k} & p_{k+2,k} & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \vdots & \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \vdots & \vdots & \cdots & -1 & p_{k,r} \end{vmatrix}.$$

11° **Generalized Fibonacci numbers**. Taking in the preceding formula that each $p_{i,j}$ equals 1, we obtain k-step Fibonacci numbers dependent on the initial conditions. The standard k-step Fibonacci numbers $F^{(k)}_{n+k}$ are obtained for $b_1 = b_2 = \cdots = b_{k-1} = 0,\ b_k = 1$. We thus have

$$F^{(k)}_{r+k} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \cdots & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & -1 & 1 \end{vmatrix},$$

where the size of the determinant is $r+k$.

12° **Fibonacci numbers** (A000045) Consider the sequence given by

$$a_1 = 1,\ a_2 = 1,\ a_r = \sum_{i=1}^{r-2} a_i,\ (r > 2).$$

This, in fact, is a recursion for the Fibonacci numbers. To show this, we first replace r by $r+1$ to obtain

$$a_{r+1} = \sum_{i=1}^{r-1} a_i.$$

Subtracting two last equations yields

$$a_{r+1} = a_r + a_{r-1},$$

which is the standard recursion for the Fibonacci numbers.

Proposition 6 implies

$$F_{r-1} = \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 1 & \cdots & 1 & 1 \\ 0 & -1 & \cdots & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & \cdots & -1 & 0 \end{vmatrix}.$$

The order of the determinant equals r.

13° **Fibonacci numbers.** We define a matrix $Q_r = (q_{ij})$ of order r as follows:

$$q_{ij} = \begin{cases} -1, & \text{if } i = j+1; \\ i+j+1 \mod 2, & \text{if } i \le j; \\ 0, & \text{otherwise.} \end{cases}$$

We find the recursion which, as in Proposition 6, produces this matrix. Obviously, $a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 2$, and

$$a_{2r} = a_1 + \cdots + a_{2r-1}.$$

Also,

$$a_{2r+2} = a_1 + \cdots + a_{2r-1} + a_{2r+1}.$$

Subtracting two last equation yields $a_{2r+2} = a_{2r+1} + a_{2r}$. Similarly, $a_{2r+1} = a_{2r} + a_{2r-1}$. The recursion for the Fibonacci numbers is thus obtained. It follows that $F_{r-1} = \det Q_r,\ (r > 1)$.

14° **Fibonacci numbers with odd indices** (A001519). Define a matrix $Q_r = (q_{ij})$ of order r as follows:

$$q_{ij} = \begin{cases} -1, & \text{if } i = j+1, \\ 2, & \text{if } i = j, \\ 1, & \text{if } i < j, \\ 0, & \text{otherwise.} \end{cases}.$$

In this case, we have the recursion $a_1 = 1,\ a_2 = 2,\ a_3 = 5,\ a_{r+1} = \sum_{i=1}^{r-1} +2a_r,\ (r \geq 2)$. From this, we easily obtain the recursion

$$a_{r+2} = 3a_{r+1} - a_r.$$

The identity 7, proved in [2], shows that we have a recursion for the Fibonacci numbers with odd indices. It follows that $F_{2r+1} = \det Q_r$.

Note that we described in [3] a connection of this determinant with a particular kind of composition of natural numbers.

15° **Fibonacci numbers with even indices** (A001906). For the matrix

$$Q_r = \begin{vmatrix} 1 & 2 & 3 & \cdots & r-1 & r \\ -1 & 1 & 2 & \cdots & r-2 & r-1 \\ 0 & -1 & 1 & \cdots & r-3 & r-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix},$$

the corresponding recursion has the form:

$$a_1 = 1,\ a_2 = 3,\ a_{1+r} = \sum_{i=1}^{r} (r-i+1)a_i,\ (r \geq 2).$$

Also,

$$a_{2+r} = \sum_{i=1}^{r+1} (r-i+1)a_i + \sum_{i=1}^{r+1} a_i.$$

Subtracting this equation from the preceding one, we obtain

$$a_{2+r} - a_{1+r} = \sum_{i=1}^{r+1} a_i.$$

In the same, way we obtain

$$a_{3+r} - a_{2+r} = \sum_{i=1}^{r+2} a_i.$$

Again, we subtract this equation from the preceding one to obtain

$$a_{3+r} = 3a_{2+r} - a_{1+r}.$$

This is a recursion for Fibonacci numbers, by Identity 7 in [2]. Taking into count the initial conditions, we have $\det Q_r = F_{2r}$.

