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CHOLESKY ALGORITHM MATRICES OF FIBONACCI TYPE AND PROPERTIES OF GENERALIZED SEQUENCES*

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1. Introduction

Many properties of the generalized Fibonacci numbers U_n and the generalized Lucas numbers V_n (e.g., see [3]-[5], [8]-[10], [12], [14], [15]) have been obtained by altering their recurrence relation and/or the initial conditions. Here we offer a somewhat new matrix approach for developing properties of this nature.

The aim of this paper is to use the 2-by-2 matrix M_k determined by the Cholesky LR decomposition algorithm to establish a large number of identities involving U_n and V_n . Some of these identities, most of which we believe to be new, extend the results obtained in [6] and elsewhere.

Particular examples of the use of the matrix M_k , including summation of some finite series involving U_n and V_n , are exhibited. A method for evaluating some infinite series is then presented which is based on the use of a closed form expression for certain functions of the matrix xM_k^n .

2. Generalities

In this section some definitions are given and some results are established which will be used throughout the paper.

2.1. The Numbers U_n and the Matrix M

Letting m be a natural number, we define (see [4]) the integers $U_n(m)$ (or more simply U_n if there is no fear of confusion) by the second-order recurrence relation

$$(2.1) \quad U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0, \quad U_1 = 1 \quad \forall m.$$

The first few numbers of the sequence $\{U_n\}$ are:

$$\begin{array}{cccccccc} U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & \dots \\ 0 & 1 & m & m^2+1 & m^3+2m & m^4+3m^2+1 & m^5+4m^3+3m & \dots \end{array}$$

We recall [4] that the numbers U_n can be expressed in the closed form (Binet form)

$$(2.2) \quad U_n = (\alpha_m^n - \beta_m^n) / \Delta_m,$$

where

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$$(2.3) \quad \begin{cases} \Delta_m = \sqrt{m^2 + 4} \\ \alpha_m = (m + \Delta_m)/2 \\ \beta_m = (m - \Delta_m)/2. \end{cases}$$

From (2.3) it can be noted that

$$(2.4) \quad \begin{cases} \alpha_m \beta_m = -1 \\ \alpha_m + \beta_m = m \\ \alpha_m - \beta_m = \Delta_m. \end{cases}$$

We also recall [4] that

$$(2.5) \quad U_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} m^{n-2j-1},$$

where $[\cdot]$ denotes the greatest integer function. Moreover, as we sometimes require negative-valued subscripts, from (2.2) and (2.4) we deduce that

$$(2.6) \quad U_{-n} = (-1)^{n+1} U_n.$$

From (2.1) it can be noted that the numbers $U_n(1)$ are the Fibonacci numbers F_n and the numbers $U_n(2)$ are the Pell numbers P_n .

Analogously, the numbers $V_n(m)$ (or more simply V_n) can be defined [4] as

$$(2.7) \quad V_n = \alpha_m^n - \beta_m^n = U_{n-1} + U_{n+1}.$$

The first few numbers of the sequence $\{V_n\}$ are:

$$\begin{array}{cccccccc} V_0 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & \dots \\ 2 & m & m^2 + 2 & m^3 + 3m & m^4 + 4m^2 + 2 & m^5 + 5m^3 + 5m & m^6 + 6m^4 + 9m^2 + 2 & \dots \end{array}$$

These numbers satisfy the recurrence relation

$$(2.8) \quad V_{n+2} = mV_{n+1} + V_n; \quad V_0 = 2, \quad V_1 = m \quad \forall m.$$

From (2.7) and (2.4) we have

$$(2.9) \quad V_{-n} = (-1)^n V_n,$$

and it is apparent that the numbers $V_n(1)$ are the Lucas numbers L_n while the numbers $V_n(2)$ are the Pell-Lucas numbers Q_n [11].

Definitions (2.1) and (2.8) can be extended to an arbitrary generating parameter, leading in particular to the double-ended complex sequences $\{U_n(z)\}_{-\infty}^{\infty}$ and $\{V_n(z)\}_{-\infty}^{\infty}$. Later we shall make use of the numbers $U_n(z)$.

Let us now consider the 2-by-2 symmetric matrix

$$(2.10) \quad M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

which is governed by the parameter m and of which the eigenvalues are α_m and β_m . For n a nonnegative integer, it can be proved by induction [6] that

$$(2.11) \quad M^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix}.$$

Also, the matrix M can obviously be extended to the case where the parameter m is arbitrary (e.g., complex).

