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q-Analogues of Freud weights and nonlinear difference equations ☆

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ABSTRACT

In this paper we derive the nonlinear recurrence relation for the recursion coefficients β_n of polynomials orthogonal with respect to q-analogues of Freud exponential weights. An asymptotic relation for β_n is given under assuming a certain smoothing condition and the Plancherel–Rotach asymptotic for the corresponding orthogonal polynomials is derived. Special interest is paid to the case $m = 2$. We prove that the nonlinear recurrence relation of β_n in this case obeys the discrete Painlevé property. Motivated by Lew and Quarles, we study possible periodic solutions for a class of nonlinear difference equations of second order. Finally we prove that the Bernstein approximation problem associated to the weights under consideration has a positive solution.

1. Introduction

Let $\{P_n(x)\}$ be monic orthogonal polynomials associated with a weight function $w(x)$ on $(0, \infty)$, that is

$$\int_0^\infty P_m(x)P_n(x)w(x)\,dx = \zeta_n\delta_{m,n}. \tag{1.1}$$

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Following the notation in [15] the P_n's must satisfy a three term recurrence relation of the form

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \tag{1.2}$$

with $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$, $n > 0$. The norms $\{\zeta_n\}$ are related to the recursion coefficients via [15]

$$\zeta_n = \beta_1 \beta_2 \cdots \beta_n. \tag{1.3}$$

A weight function w leads to two potential functions u and v defined by

$$u(x) = -\frac{D_{q^{-1}} w(x)}{w(x)}, \qquad v(qx) = -\frac{D_q w(x)}{w(x)}, \tag{1.4}$$

where D_q is the q-difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad x \neq 0. \tag{1.5}$$

The two potential functions are related via

$$v(x) = \frac{u(x)}{1 + (1 - 1/q)xu(x)}. \tag{1.6}$$

Ismail, Johnston and Mansour [16] proved the following theorem.

Theorem 1.1. *Let u and v be as in* (1.4) *and define*

$$A_n(x) := \frac{q \log q P_n^2(0) w(0)}{(1-q)\zeta_n x} + \frac{1}{\zeta_n} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y)\, dy, \tag{1.7}$$

$$B_n(x) := \frac{q \log q P_n(0) P_{n-1}(0) w(0)}{(1-q)\zeta_{n-1} x} + \frac{1}{\zeta_{n-1}} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y)\, dy, \tag{1.8}$$

$$C_n(x) := \frac{q \log q P_n^2(0) w(0)}{(1-q)\zeta_n x} + \frac{q}{\zeta_n} \int_0^\infty \frac{v(x) - v(qy)}{x - qy} P_n(y) P_n(qy) w(y)\, dy, \tag{1.9}$$

$$D_n(x) := \frac{q \log q P_n(0) P_{n-1}(0) w(0)}{(1-q)\zeta_{n-1} x} + \frac{q}{\zeta_{n-1}} \int_0^\infty \frac{v(x) - v(qy)}{x - qy} P_n(y) P_{n-1}(qy) w(y)\, dy, \tag{1.10}$$

if w is defined on $(0, \infty)$ and continuous at $x = 0$ from the right. If w is defined on $\mathbb{R}$ and is continuous at $x = 0$, then we replace $\int_0^\infty$ by $\int_{\mathbb{R}}$ and delete the boundary terms in (1.7)–(1.10)*. We then have the structure relations*

$$D_{q^{-1}} P_n(x) = \beta_n C_n(x) P_{n-1}(x) - D_n(x) P_n(x), \tag{1.11}$$

$$D_q P_n(x) = \beta_n A_n(x) P_{n-1}(x) - B_n(x) P_n(x). \tag{1.12}$$

Moreover the functions $A_n(x)$, $B_n(x)$, $C_n(x)$ and $D_n(x)$ satisfy the pairs of recursions

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) + x(q-1)\sum_{j=0}^{n} A_j(x) - u(qx), \tag{1.13}$$

$$1 + (x - \alpha_n)B_{n+1}(x) - (qx - \alpha_n)B_n(x) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x), \tag{1.14}$$

$$D_{n+1}(x) + D_n(x) = (x - \alpha_n)C_n(x) + x(1/q - 1)\sum_{j=0}^{n} C_j(x) - v(x), \tag{1.15}$$

$$(x - \alpha_n)D_{n+1}(x) - (x/q - \alpha_n)D_n(x) = -1 + \beta_{n+1}C_{n+1}(x) - \beta_n C_{n-1}(x), \tag{1.16}$$

where α_n and β_n are as in (1.2)*.*

This generalizes earlier results in [4,5,14]. We now define a discrete analogue of the Painlevé property [7].

Definition 1.2. Let $f(x_n, x_{n+1}, \ldots, x_{n+m}) = 0$, $n = 1, 2, \ldots,$ be a difference equation. We say that the equation has the discrete Painlevé property (singularity confinement) if x_n is such that it results in a singularity for x_{n+1}, then there is a natural number p such that this singularity is confined to $x_{n+1}, x_{n+2}, \ldots, x_{n+p}$. Furthermore x_{n+p+1} depends only on $x_n, x_{n-1}, \ldots$.

Géza Freud considered the class of weight functions

$$w_{\rho m}(x) = C|x|^{\rho} \exp(-x^{2m}), \quad \rho > -1,\ m > 0,\ -\infty < x < \infty, \tag{1.17}$$

where C is a positive constant. During the period 1973–1976, Freud formulated two conjectures that have generated considerable interest till the present day. These conjectures led to many developments in orthogonal polynomials and approximation theory and were proved using potential theoretic and weighted polynomial approximation techniques. The first conjecture, see [8,11,9], concerns the largest zero $X_n(w_{\rho m})$ of $P_n(x)$. He conjectured that $X_n(w_{\rho m})$ has the asymptotic behavior

$$\lim_{n\to\infty} n^{-1/2m} X_n(w_{\rho m}) = \left[\sqrt{\pi}\,\frac{\Gamma(m)}{\Gamma((2m+1)/2)}\right]^{1/2m}. \tag{1.18}$$

The second conjecture was formulated in 1976, [12] and states that

$$\lim_{n\to\infty} n^{-1/m}\beta_n = \left[\frac{\Gamma(2m+1)}{\Gamma(m)\Gamma(m+1)}\right]^{-1/m} = \left[\frac{2\Gamma(2m)}{\Gamma^2(m)}\right]^{-1/m}, \quad m \in \mathbb{N}, \tag{1.19}$$

where $\{\beta_n\}$ are the recursion coefficients of the corresponding monic orthogonal polynomials. It was suggested by Ullman that the same conjectures hold for all $m > 1$ (not necessarily an integer). These conjectures were proved in several steps. The first breakthrough was introduced by Magnus in 1985–1986 when $w(x) = \exp(-P(x))$ and $P(x)$ is a polynomial of degree $2m$, see [24,25]. Mhaskar, Lubinsky and Saff [22] proved (1.19) for the general class of weights of the form $g(x)e^{-P(x)}$ where g is a Jacobi type weight and P is not necessarily a polynomial but certain assumptions are made on P. As for the conjecture (1.18), it was proved in several steps by many mathematicians and Rakhmanov in [31] completed the last step. Surveys of this topic are available in [21] and [23].

A q-analogue of the exponential function is

$$e_q(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{(1-q)(1-q^2)\cdots(1-q^n)}, \quad |q| < 1,\ |x| < 1. \tag{1.20}$$

Euler proved that [1,13]

$$e_q(x) = 1 \Big/ \prod_{n=0}^{\infty} \big(1 - xq^n\big). \tag{1.21}$$

It is clear from (1.20) that

$$\lim_{q \to 1^-} e_q\big((1-q)x\big) = e^x. \tag{1.22}$$

In this paper we study the recursion coefficients β_n of the polynomials orthogonal with respect to a q-analogue of the Freud weights. Precisely we study the β_n's corresponding to the weight functions

$$w_m(x;q) = C_m \prod_{n=0}^{\infty} \frac{1}{1 + a(1-q)q^{2mn+2m-1}x^{2m}}, \quad x \in \mathbb{R},\ m \geqslant 1, \tag{1.23}$$

where $0 < q < 1$, $a > 0$ and $m = 1, 2, \ldots$. The normalization constant C_m makes $\int_{\mathbb{R}} w_m(x;q)\,dx = 1$. In view of Eqs. (1.20)–(1.22) the weight function $w_m(x;q)$ is indeed a q-analogue of the Freud weight $C_m \exp(-ax^{2m}/2m)$. A calculation for the moment sequence and the constant C_m will be given in Section 2. From (1.4), the weight $w_m(x;q)$ leads to the potential function $v_m(x)$ defined by

$$D_q w_m(x;q) = -v_m(qx) w_m(x;q), \quad \text{i.e. } v_m(x) = ax^{2m-1}. \tag{1.24}$$

The structure relations (1.11)–(1.12) or the recursions (1.13)–(1.14) lead to nonlinear recurrence relations satisfied by the β_n's. The nonlinear difference equations which arise this way always seem to have the discrete Painlevé property and we believe this may be a general phenomenon.