4 2-determinants

In this section we consider the case $n = 2$. We first investigate the case that $p_{i,j} = 0,\ (j > i)$. Then, P has at most three nonzero diagonals, and may be written in the form:

$$P = \begin{vmatrix} b_1 & 0 & 0 & \cdots & 0 & 0 \\ c_2 & b_2 & 0 & \cdots & 0 & 0 \\ -1 & c_3 & b_3 & \cdots & 0 & 0 \\ 0 & -1 & c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & \cdots & c_r & b_r \\ 0 & 0 & \vdots & \cdots & -1 & c_{r+1} \end{vmatrix}. \tag{9}$$

The corresponding 2-determinant $\det Q$ is a lower triangular block determinant of the form:

$$\left|\begin{array}{c|c} Q_{11} & 0 \\ \hline Q_{12} & Q_{22} \end{array}\right|. \tag{10}$$

Here, Q_{11} is a lower triangular determinant lying in the first k rows and the first k columns of Q. It follows that $Q_{11} = b_1 \cdots b_k$. The order of Q_{22} is $r - k$ and it is of the same form as the determinant of the matrix P in (6).

As a consequence of Theorem 2, we obtain

Proposition 8. *Let* $(b_1, b_2, \ldots),\ (c_2, c_3, \ldots)$ *be any two sequences. Let*

$$(a_1^{(i)}, a_2^{(i)}, \ldots)\ (i = 1, 2)$$

be two sequences defined by the same recurrence equation of the second order:

$$a_r^{(i)} = b_{r-2}a_{r-2}^{(i)} + c_{r-1}a_{r-1}^{(i)},\ (r > 2),\ (i = 1, 2)..$$

Then,

$$\begin{vmatrix} a_{k+1}^{(1)} & a_{r+2}^{(1)} \\ a_{k+1}^{(2)} & a_{r+2}^{(2)} \end{vmatrix} = (-1)^k b_1 \cdots b_k \cdot d_{r-k+1} \cdot \begin{vmatrix} a_1^{(1)} & a_2^{(1)} \\ a_1^{(2)} & a_2^{(2)} \end{vmatrix},$$

where

$$d_1 = 1,\ d_2 = c_{k+2},\ d_i = b_{k+i-1}d_{i-2} + c_{k+i}d_{i-1},\ (i > 2)..$$

We illustrate the preceding proposition with some examples.

1° **Fibonacci polynomials**. Take $x_1^{(1)} = F_u(x),\ x_2^{(1)} = F_{u+1}(x),\ x_1^{(2)} = F_v(x),\ x_2^{(2)} = F_{v+1}(x),\ b_i = 1,\ c_{i+1} = x,\ (i = 1, 2, \ldots)$.

Note that, in this case, the 2-determinant equals the Fibonacci polynomial $F_{r-k}(x)$. From Proposition 8, we obtain the following identity:

$$\begin{vmatrix} F_{u+k}(x) & F_{u+r}(x) \\ F_{v+k}(x) & F_{v+r}(x) \end{vmatrix} = (-1)^k F_{r-k}(x) \cdot \begin{vmatrix} F_u(x) & F_{u+1}(x) \\ F_v(x) & F_{v+1}(x) \end{vmatrix}. \tag{11}$$

Several well-known formulas may be obtained from this.