2.2 A Cholesky Decomposition of the Matrix M : The Matrix M_k

Let us put

$$(2.12) \quad M_1 = M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

and decompose M_1 as

$$(2.13) \quad M_1 = T_1 T_1' = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where T_1 is a lower triangular matrix and the superscript ($'$) denotes transposition, so that T_1' is an upper triangular matrix. The values of the entries a_1 , b_1 , and c_1 of T_1 can be readily obtained by applying the usual matrix multiplication rule on the right-hand side of the matrix equation (2.13). In fact, the system

$$(2.14) \quad \begin{cases} a_1^2 = m \\ a_1 c_1 = 1 \\ b_1^2 + c_1^2 = 0 \end{cases}$$

can be written, whose solution is

$$(2.15) \quad \begin{cases} a_1 = \pm\sqrt{m} \\ c_1 = 1/a_1 \\ b_1 = \pm ic_1, \end{cases}$$

where $i = \sqrt{-1}$.

Any of these four solutions leads to a *Cholesky LR decomposition* [17] of the symmetric matrix M_1 .

On the other hand, it is known [7] that a lower triangular matrix and an upper triangular matrix do not commute, so that the reverse product $T_1' T_1$ leads to the symmetric matrix M_2 which is similar to but different from M_1 . If we take $b_1 = +ic_1$ [cf. (2.15)], we have

$$(2.16) \quad M_2 = \frac{1}{m} \begin{bmatrix} m^2 + 1 & i \\ i & -1 \end{bmatrix},$$

while, if we take $b_1 = -ic_1$, the off diagonal entries of M_2 become negative.

In turn, M_2 can be decomposed in a similar way, thus getting

$$M_2 = T_2 T_2' = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{cases} a_2 = \pm\sqrt{(m^2 + 1)/m} \\ c_2 = 1/a_2 \\ b_2 = \pm ic_2. \end{cases}$$

The reverse product $T_2' T_2$ leads to a matrix M_3 with sign of the off diagonal entries depending on whether $b_2 = +ic_2$ or $-ic_2$ has been considered.

If we repeat such a procedure *ad infinitum*, we obtain the set $\{M_k\}_1^\infty$ of 2-by-2 symmetric matrices

$$(2.17) \quad M_k = \frac{1}{U_k} \begin{bmatrix} U_{k+1} & i^{k-1} \\ i^{k-1} & -U_{k-1} \end{bmatrix} \quad (k \geq 1).$$

Because of the ambiguity of signs that arises in the Cholesky factorization, (2.17) is not the only possible result of k applications of the LR algorithm to M . However, the only other possible result differs from that shown in (2.17) only in the sign of the off diagonal entries. From here on, we will consider only the sequence $\{M_k\}$ given by equation (2.17).

Since the matrices M_k are similar, their eigenvalues coincide. M_k tends to a diagonal matrix containing these eigenvalues (namely, α_m and β_m) as k tends to infinity.

To establish the general validity of (2.17), consider the Cholesky decomposition

$$M_k = \frac{1}{U_k U_{k+1}} \begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix} \begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}$$

where *Simson's formula*

$$(2.18) \quad U_{k+1}U_{k-1} - U_k^2 = (-1)^k$$

has been invoked. Simson's formula may be quite readily established by using the Binet form (2.2) for U_n and the properties (2.4) of α_m and β_m . On the other hand, from (2.11) and (2.10), it is seen that

$$U_{k+1}U_{k-1} - U_k^2 = \det(M^k) = (\det M)^k = (-1)^k.$$

Reversing the order of multiplication, we get

$$\frac{1}{U_k U_{k+1}} \begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix} \begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix} = \frac{1}{U_{k+1}} \begin{bmatrix} U_{k+2} & i^k \\ i^k & -U_k \end{bmatrix} \begin{matrix} \\ \text{[by (2.17)]} \end{matrix}$$

[by (2.18)]

Thus, if the matrix for M_k is valid, then so is the matrix for M_{k+1} .

For convenience, M_k may be called the *Cholesky algorithm matrix of Fibonacci type* of order k .