To denote the dependence on a weight function w, we shall use the notations $\beta_n(w)$, $P_n(x,w)$, and $X_n(w)$ for the recursion coefficients β_n, the orthogonal polynomials $P_n(x)$ associated with the weight w, and the largest zero X_n of $P_n(x)$, respectively. In Section 3 we derive the nonlinear difference equation satisfied by $\beta_n(w_m)$ which is a q-analogue of the celebrated Freud's equation introduced by Magnus in [24–26]. In Section 4 we introduce upper bounds for the L_p norm of a weighted polynomial Pw_m in terms of the L_p norm of Pw_m over some finite interval. As an application we introduce upper and lower bounds for $\beta_n(w_m)$ and $X_n(w_m)$, where $X_n(w_m)$ denotes the largest zero of $P_n(w_m;x)$. The determination of the large n asymptotic behavior of the recursion coefficients $\beta_n(w_m)$ as well as the Plancherel–Rotach asymptotics of the polynomials $P_n(x,w_m)$ and the largest zeros $X_n(w_m)$ will be given in Section 5. In Section 6, we give special interest to the case $m = 2$. In this case, the nonlinear difference equation is

$$\frac{1-q^n}{1-q}\frac{q^{-2n}}{ax_n} = q^2[x_n + x_{n+1}] + x_{n-1} - \big(1-q^2\big)\sum_{k=1}^{n-1} x_k, \tag{1.25}$$

for $n = 1, 2, \ldots$, where the empty sum is zero and $x_0 := 0$. Eq. (1.25) implies the third order difference equation

$$q^2 x_{n+2} = x_{n-1} + \frac{1-q^{n+1}}{1-q} \frac{q^{-2n-2}}{a x_{n+1}} - \frac{1-q^n}{1-q} \frac{q^{-2n}}{a x_n}, \tag{1.26}$$

for $n = 2, 3, \ldots$. Observe that when $q \to 1^-$ Eq. (1.25) tends to the discrete DPI equation

$$n = a x_n [x_n + x_{n+1} + x_{n-1}], \tag{1.27}$$

originally derived by Freud and later appeared in the physics literature as the string equation. Lew and Quarles [19] studied the more general nonlinear difference equation

$$c_n = x_n [x_n + x_{n+1} + x_{n-1}], \quad n = 2, 3, \ldots, \tag{1.28}$$

for a given positive sequence c_n. Motivated by their work we study, in Section 7, the difference equation

$$\begin{gathered} \frac{c_n}{x_n} = q^2 [x_n + x_{n+1}] + x_{n-1} - \left(1-q^2\right) \sum_{k=1}^{n-1} x_k, \quad n = 2, 3, \ldots, \\ \frac{c_1}{x_1} = q^2 [x_1 + x_2], \end{gathered} \tag{1.29}$$

which implies

$$\begin{gathered} q^2 x_{n+2} = x_{n-1} + \frac{c_{n+1}}{x_{n+1}} - \frac{c_n}{x_n}, \quad n = 2, 3, \ldots, \\ q^2 x_3 = \frac{c_2}{x_2} - \frac{c_1}{x_1}, \end{gathered} \tag{1.30}$$

where c_n is a given sequence of positive terms. It is important to note that (1.30) implies (1.29) only if we assume the initial condition

$$c_1 = q^2 x_1 [x_1 + x_2]. \tag{1.31}$$

Finally, Section 8 contains some remarks about some approximation problems related to the weights $w_m(x; q)$ and more generalized weights.

2. Notation and moments

Throughout this paper we shall use the following notations. If $f(z)$ is an entire function, then $M(r, f)$ will denote the maximum of the function $f(z)$ on $|z| = r$, $r > 0$. We write $\phi(r) \approx \psi(r)$ if $\phi(r)/\psi(r) \to 1$ as $r \to \infty$.

We also follow the standard notation for q-series as in [1,13],

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=0}^{n-1} \left(1 - a q^k\right), \quad n > 0. \tag{2.1}$$

We shall also use the notation

$$P_n(x) = \sum_{k=0}^{n} c_{n,k} x^{n-k}, \quad c_{n,0} := 1. \tag{2.2}$$

Note that the three term recurrence relation implies that

$$\beta_0 := 0, \qquad c_{n,1} = -\sum_{k=0}^{n-1} \alpha_k, \quad n > 0, \qquad c_{0,1} := 0. \tag{2.3}$$

Moreover we have

$$c_{n,2} = -\sum_{k=1}^{n-1} \beta_k - \sum_{k=1}^{n-1} \alpha_k c_{k,1}, \quad n > 1, \qquad c_{1,2} = 0. \tag{2.4}$$

The q-analogue of the product rule is

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x). \tag{2.5}$$

The following lemma from [16] will be used in the sequel.

Lemma 2.1. *Let the functions f and g be defined and continuous on $\mathbb{R}$. Assume that the improper Riemann integrals of the functions $f(x)g(x)$, $f(x)g(qx)$ and $f(x/q)g(x)$ exist on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} f(x)D_q g(x)\,dx = -\frac{1}{q}\int_{\mathbb{R}} g(x)D_{q^{-1}} f(x)\,dx. \tag{2.6}$$

We shall consider only even weight functions, hence $P_n(-x) = (-1)^n P_n(x)$ and we conclude that $\alpha_n = 0$. Recall the integral evaluation [13]

$$I(\alpha; q) := \int_0^\infty \frac{y^{\alpha-1}}{(-y;q)_\infty}\,dy = \frac{(q^{1-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin \pi\alpha}, \quad \alpha > 0. \tag{2.7}$$

We now evaluate the moments

$$\mu_n(w_m) := \int_{\mathbb{R}} x^n w_m(x;q)\,dx. \tag{2.8}$$

It is clear that $\mu_{2n+1}(w_m) = 0$, while

$$\mu_{2n}(w_m) = 2C_m \int_0^\infty \frac{x^{2n}}{(-aq^{2m-1}(1-q)x^{2m};q^{2m})_\infty}\,dx.$$

A change of variable and the use of (2.7) identify the above integral as

$$\mu_{2n}(w_m) = C_m\big(a(1-q)q^{2m-1}\big)^{-\frac{2n+1}{2m}} \frac{(q^{2m-2n-1};q^{2m})_\infty}{(q^{2m};q^{2m})_\infty}\frac{\pi/m}{\sin (2n+1)\pi/2m}.$$

Thus

$$C_m = \big(a(1-q)q^{2m-1}\big)^{1/(2m)} \frac{(q^{2m};q^{2m})_\infty}{(q^{2m-1};q^{2m})_\infty} \frac{\sin\pi/2m}{\pi/m}, \tag{2.9}$$

and

$$\mu_{2n}(w_m) = \big(a(1-q)q^{2m-1}\big)^{-n/m} \frac{(q^{2m-2n-1};q^{2m})_\infty}{(q^{2m-1};q^{2m})_\infty} \frac{\sin\pi/2m}{\sin(2n+1)\pi/2m}. \tag{2.10}$$

It is clear that

$$\beta_1(w_m) = \mu_2(w_m). \tag{2.11}$$

Remark 2.2. Let H be the space of all polynomials. Since, for each $m \in \mathbb{N}$, the moment functional

$$\mathcal{L}_m(f) := \int_{-\infty}^{\infty} f(x) w_m(x;q)\,dx, \quad f \in H,$$

is positive definite, then the sequence $\mu_{2n}(w_m)$ should be positive. This implies

$$\big(q^{2m-2n-1};q^{2m}\big)_\infty \sin\big[(2n+1)\pi/2m\big] > 0, \quad \forall n, m \in \mathbb{N}. \tag{2.12}$$

One can verify that (2.12) holds by direct calculations.

3. *q*-Analogue of Freud's equation

Let $\beta = (a_1, a_2, \ldots)$ be the recurrence coefficients that appear in the three term recurrence relations

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n(x)p_{n-1}(x), \quad a_n := \sqrt{\beta_n}, \tag{3.1}$$

where $\{p_n(x)\}$ are the orthonormal polynomial sequence associated with the weights $w_{\rho m}(x)$ defined in (1.17). In [24], Magnus proved that the sequence β satisfies what he called Freud's equation. Indeed, he proved that the sequence β satisfies the equation

$$F_n(\beta) = n + \rho\ \text{odd}\ (n), \qquad F_n(\beta) := 2ma_n \int_{-\infty}^{\infty} x^{2m-1} P_{n-1}(x) P_n(x) w_{\rho m}(x)\,dx, \tag{3.2}$$

where $n = 1, 2, \ldots$, and odd $(n) = 1$ if n is odd; 0 if n is even. Moreover the integral on the left-hand side of (3.2) is equal to $(A^{2m-1})_{n,n-1}$, which represents the element at row n and column $n-1$ of the infinite matrix A^{2m-1}, where A is the Jacobi matrix

$$\begin{pmatrix} 0 & a_1 & 0 & \cdots \\ a_1 & 0 & a_2 & \\ 0 & a_2 & 0 & \ddots \\ & \ddots & \ddots & \end{pmatrix}$$

associated with (3.1). Magnus derived also Freud's equation for weights of the form $\exp(-P(x))$ and $P(x)$ is a polynomial of degree $2m$ in [25,26].

In this section we shall derive a q-analogue of Freud's equation for orthogonal polynomials associated to the weights $w_m(x;q)$. We start with calculating the value of the integration $A_{n,k}$ defined by

$$A_{n,k} := \int_{-\infty}^{\infty} x^{n+k} P_n(x) w(x)\, dx, \tag{3.3}$$

where $w(x)$ is an even weight function and $P_n(x)$ are the sequence of orthogonal polynomials satisfy (1.2). Then

$$A_{n,k} = \begin{cases} 0, & k < 0, \\ \zeta_n, & k = 0. \end{cases} \tag{3.4}$$

In the following we shall calculate the value when $k > 0$. Since $w(x)$ is even, then $P_n(x)$ are symmetric polynomials and $A_{n,2k+1} = 0$ for $k \in \mathbb{N}$. If $k = 2$, then $A_{n,2}$ satisfies the recurrence relation

$$\frac{A_{n,2}}{\zeta_n} - \frac{A_{n-1,2}}{\zeta_{n-1}} = \beta_{n+1}. \tag{3.5}$$

Solving the previous first order difference equation gives

$$\frac{A_{n,2}}{\zeta_n} = \sum_{j=1}^{n+1} \beta_j. \tag{3.6}$$

Lemma 3.1. *If in* (3.3) $w(x)$ *is even, then*

$$\frac{A_{n,2k}}{\zeta_n} = \sum_{j_k=1}^{n+1} \sum_{j_{k-1}=1}^{j_k+1} \cdots \sum_{j_1=1}^{j_2+1} \beta_{j_k} \cdots \beta_{j_1}. \tag{3.7}$$

Proof. We prove this lemma by induction on k. From (3.6) it holds for $k = 1$. Assume that (3.7) holds for all $n \in \mathbb{N}$ at $k = m$. Since

$$A_{j,2m+2} = \int_{-\infty}^{\infty} x^{2m+2+j} P_j(x) w(x)\, dx, \tag{3.8}$$

then substituting with $xP_j(x) = P_{j+1}(x) + \beta_j P_{j-1}(x)$ in (3.8) gives

$$\begin{aligned} A_{j,2m+2} &= \int_{-\infty}^{\infty} x^{2m+1+j} P_{j+1}(x) w(x)\, dx + \beta_j \int_{-\infty}^{\infty} x^{2m+1+j} P_{j-1}(x) w(x)\, dx, \\ &= A_{j+1,2m} + \beta_j A_{j-1,2m+2}. \end{aligned} \tag{3.9}$$