Taking $u = 1, v = 0$ yields

$$F_{k+1}(x)F_r(x) - F_k(x)F_{r+1}(x) = (-1)^k F_{r-k}(x),$$

which is the Ocagne's identity for the Fibonacci polynomials. Applying this identity on the right-hand side of (11), we obtain

$$\begin{vmatrix} F_{u+m}(x) & F_{u+r}(x) \\ F_{v+m} & F_{v+r}(x) \end{vmatrix} = (-1)^{m+u+1} F_{r-m}(x)F_{v-u}(x).$$

We now may easily derive the index-reduction formula for the Fibonacci polynomials. Namely, replacing m by $m - t$ and r by $r - t$, we get

$$\begin{vmatrix} F_{u+k-t}(x) & F_{u+r-t}(x) \\ F_{v+k-t}(x) & F_{v+r-t}(x) \end{vmatrix} = (-1)^{k-t+u+1} F_{r-k}(x)F_{v-u}(x).$$

Comparing the last two equations produces

$$\begin{vmatrix} F_{u+k-t}(x) & F_{u+r-t}(x) \\ F_{v+k-t}(x) & F_{v+r-t}(x) \end{vmatrix} = (-1)^t \begin{vmatrix} F_{u+k}(x) & F_{u+r}(x) \\ F_{v+k}(x) & F_{v+r}(x) \end{vmatrix},$$

which is the index reduction formula.

Note that such a formula for the Fibonacci numbers is proved in [4].

2° **Fibonacci and Lucas polynomials**. The Lucas polynomials $L_r(x)$ satisfy the same recurrence relation as do the Fibonacci polynomials with different initial conditions. In this case, also, the 1-determinant equals a Fibonacci polynomial. We state two equations, one for mixed Lucas and Fibonacci polynomials, another for Lucas polynomials:

$$\begin{vmatrix} F_{u+k}(x) & F_{u+r}(x) \\ L_{v+k}(x) & L_{v+r}(x) \end{vmatrix} = (-1)^k F_{r-k}(x) \cdot \begin{vmatrix} F_u(x) & F_{u+1}(x) \\ L_v(x) & L_{v+1}(x) \end{vmatrix},$$

and

$$\begin{vmatrix} L_{u+k}(x) & L_{u+r}(x) \\ L_{v+k}(x) & L_{v+r}(x) \end{vmatrix} = (-1)^k F_{r-k}(x) \cdot \begin{vmatrix} L_u(x) & L_{u+1}(x) \\ L_v(x) & L_{v+1}(x) \end{vmatrix}.$$

3° **Tchebychev polynomials.** Tchebychev polynomials of the first and second kind also satisfy the same recursion with different initial conditions. Here, the 1-determinant equals a Tchebychev polynomial of the second kind. We state the following three identities, which are a consequence of Proposition 8.

$$\begin{vmatrix} U_{u+k}(x) & U_{u+r}(x) \\ U_{v+k}(x) & U_{v+r}(x) \end{vmatrix} = U_{r-k-1}(x) \begin{vmatrix} U_u(x) & U_{u+1}(x) \\ U_v(x) & U_{v+1}(x) \end{vmatrix},$$

$$\begin{vmatrix} T_{u+k}(x) & T_{u+r}(x) \\ T_{v+k}(x) & T_{v+r}(x) \end{vmatrix} = U_{r-k-1}(x) \begin{vmatrix} T_u(x) & T_{u+1}(x) \\ T_v(x) & T_{v+1}(x) \end{vmatrix},$$

$$\begin{vmatrix} U_{u+k}(x) & U_{u+r}(x) \\ T_{v+k}(x) & T_{v+r}(x) \end{vmatrix} = U_{r-k-1}(x) \begin{vmatrix} U_u(x) & U_{u+1}(x) \\ T_v(x) & T_{v+1}(x) \end{vmatrix}.$$

4° **Continued fractions**. Up until now, the division was not used. We might therefore assume that the elements of the concerned sequences belong to any commutative ring with 1. In this part, we suppose that they are positive real numbers. Let A_2 be the identity matrix of order 2, and let $(c_1, c_2, \ldots)$ be an arbitrary sequence of positive real numbers. Form the matrix A_r by the following rule:

$$A_{2+k} = A_k + c_k A_{k+1},\ (k = 1, 2, \ldots, r).$$

It is easy to see that A_r has the form:

$$A_r = \begin{pmatrix} 1 & 0 & 1 & c_2 & \ldots & (c_2, c_3, \ldots, c_r) \\ 0 & 1 & c_1 & (c_1, c_2) & \ldots & (c_1, c_2, c_3, \ldots, c_r) \end{pmatrix},$$

where $(c_m, c_{m+1}, \ldots, c_p)$ are the continuants. The 1-determinant equals the continuant $(c_{k+2}, c_{k+3}, \ldots, c_r)$.