Furthermore, if we apply the Cholesky algorithm to M^n [see (2.11)] rather than to M , we obtain

$$(2.19) \quad (M^n)_k = \frac{1}{U_k} \begin{bmatrix} U_{k+n} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^n U_{k-n} \end{bmatrix} = \frac{1}{U_k} \begin{bmatrix} U_{n+k} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^{k-1}U_{n-k} \end{bmatrix}.$$

Observe that

$$(2.20) \quad (M^n)_k = (M_k)^n = M_k^n.$$

Validation of this statement may be achieved through an inductive argument. Assume (2.20) is true for some value of n , say N . Thus,

$$(2.21) \quad (M_k)^N = (M^N)_k.$$

Then,

$$(M_k)^{N+1} = M_k(M_k)^N = M_k(M^N)_k = (M^{N+1})_k$$

[by (2.21)]

after a good deal of calculation, so that if (2.20) is true for N , it is also true for $N + 1$. In the calculations, it is necessary to derive certain identities among the U_n by using (2.2) and (2.4).

2.3 Functions of the Matrix xM_k^n

From the theory of functions of matrices [7], it is known that if f is a function defined on the spectrum of a 2-by-2 matrix $A = [a_{ij}]$ with distinct eigenvalues λ_1 and λ_2 , then

$$(2.22) \quad f(A) = X = [x_{ij}] = c_0I + c_1A,$$

where I is the 2-by-2 identity matrix and the coefficients c_0 and c_1 are given by the solution of the system

$$(2.23) \quad \begin{cases} c_0 + c_1\lambda_1 = f(\lambda_1) \\ c_0 + c_1\lambda_2 = f(\lambda_2). \end{cases}$$

Solving (2.23) and using (2.22), after some manipulations we obtain

$$(2.24) \quad x_{11} = [(\alpha_{11} - \lambda_1)f(\lambda_2) - (\alpha_{11} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

$$(2.25) \quad x_{12} = \alpha_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

$$(2.26) \quad x_{21} = \alpha_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

$$(2.27) \quad x_{22} = [(\alpha_{22} - \lambda_1)f(\lambda_2) - (\alpha_{22} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1).$$

For x an arbitrary quantity, let us consider the matrix xM_k^n having eigenvalues

$$(2.28) \quad \begin{cases} \lambda_1 = x\alpha_m^n \\ \lambda_2 = x\beta_m^n \end{cases}$$

and let us find closed form expressions for the entries y_{ij} of

$$Y = [y_{ij}] = f(xM_k^n).$$

by (2.24)-(2.27), after some tedious manipulations involving the use of certain identities easily derivable from (2.2) and (2.3), we get

$$(2.29) \quad y_{11} = [\alpha_m^k f(x\alpha_m^n) - \beta_m^k f(x\beta_m^n)]/(\Delta_m U_k)$$

$$(2.30) \quad y_{12} = y_{21} = i^{k-1}[f(x\alpha_m^n) - f(x\beta_m^n)]/(\Delta_m U_k)$$

$$(2.31) \quad y_{22} = [\alpha_m^k f(x\beta_m^n) - \beta_m^k f(x\alpha_m^n)]/(\Delta_m U_k).$$

As an illustrative example, let f be the inverse function. Then, from (2.29)-(2.31) we obtain

$$(2.32) \quad (xM_k^n)^{-1} = \frac{(-1)^n}{xU_k} \begin{bmatrix} (-1)^{k-1}U_{n-k} & -i^{k-1}U_n \\ -i^{k-1}U_n & U_{n+k} \end{bmatrix}$$

$$(2.33) \quad = \frac{1}{x} M_k^{-n} \quad [\text{using (2.19) and (2.6)}].$$

3. Some Applications of the Matrix M_k

In this and later sections some identities involving U_n and V_n are worked out as illustrations of the use of our Cholesky algorithm matrix of Fibonacci type M_k .

Example 1: From (2.19) we can write

$$(3.1) \quad M_k^k = \frac{1}{U_k} \begin{bmatrix} U_{2k} & i^{k-1}U_k \\ i^{k-1}U_k & 0 \end{bmatrix} = \begin{bmatrix} V_k & i^{k-1} \\ i^{k-1} & 0 \end{bmatrix} = R_k = [r_{ij}] (= [r_{ij}^{(1)}]),$$

whence

$$(3.2) \quad M_k^{nk} = R_k^n = [r_{ij}^{(n)}].$$

Thus, $r_{11}^{(n)} = V_k$, $r_{12}^{(n)} = r_{21}^{(n)} = i^{k-1}$, $r_{22}^{(n)} = 0$. Take $r_{11}^{(0)} = 1$. By induction on n , with the aid of Pascal's formula for binomial coefficients, it can be proved that

$$(3.3) \quad \begin{cases} r_{11}^{(n)} = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j} \\ r_{12}^{(n)} = r_{21}^{(n)} = i^{k-1} r_{11}^{(n-1)} \\ r_{22}^{(n)} = (-1)^{k-1} r_{11}^{(n-2)} \quad (n \geq 2). \end{cases}$$