Thus $\frac{A_{j,2m+2}}{\zeta_j}$ satisfies the difference equation

$$\frac{A_{j,2m+2}}{\zeta_j} - \frac{A_{j-1,2m+2}}{\zeta_{j-1}} = \beta_{j+1} \frac{A_{j+1,2m}}{\zeta_{j+1}}. \tag{3.10}$$

Then summing on the two sides over j, $j = 1, 2, \dots, n$, gives

$$\frac{A_{n,2m+2}}{\zeta_n} = \frac{A_{0,2m+2}}{\zeta_0} + \sum_{j=1}^{m} \beta_{j+1} \frac{A_{j+1,2m}}{\zeta_{j+1}}. \tag{3.11}$$

Since $P_1(x) = x$ and $\xi_0 = 1$, then

$$\frac{A_{0,2m+2}}{\zeta_0} = \int_{-\infty}^{\infty} x^{2m+1} P_1(x) w(x)\, dx = \beta_1 \frac{A_{1,2m}}{\zeta_1}.$$

Thus

$$\frac{A_{n,2m+2}}{\zeta_n} = \sum_{k=0}^{n} \beta_{k+1} \frac{A_{k+1,2m}}{\zeta_{k+1}} = \sum_{k=1}^{n+1} \beta_k \frac{A_{k,2m}}{\zeta_k}. \tag{3.12}$$

Consequently the lemma follows from the induction hypothesis. □

Lemma 3.2. *If in* (1.1) $w(x)$ *is even, then* $P_n(x) = \sum_{k=0}^{[n/2]} c_{n,2k} x^{n-2k}$,

$$c_{n,2k} = (-1)^k \sum_{j_k=2k-1}^{n-1} \beta_{j_k} \sum_{j_{k-1}=2k-3}^{j_k-2} \beta_{j_{k-1}} \cdots \sum_{j_2=3}^{j_3-2} \beta_{j_2} \sum_{j_1=1}^{j_2-2} \beta_{j_1}, \tag{3.13}$$

$k = 0, 1, \dots, [n/2]$.

Proof. Since $w(x)$ is even, then $\alpha_n \equiv 0$ in (1.2). Hence by equating the coefficients of x^{n-2k+1} in (1.2) gives

$$c_{n,2k} - c_{n+1,2k} = -\beta_n c_{n-1,2k-2}, \quad k = 1, 2, \dots. \tag{3.14}$$

So the lemma follows by induction since from (2.4) with $\alpha_i \equiv 0$ it holds at $k = 1$. □

Theorem 3.3. *For each* $m \in \mathbb{N}$, *the sequence* $\beta_n(w_m)$ *satisfies the q-Freud equation*

$$\frac{1-q^n}{1-q} q^{-n} = \frac{aq^{2m-1}}{\zeta_{n-1}} \int_{-\infty}^{\infty} x^{2m-1} P_n(x) P_{n-1}(qx) w_m(x; q)\, dx. \tag{3.15}$$

Proof. The polynomial $D_{q^{-1}} P_n(x)$ can be expanded in terms of the polynomials $\{P_j(x)\}_{j=0}^{n-1}$ as

$$D_{q^{-1}} P_n(x) = \sum_{j=0}^{n-1} a_j P_j(x), \tag{3.16}$$

for some real constants $\{a_j\}_{j=0}^{n-1}$. A direct calculation gives

$$a_{n-1} = \frac{1-q^{-n}}{1-q^{-1}} = q^{-n+1} \frac{1-q^n}{1-q}. \tag{3.17}$$

On the other hand we have

$$
\begin{aligned}
a_{n-1} &= \frac{1}{\zeta_{n-1}} \int_{-\infty}^{\infty} D_{q^{-1}} P_n(x) P_{n-1}(x) w_m(x;q)\, dx \\
&= \frac{-q}{\zeta_{n-1}} \int_{-\infty}^{\infty} P_n(x) D_q\big(P_{n-1}(x) w_m(x;q)\big)\, dx \\
&= \frac{aq^{2m}}{\zeta_{n-1}} \int_{-\infty}^{\infty} x^{2m-1} P_n(x) P_{n-1}(qx) w_m(x;q)\, dx,
\end{aligned}
$$

where we used the orthogonality, the q-product rule (2.5), (1.24), and Lemma 2.1. □

Corollary 3.4. *The q-analogue of Freud's equation* (3.15) *can be written as*

$$\frac{1-q^n}{1-q} q^{-2n} = F_n(\beta), \tag{3.18}$$

$$F_n(\beta) = aq^{2m-2} \beta_n \sum_{k=0}^{m-1} q^{-2k} c_{n-1,2k} \frac{A_{n,2m-2k-2}}{\zeta_n}, \quad m \in \mathbb{N}. \tag{3.19}$$

Proof. Since

$$P_{n-1}(qx) = \sum_{k=0}^{n-1} c_{n-1,2k} (qx)^{n-1-2k}, \tag{3.20}$$

then from (3.15) and the orthogonality we obtain

$$\frac{1-q^n}{1-q} q^{-n} = \frac{q^{n+2m-2}}{\zeta_{n-1}} \sum_{k=0}^{n-1} q^{-2k} c_{n-1,2k} \int_{-\infty}^{\infty} x^{n+2m-2k-2} P_n(x) w_m(x;q)\, dx. \tag{3.21}$$

Consequently,

$$
\begin{aligned}
\frac{1-q^n}{1-q} q^{-2n} &= \frac{q^{2m-2}}{\zeta_{n-1}} \sum_{k=0}^{m-1} q^{-2k} c_{n-1,2k} \int_{-\infty}^{\infty} x^{n+2m-2k-2} P_n(x) w_m(x;q)\, dx \\
&= \frac{q^{2m-2}}{\zeta_{n-1}} \beta_n \sum_{k=0}^{m-1} q^{-2k} c_{n-1,2k} \frac{A_{n,2m-2k-2}}{\zeta_n},
\end{aligned} \tag{3.22}
$$

completing the proof. □

Example. For the weight $w_1(x;q)$, $F_n(\beta) = a\beta_n$ for all $n \in \mathbb{N}$.

For the weight $w_2(x;q)$,

$$F_n(\beta) = \begin{cases} aq^2 \beta_n (\beta_{n+1} + \beta_n + \beta_{n-1}), & n = 1, 2, \\ aq^2 \beta_n (\sum_{j=1}^{n+1} \beta_j - q^{-2} \sum_{j=1}^{n-2} \beta_j), & n > 2, \end{cases}$$

which coincides with (1.25).

For $w_3(x;q)$, we have for $n \geqslant 3$,

$$F_n(\beta) = aq^4\beta_n\Bigg[\sum_{j=1}^{n+1}\beta_j^2 + 2\sum_{j=1}^{n-3}\beta_j\beta_{j+1} + \big(1+q^{-4}\big)\sum_{j=3}^{n-2}\beta_j\sum_{r=1}^{j-2}\beta_r + \beta_{n+1}\beta_{n+2} + \beta_{n-1}\beta_{n-2} - \big(q^{-2}-1\big)(\beta_{n+1}+\beta_n+\beta_{n-1})\sum_{j=1}^{n-2}\beta_j - q^{-2}\Bigg(\sum_{j=1}^{n-2}\beta_j\Bigg)^2\Bigg],$$

and for $n = 1, 2$,

$$F_n(\beta) = aq^4\beta_n\Bigg[\sum_{j=1}^{n+1}\beta_j^2 + 2\sum_{j=1}^{n}\beta_j\beta_{j+1} + \beta_{n+1}\beta_{n-1} + \beta_{n+1}\beta_{n+2}\Bigg]. \tag{3.23}$$

Remark 3.5. The matrix

$$J(\beta) := \left[a_k\frac{\partial F_n(\beta)}{\partial a_k}\right], \quad n, k = 1, 2, \ldots,$$

associated with the Freud's equation of the weights $w_{\rho m}$ is symmetric and positive definite. These properties play an essential role in the proof of Freud conjecture (1.19) introduced by Magnus in [24]. Unfortunately this is not the case for the matrix $J(\beta)$ associated with the weights $w_m(x;q)$. For example, if we take $m = 2$ and calculate the corresponding matrix $J(\beta)$ we shall find that the entity

$$a_1\frac{\partial F_n(\beta)}{\partial a_1} = -2a\big(1-q^2\big)a_1^2a_3^2 < 0.$$

Hence $J(\beta)$ cannot be positive definite.

4. Bounds for $\beta_n(w_m)$ and $X_n(w_m)$

In this section we give upper and lower bounds for the sequence $\{\beta_n(w_m)\}$. The proof depends on a q-analogue of the following theorem introduced by Lubinsky in [20, Lemma 6.1].

Theorem 4.1. *Let $W(x)$ denote a function positive in $\mathbb{R}$ and $Q(x) = -\log W(x)$. Assume that:*

i $Q(x)$ *is bounded in each finite interval and* $\lim_{|x|\to\infty} Q(x)/\log|x| = \infty$.
ii $Q''(x)$ *is continuous in* $\mathbb{R}/(-A, A)$ *for some* $A > 0$ *and* $Q''(x)$ *is not identically zero there.*
iii *There exist* $C_1 > 0$ *and* $C_2 > 0$ *such that for* $\xi \geqslant C_1$,

$$3\xi M_1(\xi)\big\{\log|x|/\xi\big\}/Q(x) \leqslant 1, \quad |x| \geqslant C_2\xi,$$

where

$$M_1(\xi) = \max\big\{\big|Q'(u)\big|:\ A \leqslant |u| \leqslant \xi\big\}, \quad \xi > A.$$

If q_n denotes the roots of the equation

$$q_nM_1(q_n) = n,$$

then

$$\|PW\|_{L_p(\mathbb{R})} \leqslant \left(1 + CK^{-n}\right)\|PW\|_{L_p(-Kq_n, Kq_n)}, \tag{4.1}$$

for all polynomials P of degree $\leqslant n$, and for $p_1 \leqslant p \leqslant \infty$, while $K \neq K(P_1, P_2)$.