The fundamental recurrence relation for the continued fractions gives an expression for the difference between two consecutive convergents. Equation 8 allows us to derive a formula for the difference between two arbitrary convergents. The following formula holds:

$$\begin{vmatrix} (c_2, c_3, \ldots, c_k) & (c_2, c_3, \ldots, c_r) \\ (c_1, c_2, c_3, \ldots, c_k) & (c_1, c_2, c_3, \ldots, c_r) \end{vmatrix} = (-1)^{k+1}(c_{k+2}, c_{k+3}, \ldots, c_r),$$

or, equivalently,

$$\frac{(c_1, c_2, c_3, \ldots, c_r)}{(c_2, c_3, \ldots, c_r)} - \frac{(c_1, c_2, c_3, \ldots, c_k)}{(c_2, c_3, \ldots, c_k)} = (-1)^{k+1} \frac{(c_{k+2}, c_{k+3}, \ldots, c_r)}{(c_2, c_3, \ldots, c_k) \cdot (c_2, c_3, \ldots, c_r)},$$

with the convention that for $r = k + 1$ the expression $(c_{k+2}, c_{k+3}, \ldots, c_r)$ equals 1.

If $r < k + 1$, then the proof follows from Proposition 8. If $r = k + 1$, then we take $(c_{k+2}, c_{k+3}, \ldots, c_m) = 1$, as then there is no matrix Q_{22}. Hence, our formula becomes the continued fraction fundamental recurrence relation.

5° **Derangements.** Take $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let the matrix A_r be formed by the recursion

$$A_{2+r} = r(A_{1+r} + A_r), \ (r \geq 1).$$

It is obvious that the rth element of the first row of A_r equals D_{r-1}. Also, the rth term of the second row of A_r equals $(r-1)!$. It follows that

$$M(k+1, r+1) = \begin{vmatrix} D_k & D_r \\ k! & r! \end{vmatrix}.$$

We thus obtain the following identity:

$$\begin{vmatrix} D_k & D_r \\ k! & r! \end{vmatrix} = (-1)^k k! \begin{vmatrix} k+1 & k+2 & 0 & \cdots & 0 & 0 \\ -1 & k+2 & k+3 & \cdots & 0 & 0 \\ 0 & -1 & k+3 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & r-1 & r \\ 0 & 0 & 0 & \cdots & -1 & r \end{vmatrix}.$$

In particular, for $r = k+1$, we have the standard recursion $D_{k+1} = kD_k + (-1)^k$ for the derangements.

The preceding identities are derived from homogenous recurrence equations of order two. We shall now consider the case of a recurrence equation of order three.

Theorem 9. *Let r be a positive integer, and let $(a_i), (b_{i+1}), (c_{i+2}), \ (i = 1, 2, \ldots)$ be arbitrary sequences. Let A be a matrix of order 2, and let the matrix A_r be defined as follows:*

$$A_3 = b_1 A_1 + c_2 A_2, \ A_{3+j} = a_j A_j + b_{j+1} A_{j+1} + c_{j+2} A_{j+2}, \ (1 < j < r-2).$$

Then, the corresponding 2-determinant Q is determined by two homogenous recurrence equations of order 3.

