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Numbers Generated by the Reciprocal of $e^x - x - 1$

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Abstract. In this paper we examine the polynomials $A_n(z)$ and the rational numbers $A_n = A_n(0)$ defined by means of

$$e^{xz}x^2(e^x - x - 1)^{-1} = 2 \sum_{n=0}^{\infty} A_n(z)x^n/n!.$$

We prove that the numbers A_n are related to the Stirling numbers and associated Stirling numbers of the second kind, and we show that this relationship appears to be a logical extension of a similar relationship involving Bernoulli and Stirling numbers. Other similarities between A_n and the Bernoulli numbers are pointed out. We also reexamine and extend previous results concerning A_n and $A_n(z)$. In particular, it has been conjectured that A_n has the same sign as $-\cos n\theta$, where $re^{i\theta}$ is the zero of $e^x - x - 1$ with smallest absolute value. We verify this for $1 \leqslant n \leqslant 14329$ and show that if the conjecture is not true for A_n, then $|\cos n\theta| < 10^{-(n-1)/5}$. We also show that $A_n(z)$ has no integer roots, and in the interval $[0, 1]$, $A_n(z)$ has either two or three real roots.

1. Introduction. Define the rational numbers $A_0, A_1, A_2, \ldots$ by means of

$$\left(\sum_{n=0}^{\infty} \frac{2x^n}{(n+2)!}\right)^{-1} = \frac{x^2/2}{e^x - x - 1} = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}. \tag{1.1}$$

This definition is apparently due to L. Carlitz [4], who raised the question of whether a theorem like the Staudt-Clausen theorem holds for the numbers A_n. Because of the obvious similarity of (1.1) to the definition of the Bernoulli numbers B_n, i.e.

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{1.2}$$

this seems to be a reasonable question. The writer [6] has shown, however, that evidently such a theorem does not hold: If p is any odd prime, then

$$p^m A_{m(p-2)}/[m(p-2)]! \equiv 2^m \pmod{p}, \tag{1.3}$$

which implies that arbitrarily large powers of p will divide the denominator of some A_n. However, for $n > 1$,

$$2A_n \equiv 1 \pmod{4}, \tag{1.4}$$

so the denominator of A_n, for $n > 1$, is even and not divisible by 4. This last property is also true of the Bernoulli numbers B_{2n}.

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