It is worthy to mention that Freud in [10, Theorem 2.6] derived the inequality (4.1) by using different technique for the special case $p = 2$ for the class of weights $w(x) = \exp(-Q(x))$, where $Q(x)$ is an even function, positive, and differentiable and $Q'(x)$ is positive and increasing to ∞. Then he gave an upper bound for the recursion coefficients β_n of the corresponding orthogonal polynomials. Actually, he proved that there exists a constant C_0 such that

$$\beta_n \leqslant C_0 q_{2n}^2,$$

where q_n is the unique positive number that satisfy $q_n Q'(q_n) = n$. He also gave estimates for the largest zero for $P_n(w^2)$ on [9]. Magnus in [25] applied Theorem 4.1 in case of $p = 2$ to calculate an upper bound for the sequence $\beta_n(w)$, where $w = \exp(-Q(x))$ and $Q(x)$ is a polynomial of degree $2m$.

The proof of Theorem 4.1 and its q-analogue depends on Cartan's lemma:

Lemma 4.2 *(Cartan's lemma). Let $z_1, \ldots, z_n$ be given points in $\mathbb{C}$ and $H > 0$ be given. Then there exist closed disks $\Delta_1, \ldots, \Delta_m$, $m \leqslant n$, such that the sum of the radii of the disks $\Delta_1, \ldots, \Delta_m$ is not bigger than $2H$ and*

$$\prod_{j=1}^{n} |z - z_j| > (H/e)^n, \quad z \notin \bigcup_{j=1}^{m} \Delta_j. \tag{4.2}$$

Ramis [33] defined the q-exponential growth of a certain class of entire functions. He assumed that q is a real number satisfies $|q| > 1$. But his definition can be modified for $|q| < 1$.

Definition 4.3. Let σ be a positive real number. An entire function f has q-exponential growth of order σ and a finite type, if there exist constants $K > 0$, α, such that

$$\left|f(x)\right| < K|x|^{\alpha} q^{-\frac{\sigma}{2}(\frac{\log|x|}{\log q})^2}. \tag{4.3}$$

Lemma 4.4. *The reciprocal of the weight functions $w_m(x; q)$ have q-exponential growth of order $2m$.*

Proof. Since $1/w_m(x; q) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)m}}{(q^{2m}; q^{2m})_k}(a(1-q)x^{2m})^k$, then

$$1/w_m(x; q) = \sum_{n=0}^{\infty} a_n x^n, \quad a_n = \begin{cases} 0, & n \neq 2mk, \ k \in \mathbb{N}, \\ \frac{q^{n/2m(n/2m-1)}}{(q^{2m}; q^{2m})_{n/2m}}(a(1-q))^{n/2m}. \end{cases}$$

Consequently

$$|a_n| \leqslant \frac{q^{-m/4}}{(q^{2m}; q^{2m})_\infty} q^{\frac{(n-m)^2}{4m}} a^{n/2m}, \quad n \in \mathbb{N}.$$

This latter inequality is equivalent to that $1/w_m$ is an entire function of q-exponential growth of order $2m$, see [33, p. 60]. $\square$

Remark 4.5. Since for each $m \in \mathbb{N}$, the function $1/w_m(x;q)$ is a solution of the functional equation

$$y(x) - \left(1 + aq^{2m-1}(1-q)x^{2m}\right)y(qx) = 0. \tag{4.4}$$

Bergweiler, Ishizaki, and Yanagihara in [2] proved, in general, if $y(x)$ is a solution of the functional equation

$$y(x) + a_1(x)y(qx) = 0, \quad x \in \mathbb{C}$$

where $a_1(x)$ is a polynomial of degree p, then for sufficiently large r

$$\log\max\left\{\left|y(x)\right|\colon |x| = r\right\} \approx \frac{-\sigma}{2\log q}(\log r)^2, \quad \sigma = \text{degree } a_1(x).$$

Then Lemma 4.4 follows since degree $a_1(x) = 2m$ and

$$\log\max\left\{\left|y(x)\right|\colon |x| = r\right\} = \max\left\{\log\left|y(x)\right|\colon |x| = r\right\}.$$

In the following by $\mathcal{P}_n$ we mean the space of all polynomials of degree less than or equal to n.

Theorem 4.6. *Let W be a function of q-exponential growth of order σ, $\sigma > 0$, and let $Q = -\log|W|$. Assume that Q is an increasing function on $\mathbb{R}^+$. If $s_n := q^{-\frac{n}{\sigma}}$, then for $0 < p < \infty$ there exist positive constants $C := C(p)$ and $K > 1$ such that for all $n \geqslant [1/\sqrt{p}\,] + 1$*

$$\|PW\|^p_{L_{p(\mathbb{R})}} \leqslant \left(1 + C(p)K^{-n^2 p}\right)\|PW\|^p_{L_p(-Ks_n, Ks_n)}, \tag{4.5}$$

for all $P \in \mathcal{P}_n$.

Proof. Let $P \in \mathcal{P}_n$ and write

$$P(x) = c\prod_{i=1}^{k}(x - x_i),$$

where $k \leqslant n$ and $c \neq 0$. We group the zeros of P as follows. For $1 \leqslant i \leqslant j$, $|x_i| \leqslant 2s_n$, and for $j < i \leqslant k$, $|x_j| > 2s_n$. Let $|x| \geqslant 2s_n$ and $|u| \leqslant s_n$. Then if $1 \leqslant i \leqslant j$,

$$\frac{|x - x_i|}{|u - x_i|} \leqslant \frac{|x| + |x_i|}{|u - x_i|} \leqslant \frac{|x| + 2s_n}{|u - x_i|} \leqslant \frac{2|x|}{|u - x_i|}, \tag{4.6}$$

while if $j < i \leqslant k$,

$$\frac{|x - x_i|}{|u - x_i|} \leqslant \frac{1 + |x|/|x_i|}{|1 - u/x_i|} \leqslant \frac{1 + (|x|/2s_n)}{1 - (|u|/2s_n)} \leqslant 2\frac{|x|}{s_n}. \tag{4.7}$$

Let $H := s_n/2$, from Cartan's lemma, there exist intervals $I_1, \ldots, I_l$, $l \leqslant k$, and the sum of the radii of these intervals is less than $2H$ such that

$$\prod_{i=1}^{j}|u - x_i| > \left(\frac{H}{e}\right)^j, \quad u \in \mathcal{M} := (-s_n, s_n) - \bigcup_{r=1}^{l} I_r. \tag{4.8}$$

Obviously the Lebesgue measure of $\mathcal{M}$ is at least s_n. Consequently from (4.6)–(4.8), we obtain

$$\begin{aligned}\frac{|P(x)|}{|P(u)|} &= \prod_{i=1}^{j}\frac{|x-x_i|}{|u-x_i|}\prod_{i=j+1}^{k}\frac{|x-x_i|}{|u-x_i|}\\ &\leqslant \frac{(2|x|)^k}{s_n^{k-j}}(e/H)^j = \left(\frac{2|x|}{s_n}\right)^k(2e)^j\\ &\leqslant \left(\frac{4e|x|}{s_n}\right)^k. \end{aligned} \tag{4.9}$$

Since $Q(x)$ is an increasing function on $[0,\infty)$, then for $0 \leqslant u \leqslant s_n$,

$$Q(u) \leqslant Q(s_n) \leqslant M\big(s_n, Q(\cdot)\big) \approx -\frac{\sigma(\log s_n)^2}{2\log q}, \tag{4.10}$$

for sufficiently large n. Then there exists a constant C_0 such that for all $n \in \mathbb{N}$ we have

$$M\big(s_n, Q(\cdot)\big) \leqslant C_0\frac{\sigma(\log s_n)^2}{2\log(1/q)} = n^2\left(\log \frac{1}{q}^{\frac{C_0}{2\sigma}}\right), \tag{4.11}$$

for all $n \in \mathbb{N}$. Hence for all $u \in \mathcal{M}$ and $|x| > (1/q)^{\frac{C_0}{2\sigma}} s_n$ we obtain

$$\exp\big(Q(u)\big) \leqslant \exp\left(n^2 \log\frac{|x|}{s_n}\right). \tag{4.12}$$

Since, from (4.10),

$$\lim_{|x|\to\infty}\frac{|Q(x)|}{(\log|x|)} = \infty,$$

then for fixed s_n

$$\lim_{|x|\to\infty}\frac{4\sigma^2\log^2 s_n \log(|x|/s_n)}{Q(x)} = 0.$$

Therefore we can choose positive constants C_1 and C_2 such that for $s_n \geqslant C_1$,

$$\frac{4\sigma^2(\log^2 s_n)\log(|x|/s_n)}{Q(x)} \leqslant \log^2(1/q), \quad |x| \geqslant C_2 s_n. \tag{4.13}$$

Consequently

$$\exp\big(-Q(x)\big) \leqslant \exp\left(-4n^2\log\frac{|x|}{s_n}\right). \tag{4.14}$$

Combining (4.9), (4.12), and (4.14) we obtain for fixed p, $0 < p < \infty$, $u \in \mathcal{M}$ and

$$|x| > \max\big\{4e, (1/q)^{\frac{C_0}{m}}, C_2\big\}s_n$$

that

$$\begin{aligned}\frac{|P(x)W(x)|}{|P(u)W(u)|} &\leqslant \left(\frac{4e|x|}{s_n}\right)^k \exp\left(-3n^2 \log\frac{|x|}{s_n}\right)\\ &\leqslant \left(\frac{4e|x|}{s_n}\right)^n \exp\left(-3n^2 \log\frac{|x|}{s_n}\right)\\ &\leqslant \exp(n\log 4e)\exp\left(n\log\frac{|x|}{s_n}\right)\exp\left(-3n^2 \log\frac{|x|}{s_n}\right)\\ &\leqslant \exp\left(2n\log\frac{|x|}{s_n}\right)\exp\left(-3n^2 \log\frac{|x|}{s_n}\right) \leqslant \exp\left(-n^2 \log\frac{|x|}{s_n}\right),\end{aligned} \tag{4.15}$$

where we used that $\log(|x|/s_n) \geqslant \log 4e$. Thus

$$\frac{|P(x)W(x)|}{|P(u)W(u)|} \leqslant \exp\left(-n^2 \log\frac{|x|}{s_n}\right) = \left(\frac{s_n}{|x|}\right)^{n^2}. \tag{4.16}$$