Proof. In this case the matrix P has the following form:

$$P = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & b_2 & a_2 & \cdots & 0 & 0 & 0 \\ -1 & c_3 & b_3 & \ldots & 0 & 0 & 0 \\ 0 & -1 & c_4 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{r-1} & b_{r-1} & a_{r-1} \\ 0 & 0 & 0 & \cdots & -1 & c_r & b_r \\ 0 & 0 & 0 & \cdots & 0 & -1 & c_{r+1} \end{pmatrix}.$$

The 2-determinant $\det Q$ is obtained from P by deleting the $(k+1)$th row of P, where $(0 \le k < r)$. We let $D(i_1, \dots, i_t)$ denote the minor of P, the main diagonal of which is $(i_1, i_2, \dots, i_t)$. Hence,

$$\det Q = D(b_1, \dots, b_k, c_{k+2}, \dots, c_{r+1}).$$

In the expansion of $\det Q$ across the first k rows all terms are zero, except eventually two. Hence,

$$\det Q = D(b_1, \dots, b_k)D(c_{k+2}, \dots, c_{r+1}) + a_k D(b_1, \dots, b_{k-1})D(c_{k+3}, \dots, c_{r+1}).$$

Furthermore, we have

$$D(b_1) = b_1,\ D(b_1, b_2) = \begin{vmatrix} b_1 & a_1 \\ c_2 & b_2 \end{vmatrix},\ D(b_1, b_2, b_3) = \begin{vmatrix} b_1 & a_1 & 0 \\ c_2 & b_2 & a_2 \\ -1 & c_3 & b_3 \end{vmatrix}.$$

Assume $k > 3$. Expanding $D(b_1, \dots, b_k)$ by the elements of the last column, we obtain

$$D(b_1, \dots, b_k) = b_k D(b_1, \dots, b_{k-1}) - a_{k-1}c_k D(b_1, \dots, b_{k-2}) - a_{k-1}a_{k-2}D(b_1, \dots, b_{k-3}). \quad (12)$$

Also,

$$D(c_{k+2}) = c_{k+2},\ D(c_{k+2}, c_{k+3}) = \begin{vmatrix} c_{k+2} & b_{k+2} \\ -1 & c_{k+3} \end{vmatrix},$$

and

$$D(c_{k+2}, c_{k+3}, c_{k+4}) = \begin{vmatrix} c_{k+2} & b_{k+2} & a_{k+2} \\ -1 & c_{k+3} & b_{k+3} \\ 0 & -1 & c_{k+4} \end{vmatrix}.$$

If $k > 3$, then by expanding $D(c_{k+2}, \dots, c_{r+1})$ along the first column, we obtain

$$\begin{aligned} &D(c_{k+2}, \dots, c_{r+1}) \\ &\qquad = c_{k+3}D(c_{k+3}, \dots, c_{r+1}) + b_{k+2}D(c_{k+4}, \dots, c_{r+1}) \\ &\qquad + a_{k+2}D(c_{k+5}, \dots, c_{r+1}). \end{aligned} \quad (13)$$

It follows that the 2-determinant $\det Q$ is uniquely determined by the recurrence equations (12) and (13). □

We illustrate the preceding considerations with two particular cases. We first assume that all a's, b's, and c's equal 1. Then,

$$D(b_1) = 1,\ D(b_1, b_2) = 0,\ D(b_1, b_2, b_3) = -2,$$

and, for $s > 3$,

$$D(b_1, b_2, \dots, b_s) = D(b_1, \dots, b_{s-1}) - D(b_1, \dots, b_{s-2}) - D(b_1, \dots, b_{s-3}).$$

This recursion designates the so-called **reflected Tribonacci numbers** (A057597). We denote these numbers by RT_i $(i = 1, 2, \dots)$. Also,

$$D(c_1) = 1,\ D(c_1, c_2) = 2,\ D(c_1, c_2, c_3) = 4,$$

and, for $s > 3$,

$$D(c_1, c_2, \ldots, c_s) = D(c_1, \ldots, c_{s-1}) + D(c_1, \ldots, c_{s-2}) + D(c_1, \ldots, c_{s-3}).$$