On the other hand, the matrix M_k^{nk} can be expressed also [cf. (2.19)] as

$$(3.4) \quad M_k^{nk} = \frac{1}{U_k} \begin{bmatrix} U_{k(n+1)} & i^{k-1} U_{nk} \\ i^{k-1} U_{nk} & (-1)^{k-1} U_{k(n-1)} \end{bmatrix}.$$

Equating the upper left entries on the right-hand sides of (3.2) and (3.4), by (3.3) we obtain the identity

$$(3.5) \quad U_{k(n+1)}/U_k = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j},$$

i.e., $U_k | U_{k(n+1)}$, as we would expect.

Furthermore, from (3.1) we can write

$$(3.6) \quad [(-i)^{k-1} M_k^{k,n}]^n = \begin{bmatrix} (-i)^{k-1} V_k & 1 \\ 1 & 0 \end{bmatrix}^n = Z_k^n = [z_{ij}^{(n)}],$$

where $Z_k = [z_{ij}]$ is an extended M matrix depending on the complex parameter

$$(3.7) \quad z = (-i)^{k-1} V_k(m).$$

From (2.11) we have

$$(3.8) \quad z_{12}^{(n)} = U_n(z),$$

and by equating $z_{12}^{(n)}$ and the upper right entry of $[(-i)^{k-1} M_k^{k,n}]^n$ obtained by (3.4) we can write

$$(3.9) \quad (-i)^{n(k-1)} i^{k-1} U_{nk}(m)/U_k(m) = (-1)^{n(k-1)} i^{(n+1)(k-1)} U_{nk}(m)/U_k(m) = U_n(z).$$

From (2.5) and (3.7) it can be verified that

$$(3.10) \quad U_n(z) = \begin{cases} U_n(V_k(m)) & (k \text{ odd}, n \text{ odd}) \\ (-1)^{(k-1)/2} U_n(V_k(m)) & (k \text{ odd}, n \text{ even}). \end{cases}$$

Therefore, from (3.9) and (3.10) we obtain the noteworthy identity

$$(3.11) \quad U_{nk}(m)/U_k(m) = U_n(V_k(m)) \quad (k \text{ odd})$$

which connects numbers defined by (2.1) having different generating parameters.

For instance,

$$\begin{aligned} (m^3 + 3m)^2 + 1 &= m^6 + 6m^4 + 9m^2 + 1 = (m^8 + 7m^6 + 15m^4 + 10m^2 + 1) / (m^2 + 1) \\ &= U_3(V_3(m)) = U_9(m) / U_3(m) \end{aligned}$$

which simultaneously verifies (3.5) and (3.11).

Example 2: Following [2], from (2.19) we can write either

$$(3.12) \quad M_k^r M_k^s = \frac{1}{U_k^2} \begin{bmatrix} U_{r+k}U_{s+k} - (-1)^k U_r U_s & i^{k-1} [U_{r+k}U_s - (-1)^k U_r U_{s-k}] \\ i^{k-1} [U_{s+k}U_r - (-1)^k U_s U_{r-k}] & U_{r-k}U_{s-k} - (-1)^k U_r U_s \end{bmatrix}$$

or

$$(3.13) \quad M_k^{r+s} = \frac{1}{U_k} \begin{bmatrix} U_{r+s+k} & i^{k-1} U_{r+s} \\ i^{k-1} U_{r+s} & (-1)^{k-1} U_{r+s-k} \end{bmatrix}.$$

By equating the upper right entries on the right-hand sides of (3.12) and (3.13) we obtain

$$(3.14) \quad \begin{aligned} U_k U_{r+s} &= U_{r+k}U_s - (-1)^k U_r U_{s-k} \\ &= U_{s+k}U_r - (-1)^k U_s U_{r-k} \text{ also.} \end{aligned}$$

4. Evaluation of Some Finite Series

In this section the sums of certain finite series involving U_n and V_n are found on the basis of some properties of the Fibonacci-type Cholesky algorithm matrix M_k .