Now assume that $|x| \geqslant Ks_n$, with K is large enough. Let $0 < p < \infty$, we have from (4.16) that

$$\int\limits_{|x|\geqslant Ks_n} \left|P(x)W(x)\right|^p dx \leqslant \inf\left\{\left|P(u)W(u)\right|^p\colon u \in \mathcal{M}\right\} \int\limits_{|x|\geqslant Ks_n} \left(\frac{s_n}{|x|}\right)^{n^2 p} dx.$$

Since $\mathcal{M}$ has a Lebesgue measure at least s_n, then

$$\int\limits_{\mathcal{M}} \left|P(u)W(u)\right|^p du \geqslant s_n \inf\left\{\left|P(u)W(u)\right|^p\colon u \in \mathcal{M}\right\}, \tag{4.17}$$

and since for $n > [1/\sqrt{p}\,] + 1$,

$$\begin{aligned}\int\limits_{|x|\geqslant Ks_n} \left(\frac{s_n}{|x|}\right)^{n^2 p} dx &= 2s_n^{n^2 p} \int\limits_{Ks_n}^{\infty} \frac{1}{x^{n^2 p}}\, dx\\ &= 2s_n\left(n^2 p - 1\right)^{-1} K^{1-n^2 p},\end{aligned} \tag{4.18}$$

then

$$\int\limits_{|x|\geqslant Ks_n} \left|P(x)W(x)\right|^p dx \leqslant 2\left(n^2 p - 1\right)^{-1} K^{1-n^2 p} \int\limits_{\mathcal{M}} \left|P(u)W(u)\right|^p du, \quad n > [1/\sqrt{p}\,] + 1. \tag{4.19}$$

This completes the proof of the theorem. □

Theorem 4.7. *There exists a positive constant $K_{1,m}$ such that*

$$\beta_n(w_m) \leqslant K_{1,m} q^{-2n/m}, \quad n = 1, 2, 3, \ldots. \tag{4.20}$$

Proof. Since the sequence $\{P_n(x)\}_{n=0}^{\infty}$ of the orthogonal polynomials associated with the weight w_m satisfies the three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n(w_m)P_{n-1}(x), \tag{4.21}$$

then we have

$$x^2 P_n^2(x) = P_{n+1}^2(x) + 2\beta_n(w_m)P_{n+1}(x)P_{n-1}(x) + \beta_n^2(w_m)P_{n-1}^2(x). \tag{4.22}$$

Consequently

$$\beta_n + \beta_{n+1} = \frac{1}{\zeta_n}\int_{-\infty}^{\infty} x^2 P_n^2(x) w_m(x;q)\,dx. \tag{4.23}$$

Let $W(x) = \sqrt{w_m(x;q)}$. Then, from Lemma 4.4, $1/W$ has q exponential growth of order m. Then applying Theorem 4.6 with $p = 2$, we conclude that there exist positive constants $C > 0$ and $K > 0$ such that, for all $n \geqslant 1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 P_n^2(x) w_m(x;q)\,dx &\leqslant C \int_{-Ks_{n+1}}^{Ks_{n+1}} x^2 P_n^2(x) w_m(x;q)\,dx \\ &\leqslant CK^2 s_{n+1}^2 \int_{-Ks_{n+1}}^{Ks_{n+1}} P_n^2(x) w_m(x;q)\,dx \\ &\leqslant CK^2 s_{n+1}^2 \int_{-\infty}^{\infty} P_n^2(x) w_m(x;q)\,dx = CK^2 s_{n+1}^2 \zeta_n. \end{aligned} \tag{4.24}$$

Combining Eqs. (4.23) and (4.24) we obtain

$$\beta_n \leqslant C\frac{K^2}{2} q^{(-2n+2)/m}, \quad n \in \mathbb{N}. \quad \square \tag{4.25}$$

Theorem 4.8. *For each $m \in \mathbb{N}$, there exists a positive constant $K_{2,m}$ such that*

$$\beta_n(w_m) \geqslant K_{2,m} q^{-2n/m}, \quad n = 1, 2, \ldots. \tag{4.26}$$

Proof. From Theorem 4.7, there exists a positive constant $K_{1,m}$ such that

$$\beta_j \leqslant K_{1,m} q^{-2j/m}, \quad j \in \mathbb{N}. \tag{4.27}$$

Then applying this inequality on the right-hand side of (3.13) and (3.7) we can prove by using induction on k that

$$|c_{n,2k}| \leqslant K_{1,m}^k \frac{q^{2k^2/m}}{(q^{2/m};q^{2/m})_k} q^{-2nk/m}, \tag{4.28}$$

and

$$\frac{|A_{n,2k}|}{|\zeta_n|} \leqslant K_{1,m}^k \frac{q^{-k(k+1)/m}}{(q^{2/m};q^{2/m})_k} q^{-2nk/m}. \tag{4.29}$$

Substituting with these upper bounds in (3.18) yields

$$\begin{aligned}\frac{(1-q^n)q^{-2n}}{a(1-q)|\beta_n|} &\leqslant q^{m-1}K_{1,m}^{m-1}\frac{q^{-2n}+\frac{2n}{m}}{(q^{2/m};q^{2/m})_{m-1}}\sum_{k=0}^{m-1}\begin{bmatrix} m-1 \\ k\end{bmatrix}_q q^{k(k-1)/m}q^{2k/m} \\ &\leqslant K_{1,m}^{m-1}\frac{q^{-2n}+\frac{2n}{m}}{(q^{2/m};q^{2/m})_{m-1}}\left(-q^{2/m};q^{2/m}\right)_{m-1}.\end{aligned} \tag{4.30}$$

Consequently

$$\beta_n \geqslant \frac{1-q^n}{a(1-q)K_{1,m}^{m-1}}\frac{(q^{2/m};q^{2/m})_{m-1}}{(-q^{2/m};q^{2/m})_{m-1}}q^{-2n/m}, \tag{4.31}$$

proving the existence of $K_{2,m}$ and the theorem. □

Freud in 1973 [8] proved the following theorem.

Theorem 4.9. *Let $w(x)$ be even weight function and let*

$$\Gamma_n(w) = \max_{1\leqslant\nu\leqslant n-1}\sqrt{\beta_\nu}. \tag{4.32}$$

Then

$$\Gamma_n(w) \leqslant X_n(w) \leqslant 2\Gamma_n(w). \tag{4.33}$$

Theorem 4.10. *There are constants L_1 and L_2 such that*

$$L_1 q^{-n/m} \leqslant X_n(w_m) \leqslant L_2 q^{-n/m}, \quad n = 1,2,\ldots. \tag{4.34}$$

Proof. The proof is an immediate consequence of Theorems 4.8 and 4.9. □

Another way to get a lower bound for $X_n(w)$ is based on (2.10) and the following lemma introduced by Freud in [8].

Lemma 4.11. *For every even w, we have*

$$X_n^2(w) \geqslant \frac{\mu_{2n-2}(w_m)}{\mu_{2n-4}(w)}. \tag{4.35}$$

5. Asymptotics of $\beta_n(w_m)$

In this section we shall prove by assuming a certain smoothing condition that the sequence $\{\beta_n(w_m)\}$ or simply $\{\beta_n\}$ of solutions of (3.18) satisfies

$$\beta_n \approx q^{-2n/m} \sqrt[m]{\frac{1-q^n}{aq^{m-1}(1-q)}}, \tag{5.1}$$

for sufficiently large n. First Theorem 4.8 let assuming that the solution $\{\beta_n\}$ satisfies

$$\lim_{n\to\infty} \frac{\beta_{n+1}(w_m)}{\beta_n(w_m)} = q^{-2/m}, \tag{5.2}$$

is reasonable. Unfortunately we could not prove that the sequence $\beta_n(w_m)$ should satisfy this smoothing condition.

Lemma 5.1. *Under the condition* (5.2) *we have*

$$c_{n,2k} \approx (-1)^k \frac{\prod_{j=1}^k \beta_{n-2j+1}}{(q^{2/m};q^{2/m})_k}, \quad k = 1, 2, \ldots, [n/2], \tag{5.3}$$

for all sufficiently large n.

Proof. The sequence $c_{n,2k}$ satisfies the recurrence relation

$$c_{n+1,2k} - c_{n,2k} = \beta_n c_{n-1,2k-2}, \quad n \in \mathbb{N},\ k = 1, 2, \ldots, [n/2]. \tag{5.4}$$

We shall prove this lemma by induction on k. If $k = 1$, then we have from (5.4) that

$$c_{n,2} - c_{n-1,2} = -\beta_{n-1}. \tag{5.5}$$

Dividing on β_{n-1} gives

$$\frac{c_{n,2}}{\beta_{n-1}} - \frac{\beta_{n-2}}{\beta_{n-1}} \frac{c_{n-1,2}}{\beta_{n-2}} = -1.$$

Thus from (5.2)

$$\lim_{n\to\infty} \frac{c_{n,2}}{\beta_{n-1}} = \frac{-1}{1-q^{2/m}},$$

proving (5.3) at $k = 1$. Now we assume that (5.3) holds at $k = r$, then replacing n in (5.4) with $n-1$ and setting $k = r+1$, we obtain

$$\frac{c_{n,2r+2}}{\beta_{n-1}c_{n-2,2r}} - \frac{\beta_{n-2}c_{n-3,2r}}{\beta_{n-1}c_{n-2,2r}} \frac{c_{n-1,2r+2}}{\beta_{n-2}c_{n-3,2r}} = -1. \tag{5.6}$$

From (5.2) and (5.3) we conclude

$$\lim_{n\to\infty} \frac{\beta_{n-2}c_{n-2,2r}}{\beta_{n-1}c_{n-1,2r}} = q^{2(r+1)/m}.$$

Consequently

$$\lim_{n\to\infty} \frac{c_{n,2r+2}}{\beta_{n-1} c_{n-2,2r}} = -\frac{1}{1-q^{2(r+1)/m}}. \tag{5.7}$$

Thus from the hypothesis induction we get

$$c_{n,2r+2} \approx (-1)^{r+1} \frac{\beta_{n-1}\beta_{n-3}\cdots\beta_{n-2r-1}}{(q^{2/m};q^{2/m})_{r+1}},$$

for sufficiently large n and the lemma follows. □

Lemma 5.2. *Under the condition* (5.2) *we have*

$$\frac{A_{n,2k}}{\zeta_n} \approx \frac{\beta_{n+k}\beta_{n+k-1}\cdots\beta_{n+1}}{(q^{2/m};q^{2/m})_k}, \tag{5.8}$$

for every $k \in \mathbb{N}$ and for all sufficiently large n.