Hence, if we denote by $T_i,\ i = 0, 1, 2, \ldots$ the Tribonacci numbers (A000073) we have

$$D(c_1, c_2, \ldots, c_s) = T_{s+2},\ (s = 1, 2, \ldots).$$

Hence, our 2-determinant Q consists of Tribonacci and reflected Tribonacci numbers. On the other hand, if A is the identity matrix of order 2, then the first row of A_r consists of the above Tribonacci numbers. The second rows consists of Tribonacci numbers $\tilde{T}_i,\ (i = 0, 1, \ldots)$ with the initial conditions given by $\tilde{T}(0) = 1,\ \tilde{T}(1) = 0,\ \tilde{T}(2) = 1$, (A001590). As a consequence of Theorem 2, we have

Corollary 10. *Let $k < r$ be nonnegative integers. The following identity holds*

$$\begin{vmatrix} \tilde{T}_{k+1} & \tilde{T}_{r+2} \\ T_{k+1} & T_{r+2} \end{vmatrix} = (-1)^k \begin{vmatrix} RT_{k-1} & -RT_k \\ T_{r-k} & T_{r-k-1} \end{vmatrix}.$$

Assume now that the b's are all equal zero and the a's and the c's are all equal 1. Then the recursion (12) gives the sequence $x_1, x_2, \ldots,$ such that $x_n =$ A077962(n). Also, the recursion (13) produces the sequence $y_1, y_2, \ldots,$ such that $y_n =$ A000930(n).

Next, the rows of A_r form sequences $a_1^{(i)}, a_2^{(i)}, \ldots\ (i = 1, 2)$ such that, for $n \geq 4$ we have $a_n^{(1)} =$ A000930$(n-4)$, $a_n^{(2)} =$ A000930$(n-2)$. We thus obtain

Corollary 11. *For integers $3 \leq k < r$ we have*

$$\begin{vmatrix} y_{k-3} & y_{r-2} \\ y_{k-1} & y_r \end{vmatrix} = (-1)^k \begin{vmatrix} x_{k-1} & -x_k \\ y_{r-k} & y_{r-k-1} \end{vmatrix}.$$

5 3-determinants

In this section, the order of the matrix A is 3. We investigate in detail the particular case when $p_{i,j} = 0,\ (j > i)$. Then, the matrix P is of order $(r+2) \times r$, may have at most four nonzero diagonals, and has the form:

$$P = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_2 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ c_3 & b_3 & a_3 & \ldots & 0 & 0 & 0 \\ -1 & c_4 & b_4 & \ldots & 0 & 0 & 0 \\ 0 & -1 & c_5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_r & b_r & a_r \\ 0 & 0 & 0 & \cdots & -1 & c_{r+1} & b_{r+1} \\ 0 & 0 & 0 & \cdots & 0 & -1 & c_{r+2} \end{pmatrix}.$$

The corresponding 3-determinant $\det Q$ is obtained by deleting rows $k+1$ and $m+1$ of P, where $(0 \le k < m \le r+1)$. The matrix Q is a lower triangular block matrix of the form (10), with $\det Q_{11} = a_1 \cdot a_2 \cdots a_k$. The order of the matrix Q_{22} is $r-k$.

We denote $\det Q_{22} = D_k(s,r)$, where $s = m-k$. Note that $s \ge 1$. For the columns of A_r we have the following recursion:

$$A_{3+i} = a_i A_i + b_{i+1} A_{1+i} + c_{i+2} A_{i+2}, \ (i \ge 1). \tag{14}$$

It follows that the sequences in the rows of A_r satisfy recurrence equations of order 3, with the initial conditions given by the rows of A. The set $\{j_1, \ldots, j_r\}$ equals $\{1, 2, \ldots, k, k+2, \ldots, m, m+2, \ldots, r+2\}$. A simple calculation shows that $\sigma(M) = (-1)^{m+k+1}$. As a consequence of Theorem 2, we have

Proposition 12. *Let A be any matrix of order 3, let $(a_i), (b_{i+1}), (c_{i+2}), \ (i = 1, 2, \ldots)$ be any sequences. Then,*

$$M(k+1, m+1, r+2) = (-1)^{m+k+1} a_1 \cdots a_k \det Q \cdot \det A.$$

We next prove the following Proposition 12.

Proposition 13. *The determinant $\det Q_{22}$ is determined with three recurrence relations.*

Proof. The matrix Q_{22} has at most five nonzero diagonals. Assume that $s \ge 3$. 1) The main diagonal of Q_{22} is

$$b_{k+2}, \ldots, b_{k+s}, k\widehat{+s+}1, c_{k+s+2}, \ldots, c_{r+2},$$

where there are $s-1$ of b's and $r+1-k-s$ of a's
2) The first superdiagonal is

$$a_{k+2}, \ldots, a_{k+s}, k\widehat{+s+}1, b_{k+s+2} \ldots, b_{r+1},$$

where there are $s-1$ of a's and $r-k-s$ of b's.
3) The first subdiagonal is

$$c_{k+3}, \ldots, c_{k+s}, k\widehat{+s+}1, -1, \ldots, -1,$$

where there are $s-2$ of c's and $r+1-k-s$ of -1's.
4) The second superdiagonal is

$$0, \ldots, 0, k\widehat{+s+}1, a_{k+s+2}, \ldots, a_r,$$

where there are $s-1$ of 0's and $r-k-s-1$ of a's.
5) The second subdiagonal is

$$-1, \ldots, -1, k\widehat{+s+}1, 0, \ldots, 0,$$

where there are $s-3$ of -1's and $r+1-k-s$ of 0's.

Case $s = 1$. We begin with

$$D_k(1,k) = 1,\ D_k(1,k+1) = c_{k+3},\ D_k(1,k+2) = \begin{vmatrix} c_{k+3} & b_{k+3} \\ -1 & c_{k+4} \end{vmatrix}.$$

For $t > 2$, expanding $D_k(1, k+t)$ by elements from the last row yields the following recursion:

$$\begin{aligned} &D_k(1,k+t) \\ &\qquad = c_{k+t+2} D_k(1,k+t-1) + b_{k+t+1} D_k(1,k+t-2) \\ &\qquad + a_{k+t} D_k(1,k+t-3). \end{aligned} \tag{15}$$

Case $s = 2$. We have

$$D_k(2,k+1) = b_{k+2},\ D_k(2,k+2) = \begin{vmatrix} b_{k+2} & a_{k+2} \\ -1 & c_{k+4} \end{vmatrix},$$

$$D_k(2,k+3) = \begin{vmatrix} b_{k+2} & a_{k+2} & 0 \\ -1 & c_{k+4} & b_{k+4} \\ 0 & -1 & c_{k+5} \end{vmatrix}.$$

For $t \geq 4$, we calculate $D_k(2, k+t)$ by the recursion (15).

Case $s = 3$. We now have $D_k(3, k+2) = b_{k+2}$,

$$D_k(3,k+3) = \begin{vmatrix} b_{k+2} & a_{k+2} \\ c_{k+3} & b_{k+3} \end{vmatrix},\ D_k(3,k+4) = \begin{vmatrix} b_{k+2} & a_{k+2} & 0 \\ c_{k+3} & b_{k+3} & a_{k+3} \\ 0 & -1 & c_{k+5} \end{vmatrix},$$

$$D_k(3,k+5) = \begin{vmatrix} b_{k+2} & a_{k+2} & 0 & 0 \\ c_{k+3} & b_{k+3} & a_{k+3} & 0 \\ 0 & -1 & c_{k+5} & b_{k+5} \\ 0 & 0 & -1 & c_{k+6} \end{vmatrix}.$$

For $t > 5$, we calculate $D_k(3, k+t)$ again by the recursion (15).