Proof. From (3.10), the sequence $A_{n,2k}/\zeta_n$ satisfies the recurrence relation

$$\frac{A_{n,2k}}{\zeta_n} - \frac{A_{n-1,2k}}{\zeta_{n-1}} = \beta_{n+1} \frac{A_{n+1,2k-2}}{\zeta_{n+1}}. \tag{5.9}$$

We shall prove this lemma by induction on k. If $k=1$, then we have

$$\frac{A_{n,2}}{\zeta_n} - \frac{A_{n-1,2}}{\zeta_{n-1}} = \beta_{n+1}. \tag{5.10}$$

Dividing on β_{n+1} gives

$$\frac{A_{n,2}/\zeta_n}{\beta_{n+1}} - \frac{\beta_n}{\beta_{n+1}} \frac{A_{n-1,2}/\zeta_{n-1}}{\beta_{n-1}} = 1.$$

Thus from (5.2)

$$\lim_{n\to\infty} \frac{A_{n,2}/\zeta_n}{\beta_{n+1}} = \frac{1}{1-q^{2/m}}.$$

Now we assume that (5.8) holds at $k=r$, then from (5.9) with $k=r+1$, we obtain

$$\frac{A_{n,2r+2}/\zeta_n}{\beta_{n+1}[A_{n+1,2r}/\zeta_{n+1}]} - \frac{\beta_n}{\beta_{n+1}} \frac{A_{n,2r}/\zeta_n}{A_{n+1,2r}/\zeta_{n+1}} \frac{A_{n-1,2r+2}/\zeta_{n-1}}{\beta_n[A_{n,2r}/\zeta_n]} = 1. \tag{5.11}$$

From (5.2) and (5.8) we conclude

$$\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} \frac{A_{n,2r}/\zeta_n}{A_{n+1,2r}/\zeta_{n+1}} = q^{2(r+1)/m}.$$

Consequently

$$\lim_{n\to\infty}\frac{A_{n,2r+2}/\zeta_n}{\beta_{n+1}[A_{n+1,2r}/\zeta_{n+1}]}=\frac{1}{1-q^{2(r+1)/m}}. \tag{5.12}$$

Thus from the hypothesis induction we get

$$\frac{A_{n,2r+1}}{\zeta_n}\approx\frac{\beta_{n+k}\beta_{n+k-1}\cdots\beta_{n+1}}{(q^{2/m};q^{2/m})_{r+1}},$$

for sufficiently large n and the lemma follows. □

Theorem 5.3. *The sequence defined in* (5.1) *is an approximate solution of* (3.18).

Proof. From (5.1) we can assume that

$$\beta_n\approx q^{2/m}\beta_{n+1} \tag{5.13}$$

for sufficiently large n. Substituting from (5.13) on (5.8) and (5.3) we obtain

$$\frac{A_{n,2k}}{\zeta_n}\approx\beta_n^k\frac{q^{-k(k+1)/m}}{(q^{2/m};q^{2/m})_k},\qquad c_{n-1,2k}\approx(-1)^k\beta_n^k\frac{q^{2(k^2+k)/m}}{(q^{2/m};q^{2/m})_k}.$$

Substituting from the previous two equations on (3.18), we obtain

$$F_n(\beta)\approx q^{m-1}\beta_n^m\sum_{k=0}^{m-1}(-1)^k\frac{q^{k(k-1)/m}}{(q^{2/m};q^{2/m})_{m-1-k}(q^{2/m};q^{2/m})_k}q^{2k/m}=aq^{m-1}\beta_n^m, \tag{5.14}$$

where we used the identity

$$\frac{(x;p)_n}{(p;p)_n}=\sum_{k=0}^{n}\frac{p^{\binom{k}{2}}}{(p;p)_k(p;p)_{n-k}}x^k.$$

So β_n should satisfy the asymptotics

$$aq^{m-1}\beta_n^m\approx\frac{1-q^n}{1-q}q^{-2n}, \tag{5.15}$$

then we obtain (5.1) for n large enough and the theorem follows. □

Remark 5.4. It is worth noting that in (5.1) if we calculate the limits of the two sides as $q\to 1^-$, we obtain

$$\lim_{q\to 1^-}\beta_n(w_m;q)\approx\left(\frac{n}{a}\right)^{1/m},$$

where $\lim_{q\to 1^-}\beta_n(w_m;q)$ represents the coefficients in the three term recurrence relations of the polynomials orthogonal with respect to the weight functions $\exp(-ax^{2m}/2m)$. Obviously this consistent with Freud conjecture (1.19).

In the remaining of this section we derive Plancherel–Rotach type asymptotics for the polynomials $P_n(x)$ orthogonal with respect to the weight $w_m(x;q)$. The term Plancherel–Rotach asymptotics refers to asymptotics around the largest and smallest zeros. Qui and Wong [30], and Deift et al. [6] derived the Plancherel–Rotach type asymptotics for the set of polynomials orthogonal with respect to general exponential weights.

Ismail introduced the following conjecture in [15, pp. 649–650]. If a sequence of orthogonal polynomials $\{H_n(x)\}_{n=0}^{\infty}$ satisfies the three term recurrence relation

$$xH_n(x) = H_{n+1}(x) + \alpha_n H_n(x) + q^{-n}\delta_n H_{n-1}(x),$$
$$H_0(x) = 1, \qquad H_1(x) = x - \alpha_0, \tag{5.16}$$

where

$$q^{-n/2}\alpha_n \to 0, \qquad \delta_n \to 1 \quad \text{as } n \to \infty,$$

and

$$x_n(t;q) = q^{-n/2}t - q^{n/2}/t, \quad t \neq 0,$$

then

$$\lim_{n\to\infty} \frac{q^{n^2/2}}{t^n} H_n\big(x_n(t)\big) = A_q\big(1/t^2\big), \tag{5.17}$$

where A_q is the q-Airy function defined by

$$A_q(z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(q;q)_k} z^k, \quad z \in \mathbb{C}. \tag{5.18}$$

In [17] Ismail, Li, and Rahman proved the conjecture in case $\alpha = 0$. They proved the following theorem.

Theorem 5.5. *Let* $S_n(t) = \frac{q^{n^2/2}}{t^n} H_n(x_n(t))$, $t \neq 0$. *Then*

$$\lim_{n\to\infty} S_n(t) = A_q\big(1/t^2\big), \quad t \neq 0, \tag{5.19}$$

where the convergence is locally uniform on $\overline{\mathbb{C}} \setminus \{0\}$.

Moreover if $x_{n,1} > x_{n,2} > \cdots > x_{n,n}$ *are the zeros of* $H_n(z)$, *then*

$$\lim_{n\to\infty} q^{n/2}|x_{n,k}| = \frac{1}{\sqrt{i_k(q)}}, \quad 1 \leqslant k \leqslant n, \tag{5.20}$$

where

$$0 < i_1(q) < i_2(q) < \cdots$$

are the zeros of $A_q(z)$.

As an application on Theorem 5.5 we introduce the following theorem.

Theorem 5.6. *Let* $\{P_n(x; w_m)\}_{n=0}^{\infty}$ *be the set of orthogonal polynomial corresponding to the weight function* $w_m(x; q)$*, and let*

$$x_{1,n}(w_m) > x_{2,n}(w_m) > \cdots > x_{n,n}(w_m),$$

be the zeros of $P_n(x; w_m)$*. Then*

$$\lim_{n\to\infty} \frac{q^{n^2/m}(aq^{m-1}(1-q))^{n/2m}}{t^n} P_n\left(w_m; \frac{x_n(t; q^{2/m})}{\sqrt[2m]{aq^{m-1}(1-q)}}\right) = A_{q^{2/m}}\left(\frac{1}{t^2}\right), \quad t \neq 0, \tag{5.21}$$

where the convergence is locally uniform on $\overline{\mathbb{C}} \setminus \{0\}$*. Moreover*

$$\lim_{n\to\infty} q^{n/m}|x_{n,k}| = \frac{1}{\sqrt[2m]{aq^{m-1}(1-q)}\sqrt{i_k(q^{2/m})}}. \tag{5.22}$$

Proof. From (5.1), the orthogonal polynomial sequence $\{P_n(x; w_m)\}$ satisfies the three term recurrence relation

$$xP_n(x; w_m) = P_{n+1}(x; w_m) + \frac{q^{-n}}{\sqrt{aq(1-q)}}\delta_n P_{n-1}(x; w_m), \tag{5.23}$$

where $\delta_n \to 1$ as $n \to \infty$. Set $\alpha_m := \sqrt[2m]{aq^{m-1}(1-q)}$. Then multiply the two sides of (5.23) by α_m^{n+1} and set

$$h_n(x) := \alpha_m^n P_n(x; w_m),$$

then we obtain

$$x\alpha_m h_n(x) = h_{n+1}(x) + q^{-2n/m}\delta_n h_n(x). \tag{5.24}$$

Replace x with x/α_m in the previous equation, then the sequence $H_n(x) := h_n(x/\alpha_m)$ satisfies the three term recurrence relation

$$xH_n(x) = H_{n+1}(x) + q^{-2n/m}\delta_n H_{n-1}(x), \quad \lim_{n\to\infty} \delta_n = 1. \tag{5.25}$$

Since

$$H_n\big(x_n(t)\big) = \alpha_m^n P_n\left(w_m; \frac{x_n(t; q^{2/m})}{\alpha_m}\right), \tag{5.26}$$

then applying Theorem 5.5 yields (5.21) and (5.22). □

6. The case $m = 2$

Theorem 6.1. *The nonlinear recursion* (1.25) *is satisfied by the recursion coefficients* $\{\beta_n\}$ *of the polynomials orthogonal with respect to* $w_2(x; q)$.

Proof. A proof is already given in [16] but we shall give a new proof of this result for completeness. In [16] it was proved that

$$\begin{aligned} D_n(x) &= aq^{n+1}\beta_n x, \\ C_n(x) &= aq^{n+1}\big[x^2 + \beta_n + q^2\beta_{n+1} + \big(1-q^2\big)c_{n+1,2}\big]. \end{aligned} \tag{6.1}$$

Now apply (1.16). The coefficients of x^2 in the result cancel and the constant term gives

$$\begin{aligned} 1 = {} & aq^{n+2}\beta_{n+1}\big[\beta_{n+1} + q^2\beta_{n+2} + \big(1-q^2\big)c_{n+2,2}\big] \\ & - aq^n\beta_n\big[\beta_{n-1} + q^2\beta_n + \big(1-q^2\big)c_{n,2}\big]. \end{aligned}$$

To find a first integral of the above equation use (2.4) to write it in the telescoping form

$$\begin{aligned} q^n/a = {} & q^{2n+2}\beta_{n+1}\big[q^2(\beta_{n+1} + \beta_{n+2}) + \beta_n + \big(1-q^2\big)c_{n+1,2}\big] \\ & - q^{2n}\beta_n\big[\beta_{n-1} + q^2(\beta_n + \beta_{n+1}) + \big(1-q^2\big)c_{n,2}\big], \end{aligned}$$

and the telescopy proves the theorem. □

Theorem 6.2. *The difference equation* (1.26) *has the discrete Painlevé property.*

Proof. Assume that $x_{N+1} = \epsilon$, where ϵ will eventually tend to 0. Then we get

$$\begin{aligned} q^2x_{N+2} &= x_{N-1} + \frac{(1-q^{N+1})q^{-2N-2}}{a\epsilon(1-q)} - \frac{(1-q^N)q^{-2N}}{a(1-q)x_N}, \\ q^2x_{N+3} &= x_N + \frac{1-q^{N+2}}{1-q^{N+1}}\epsilon - \frac{(1-q^{N+1})q^{-2N-2}}{a(1-q)\epsilon} + \mathcal{O}\big(\epsilon^2\big), \\ q^2x_{N+4} &= -q^{-2}\frac{(1-q^{N+4})}{1-q^{N+1}}\epsilon + \mathcal{O}\big(\epsilon^2\big), \\ q^2x_{N+5} &= \frac{q^{-2}(1-q^{N+3})}{1-q^{N+1}}\epsilon + q^{-2}x_{N-1} - \frac{(1-q^N)q^{-2n-2}}{a(1-q)x_N} + \mathcal{O}\big(\epsilon^2\big). \end{aligned}$$

Now we distinguish between two cases.

Case 1: $x_{N-1}x_N \neq \frac{(1-q^N)q^{-2N}}{a(1-q)}$. So we see that as $\epsilon \to 0$ the indeterminate form for x_{N+4} becomes 0, but it does not give a new singularity for x_{N+5} since x_{N+5} does not tend to zero as $\epsilon \to 0$.

Case 2: $x_{N-1}x_N = \frac{(1-q^N)q^{-2N}}{a(1-q)}$. We shall continue computing few more x_{N+j}'s. Indeed, we have

$$\begin{aligned}
q^2 x_{N+6} &= q^{-2}x_N + q^{-2}\frac{(1-q^{N+2})}{(1-q^{N+1})}\epsilon + \frac{(1-q^{N+1})(1-q^{N+5})}{(1-q)(1-q^{N+3})}\frac{q^{-2N-6}}{a\epsilon} + \mathcal{O}(\epsilon^2),\\
q^2 x_{N+7} &= -\frac{q^{N-1}(1-q)(1-q^2)}{(1-q^{N+1})(1-q^{N+5})}\epsilon - \frac{[N+5]_q[N+1]_q q^{-2N-6}}{a[N+3]_q\epsilon} + \mathcal{O}(\epsilon^2),\\
q^2 x_{N+8} &= -q^{-6}\frac{(1-q^{N+3})(1-q^{N+8})}{(1-q^{N+1})(1-q^{N+5})}\epsilon + \mathcal{O}(\epsilon^2),\\
q^2 x_{N+9} &= \frac{q^{-4}(1-q^{N+2})}{1-q^{N+1}}\epsilon + \frac{(1-q^{N+7})(1-q^{N+3})}{(1-q^{N+1})(1-q^{N+5})}q^{-6}\epsilon + q^{-4}x_N + \mathcal{O}(\epsilon^2). \qquad \square
\end{aligned}$$

Remark 6.3. Professor Y. Ohta has kindly pointed out that the third order difference equation (1.26) can be integrated to a second order equation. To see this multiply (1.26) by $1 - a(1-q)q^{2n+2}x_n x_{n+1}$ produces a telescoping series which leads to

$$q^2 x_{n+1} + q^{n+1}x_n + x_{n-1} = \frac{1-q^n}{1-q}\frac{q^{-2n}}{ax_n} + a(1-q)q^{2n+2}x_{n-1}x_n x_{n+1} + C, \tag{6.2}$$

for a constant C. The initial values of β_n make $C = 0$ and we are led to the second order equation

$$q^2 x_{n+1} + q^{n+1}x_n + x_{n-1} = \frac{1-q^n}{1-q}\frac{q^{-2n}}{ax_n} + a(1-q)q^{2n+2}x_{n-1}x_n x_{n+1}, \quad n > 1. \tag{6.3}$$

In (6.3) we let $y_n = q^n x_n\sqrt{aq(1-q)}$ and arrive at

$$(1 - y_n y_{n+1})(1 - y_n y_{n-1}) = q^n\big(1 + y_n^2\big). \tag{6.4}$$

Identity (6.4) is derived in [16] by a different approach. This is a special case of the q-discrete from of P_{34} equation [32], Eq. (6.3)

$$(y_{n+1}y_n - 1)(y_{n-1}y_n - 1) = q^n(y_n - c)(y_n - 1/c), \tag{6.5}$$

with $c = i$.

It is worth noting that (1.26) is a special case of the q-deformed discrete Painlevé I

$$\beta q^{-n-1}\big(x_{n+2} - q^2 x_{n-1}\big) = q\frac{(1-q^{n+1})/(1-q) + \alpha q^{n+1}}{x_{n+1}} - \frac{(1-q^n)/(1-q) + \alpha q^n}{x_n}. \tag{6.6}$$

Indeed (1.26) is (6.6) when q is replaced by q^{-1} and then we substitute with the special values $\alpha = 0$, $\beta = aq^2$. This equation was mentioned by Nijhoff in [28] who proved its integrability and introduced its corresponding Lax pairs. The authors of [18] proved that (6.5) is the integrated form of (6.6) where

$$\xi_n = \frac{q^{-n}}{\mu}, \qquad y_n = \lambda\xi_n x_n, \qquad \lambda = (1-q)\beta\mu, \qquad \mu = \alpha(1-q) - 1,$$

and the constant c plays the rule of the integration constant.

Remark 6.4. Lew and Quarles gave sufficient conditions that guarantee the uniqueness of positive solutions of (1.28). One of these condition is that $\lim_{n\to\infty} c_{n+1}/c_n > 0$. This confirms that (1.27) has a unique positive solution. Another uniqueness proof for (1.27) is given by Névai in [27]. The uniqueness question of positive solution of (1.29) and similar ones, cf. [3] has not been answered yet. The authors of [3] introduced a stable method for computing the recurrence coefficients of the difference equations

$$q^{n-\alpha}\big(-cy_ny_{n+1}+q^\alpha\big)\big(-cy_ny_{n-1}+q^\alpha\big)=\begin{cases}(q^\alpha-y_n)(q^\alpha-cy_n)q^{-\alpha}, & \text{for even } n,\\ (1-y_n)(1-cy_n), & \text{for odd } n,\end{cases} \tag{6.7}$$

where $c \leqslant 0$, $\alpha > -1$. Eq. (6.4) is slightly different from (6.7) but their stable method is still valid for it.

7. Solutions of (1.29)–(1.30)

We first consider the cases when (1.30) has constant solutions.

Theorem 7.1. *The constant sequence $x_n = X$, $n = 1, 2, \ldots,$ satisfies* (1.30) *if and only if $c_{n+1} = c_1 + n(q^2 - 1)X^2$, and c_1 is arbitrary. On the other hand $x_n = X$, $n = 1, 2, \ldots,$ satisfies* (1.29) *if and only if*

$$c_1 = 2q^2X^2 \quad \textit{and} \quad c_{n+1} = \big[2q^2 + 1 + n\big(q^2 - 1\big)\big]X^2. \tag{7.1}$$

Proof. The proof follows easily from Eqs. (1.29)–(1.30). □

We next consider the case when $c_n = A$, and A is a positive constant. The initial conditions in (1.29) and (1.30) are too restrictive so we shall only consider the equations

$$\frac{A}{x_n} = q^2\sum_{k=1}^{n+1} x_k - \sum_{k=1}^{n-2} x_k, \quad n > 2, \tag{7.2}$$

$$q^2x_{n+2} = x_{n-1} + \frac{A}{x_{n+1}} - \frac{A}{x_n}, \quad n > 1. \tag{7.3}$$

Theorem 7.2. *If the difference equation* (7.2) *or* (7.3) *has a periodic solution $\{x_k\colon 1 \leqslant k\}$ of period p, $p > 2$ then $\sum_{k=1}^{p} x_k = 0$. If in addition to Eq.* (7.3) *we assume that*

$$q^2x_3 = \frac{A}{x_2} - \frac{A}{x_1}, \tag{7.4}$$

then the resulting problem has no finite periodic solutions.