Case $s \geq 4$. The minors $D_k(s, k+s-1), \ldots, D_k(s, k+2s-1)$ may be obtained as follows:

$$D_k(s,k+s-1) = b_{k+2},\ D_k(s,k+s) = \begin{vmatrix} b_{k+2} & a_{k+2} \\ c_{k+3} & b_{k+3} \end{vmatrix},$$

$$D_k(s,k+s+1) = \begin{vmatrix} b_{k+2} & a_{k+2} & 0 \\ c_{k+3} & b_{k+3} & a_{k+3} \\ -1 & c_{k+4} & b_{k+4} \end{vmatrix}.$$

When $1 < t \leq s-1$, we have the following recursion:

$$\begin{aligned} &D_k(s,k+s+t) \\ &\qquad = b_{t+k+2} D_k(s,k+s+t-1) - a_{t+k+1} c_{t+k+2} D_k(s,k+s+t-2) \\ &\qquad - a_{t+k+1} a_{t+k} D_k(s,k+s+t-3). \end{aligned} \tag{16}$$

Next, we have

$$D_k(s, k+2s) = c_{s+k+2}D_k(s, k+2s-1) + a_{s+k}D_k(s, k+2s-2).$$

If $s < r-k$, then

$$D_k(s, k+2s+1) = c_{s+k+3}D_k(s, k+2s) + b_{s+k+2}D_k(s, k+2s-1).$$

If $s+1 < r-k$, then for t, where $s+1 < t \le r-k$, we have the recursion (15).

The recursion with respect to k is backward. The minimal value that r can take is $r = k$. Then,

$$D_k(1, k) = 1,\ D_k(2, k+1) = D_k(3, k+2) = b_{k+2}.$$

Assume that $s > 3$. Expanding $D_k(s, r)$ by elements of the first row, we obtain the following recursion:

$$\begin{aligned} D_k(s,r) = b_{k+2}D_{k+1}(s-1, r-1) & \\ & - a_{k+2}c_{k+3}D_{k+2}(s-2, r-2) \\ & - a_{k+2}a_{k+3}D_{k+3}(s-3, r-3) \end{aligned} \quad . \tag{17}$$

We have thus proved that $\det Q$ is uniquely determined by the formulas (15), (16), and (17). $\square$

We state some particular cases.

1° All a's equal 0. It follows from (12) that all minors $M(k+1, m+1, r+3)$ are zeros, except the case $k = 0$, when we have the same situation as in the preceding section.

2° All b's equal 0, and all a's and c's equal 1. The formula (14) has the form:

$$A_{3+i} = A_i + A_{i+2},\ (i \ge 1).$$

If A is the identity matrix, then the rows of A_r make the so-called **middle sequence** (A000930). For a fixed k the first three rows of the array $\det Q_{22}$ are obtained by the recursion (15), hence they are also formed from the numbers of the middle sequence. If $s > 3$, then the first $s-1$ elements in row s are obtained by the recursion (16). The remaining terms are again obtained from (15). Therefore, Proposition (12) gives an identity for the numbers of the middle sequence.

3° All c's equal 0, and all a's and b's equal 1. In this case, we have

$$A_{3+i} = A_i + A_{i+1},\ (i \ge 1),$$

which is the recursion for the **Padovan** sequence (A000931). From Proposition 12, we obtain an assertion for the Padovan numbers.

4° All a's, b's and c's equal 1. The rows of A_r are made of Tribonacci numbers, with the initial conditions given by the rows of A. Proposition 12 produces an identity for Tribonacci numbers.

Assume that A is the identity matrix of order 3. Let $T_{t_1,t_2,t_3}(n),\ (n = 1, 2, \ldots)$ denote Tribonacci numbers with initial conditions $T_{t_1,t_2,t_3}(1) = t_1, T_{t_1,t_2,t_3}(2) = t_2, T_{t_1,t_2,t_3}(3) = t_3$.

We then have

Proposition 14. *Let $0 \leq k < m < r + 2$ be integers. Then,*

$$\begin{vmatrix} T_{1,0,0}(k) & T_{1,0,0}(m) & T_{1,0,0}(r+2) \\ T_{0,1,0}(k) & T_{0,1,0}(m) & T_{0,1,0}(r+2) \\ T_{0,0,1}(k) & T_{0,0,1}(m) & T_{0,0,1}(r+2) \end{vmatrix} = (-1)^{m+k+1} D_k(m-k, r).$$

Note that the nature of numbers $D_k(m-k, r)$ depends on values of $k, m-k$ and r.

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