Proof. Assume (7.2) has a periodic solution of period p. Then

$$0 = \frac{A}{x_{n+p}} - \frac{A}{x_n} = q^2\sum_{k=n+2}^{n+p+1} x_k - \sum_{k=n-1}^{n+p-2} x_k = \big(q^2-1\big)\sum_{k=1}^{p} x_k.$$

Hence $\sum_{k=1}^{p} x_k = 0$. Similarly for (7.3), just sum the equations for $k \leqslant n \leqslant k+p-1$ and the same result follows. Under the additional assumption (7.4) we find from (7.3) that $A = q^2x_2x_2x_3/(x_1 - x_3)$. Now Eq. (7.3) implies

$$q^2 x_{n+2} = x_{n-1} + \frac{q^2 x_1 x_2 x_3}{x_n x_{n+1}} \frac{x_n - x_{n+1}}{x_1 - x_2}, \quad n > 1.$$

The choice $n = kp + 1$ in the above equation gives $x_p = 0$, so at least one element of the sequence becomes infinite. Proof is complete. □

Rewrite (1.29) in the form

$$\frac{A}{x_1} = q^2[x_1 + x_2], \qquad \frac{A}{x_n} = q^2 \sum_{k=1}^{n+1} x_k - \sum_{k=1}^{n-2} x_k, \quad n > 1, \tag{7.5}$$

with the empty sum defined as zero. On the other hand Eq. (1.30) becomes

$$q^2 x_3 = \frac{A}{x_2} - \frac{A}{x_1}, \qquad q^2 x_{n+2} = x_{n-1} + \frac{A}{x_{n+1}} - \frac{A}{x_n}, \quad n > 1. \tag{7.6}$$

Theorem 7.3. *Eq.* (7.5) *or* (7.6) *has a periodic solution* $\{x_k\colon\ 1 \leqslant k\}$ *of period* p, $p > 2$ *then* $\sum_{k=1}^{p} x_k = 0$, *provided that the initial condition* (1.31) *holds.*

Proof. Assume (7.5) has a periodic solution of period p. Then

$$0 = \frac{A}{x_{n+p}} - \frac{A}{x_n} = q^2 \sum_{k=n+2}^{n+p+1} x_k - \sum_{k=n-1}^{n+p-2} x_k = (q^2 - 1) \sum_{k=1}^{p} x_k.$$

Hence $\sum_{k=1}^{p} x_k = 0$ and the proof is complete in the case (7.5). The equivalence of Eqs. (7.5) and (7.6) under the assumption (1.31) completes the proof. □

A direct calculation easily shows that neither Eq. (7.5) nor (7.6) has any periodic solutions. There are no periodic solutions of period 3 either. To see this first evaluate A as $q^2 x_1 x_2 x_3/(x_1 - x_2)$ from the first equation in (7.6). With this value of A let $n = 4$ in the second equation in (7.6). The result is

$$(q^2 - 1) x_3 = A \frac{x_4 - x_5}{x_4 x_5} = q^2 x_3,$$

which is not possible.

8. Concluding remarks

This section includes some general remarks. It turned out for the authors the problem of considering more general weights than $w_m(x;q)$. By general here we mean class of weights which tend to $\exp(-P(x))$ and $P(x)$ as a polynomial when q tend to 1 from the left. As an example we shall consider in this section the weight function

$$w(x;a,b) = C \prod_{n=0}^{\infty} \frac{1}{[1 + a(1-q)q^{2n+1}x^2 + b(1-q)q^{4n+3}x^4]}, \tag{8.1}$$

where a and b are positive constants, $x \in \mathbb{R}$, and C is a normalization constant to make $\int_{\mathbb{R}} w(x; a, b)\,dx = 1$.

A calculation gives

$$v(x) = ax + bx^3. \tag{8.2}$$

Using the definitions of $C_n(x)$ and $D_n(x)$ in (1.9)–(1.10) we find that

$$\begin{gathered} D_n(x) = bq^{n+1}\beta_n x, \\ C_n(x) = aq^{n+1} + bq^{n+1}\left[x^2 + \beta_n + q^2\beta_{n+1} + \left(1-q^2\right)c_{n,2}\right]. \end{gathered} \tag{8.3}$$

The moments μ_n are given by

$$\mu_{2n} = \int\limits_{\mathbb{R}} x^{2n} w(x; a, b)\, dx, \qquad \mu_{2n+1} = 0. \tag{8.4}$$

Theorem 8.1. *The β_n's associated with the weight $w(x, a, b)$ satisfy the nonlinear difference equation*

$$\frac{1-q^n}{1-q}\frac{q^{-2n}}{x_n} = a + bq^2[x_n + x_{n+1}] + bx_{n-1} - b\left(1-q^2\right)\sum_{k=1}^{n-1} x_k, \tag{8.5}$$

for $n = 2, 3, \ldots,$ and the initial conditions

$$\beta_0 = 0, \qquad \beta_1 = \mu_2, \qquad bq^2(\beta_2 + \beta_1) = a + q^{-2}/\beta_1. \tag{8.6}$$

Moreover the β_n's also satisfy the third order difference equation

$$q^2 x_{n+2} = x_{n-1} + \frac{1-q^{n+1}}{1-q}\frac{q^{-2n-2}}{bx_{n+1}} - \frac{1-q^n}{1-q}\frac{q^{-2n}}{bx_n}, \tag{8.7}$$

for $n = 2, 3, \ldots.$

Proof. Applying (1.16) with $C_n(x)$ and $D_n(x)$ as in (8.3) we find that the x^2 terms cancel but the constant term implies

$$\begin{aligned} q^n = {} & bq^{2n+2}\beta_{n+1}\left[\beta_{n+1} + q^2\beta_{n+2} + \left(1-q^2\right)c_{n+2,2}\right] \\ & - b\beta_n q^{2n}\left[q^2\beta_n + \beta_{n-1} + \left(1-q^2\right)c_{n,2}\right] + aq^{2n+2}\beta_{n+1} - aq^{2n}\beta_n, \end{aligned}$$

which in view of (2.4) can be written in the telescoping form

$$\begin{aligned} q^n = {} & bq^{2n+2}\beta_{n+1}\left[q^2\beta_{n+1} + q^2\beta_{n+2} + \beta_n + \left(1-q^2\right)c_{n+1,2}\right] \\ & - b\beta_n q^{2n}\left[q^2\beta_n + +q^2\beta_{n+1} + \beta_{n-1} + \left(1-q^2\right)c_{n,2}\right] + aq^{2n+2}\beta_{n+1} - aq^{2n}\beta_n, \end{aligned}$$

and the results follow. □

We can reduce the third order difference equation (8.7) to the nonhomogeneous second order difference equation

$$q^2 x_{n+1} + q^{n+1}x_n + x_{n-1} = \frac{1-q^n}{1-q}\frac{q^{-2n}}{bx_n} + b(1-q)q^{2n+2}x_{n-1}x_n x_{n+1} + a/b. \tag{8.8}$$

The proof consists of multiplying Eq. (1.26) by $1 - b(1-q)q^{2n+2}x_n x_{n+1}$ to produce a telescoping series which leads to

$$q^2 x_{n+1} + q^{n+1}x_n + x_{n-1} = \frac{1-q^n}{1-q}\frac{q^{-2n}}{bx_n} + b(1-q)q^{2n+2}x_{n-1}x_n x_{n+1} + C, \tag{8.9}$$

for a constant C. The constant C can be evaluated from the initial conditions (8.6) and is found to be equal to a/b. Similar to (8.8) if we substitute with $y_n = q^n x_n\sqrt{bq(1-q)}$, we get the q-discrete form P_{34} Eq. (6.5) with

$$c := \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4}\right), \quad \delta = a\sqrt{\frac{1-q}{bq}}.$$

To the best we know, there are still several open questions in approximation theory concerning the weights $w_m(x;q)$. We exposed to one of them which is the weighted polynomial inequality we gave in Section 4. We shall prove in the following that the corresponding Bernstein's approximation problem has a positive solution where the Bernstein's approximation problem is described as:

Let $W : \mathbb{R} \to [0,1]$ be measurable. When is it true that for every function $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} (fW)(x) = 0,$$

there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \left\| (f - P_n)W \right\|_{L_\infty(\mathbb{R})} = 0?$$

If it is true then we say that the Bernstein's approximation problem has a positive solution. See Lubinsky survey of weighted polynomial approximation with exponential weights [21] and the reference within it.

We shall use the following theorem of Pollard [29], see also [21, p. 4].

Theorem 8.2. *Let $W : \mathbb{R} \to (0,1]$ be continuous and satisfy*

$$\lim_{|x|\to\infty} x^n W(x) = 0, \quad n = 0,1,2,\ldots.$$

There is a positive answer to Bernstein's problem if and only if both

$$\int_{-\infty}^{\infty} \frac{-\log W(t)}{1+t^2}\,dt = \infty \tag{8.10}$$

and there exists a sequence of polynomials $\{P_n\}$ such that for each x,

$$\lim_{n\to\infty} P_n(x)W(x) = 1, \tag{8.11}$$

while

$$\sup_{n\geqslant 1} \|P_n W\|_{L_\infty(\mathbb{R})} < \infty. \tag{8.12}$$

In this section, we shall prove that for the weights $w_m(x;q)$ defined in (1.23) the Bernstein's approximation problem has a positive solution.

Theorem 8.3. *Let $w_m(x;q)$ be the weights defined in (1.23). Then the corresponding Bernstein's approximation has a positive solution.*

Proof. First we shall prove (8.10). One can verify by using the calculus of residue that

$$\int_{-\infty}^{\infty} \frac{-\log w_m(t;q)}{1+t^2}\,dt = \int_{-\infty}^{\infty} \frac{\sum_{j=0}^{\infty} \log(1+aq^{2m-1}(1-q)q^{2mj}t^{2m})}{1+t^2}\,dt = \infty, \tag{8.13}$$

$m = 1, 2, \ldots$. Define $P_n(x) = \prod_{k=1}^{n}(1+aq^{2m-1}(1-q)q^{2mk}x^{2m})$. Then $P_n(x)$ is a polynomial in x^{2m} of degree n.

Now

$$\lim_{n\to\infty} P_n(x)w_m(x;q) = \lim_{n\to\infty} \frac{1}{\prod_{k=n+1}^{\infty}(1+aq^{2m-1}(1-q)q^{2mk}x^{2m})} = 1,$$

and

$$\sup_{x\in\mathbb{R}}\{\left|P_n(x)w_m(x;q)\right|\} = \sup_{x\in\mathbb{R}} \frac{1}{\prod_{k=n+1}^{\infty}(1+aq^{2m-1}(1-q)q^{2mk}x^{2m})} = 1.$$

Hence the conditions (8.11) and (8.12) are satisfied and the theorem follows. □

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