It is readily seen from (2.17) and (2.19), with the aid of (2.1), that

$$(4.1) \quad M_k^2 = mM_k + I,$$

whence

$$(4.2) \quad M_k^{-1} = M_k - mI.$$

Moreover, using the identity

$$(4.3) \quad V_n U_p - U_{n+p} = (-1)^{p-1} U_{n-p},$$

which can be easily proved using (2.2) and (2.7), we can verify that

$$(4.4) \quad (xM_k^n - I)^{-1} = \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}$$

where x is an arbitrary quantity subject by (2.28) to the restrictions

$$(4.5) \quad x \neq \begin{cases} 1/\alpha_m^n \\ 1/\beta_m^n. \end{cases}$$

A) From (4.1) we can write

$$(M_k^2 - I)^n = (mM_k)^n$$

and, therefore,

$$\sum_{j=0}^n (-1)^j \binom{n}{j} M_k^{2j} = m^n M_k^n,$$

whence, by (2.19), we obtain a set of identities which can be summarized by

$$(4.6) \quad \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} U_{2j+s} = m^n U_{n+s},$$

where n is a nonnegative integer and s an arbitrary integer. Replacing s by $s \pm 1$ in (4.6) and combining the results obtained, from (2.7) we have

$$(4.7) \quad \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} V_{2j+s} = m^n V_{n+s}.$$

Furthermore, following [13], from (4.1) we can write

$$(4.8) \quad (mM_k + I)^n M_k^s = M_k^{2n+s}.$$

Equating appropriate entries on both sides of (4.8), with the aid of (2.19), we obtain

$$(4.9) \quad \sum_{j=0}^n \binom{n}{j} m^j U_{j+s} = U_{2n+s},$$

whence, replacing s by $s \pm 1$ as earlier, we get

$$(4.10) \quad \sum_{j=0}^n \binom{n}{j} m^j V_{j+s} = V_{2n+s}.$$

B) From (4.2) we can write

$$(M_k - mI)^n = (M_k^n)^{-1},$$

whence, by (2.19) and (2.32), after some manipulations, we obtain a set of identities which can be summarized by

$$(4.11) \quad \sum_{j=0}^n (-1)^j m^{n-j} \binom{n}{j} U_{j+s} = (-1)^{s-1} U_{n-s}.$$

C) Finally, let us consider the identity

$$(4.12) \quad (xA^n - I) \sum_{j=0}^h x^j A^{nj} = x^{h+1} A^{n(h+1)} - I,$$

which holds for any square matrix A . From (4.12) and (4.4) we can write, for the Cholesky algorithm matrix M_k of Fibonacci type,

$$\begin{aligned} (4.13) \quad \sum_{j=0}^h x^j M_k^{nj} &= (xM_k^n - I)^{-1} (x^{h+1} M_k^{n(h+1)} - I) \\ &= \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1} (x^{h+1} M_k^{n(h+1)} - I) \\ &= \frac{x^{h+2} M_k^{n(h+2)} - xM_k^n - x^{h+1} (V_n x - 1) M_k^{n(h+1)} + (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}. \end{aligned}$$

After some manipulations involving the use of (4.3), from (4.13) and (2.19) we obtain a set of identities which can be summarized as

$$(4.14) \quad \sum_{j=0}^h x^j U_{nj+s} = \frac{(-1)^{n-1} x^{h+2} U_{nh+s} + x^{h+1} U_{n(h+1)+s} - (-1)^s x U_{n-s} - U_s}{(-1)^{n-1} x^2 + V_n x - 1},$$

where n is a nonnegative integer and s is an arbitrary integer. Replacing s by $s \pm 1$ in (4.14), by (2.7) we can derive

$$(4.15) \quad \sum_{j=0}^h x^j V_{nj+s} = \frac{(-1)^{n-1} x^{h+2} V_{nh+s} + x^{h+1} V_{n(h+1)+s} + (-1)^s x V_{n-s} - V_s}{(-1)^{n-1} x^2 + V_n x - 1}.$$

We point out that (4.14) and (4.15) obviously hold under the restrictions (4.5).

5. Evaluation of Some Infinite Series

In this section a method for finding the sums of certain infinite series involving U_n and V_n is shown which is based on the use of functions of the matrix xM_k^n (see Section 2.3).

Under certain restrictions, some sums can be worked out by using the results established in Section 4 above. For example, if

$$(5.1) \quad -1/\alpha_m^n < x < 1/\alpha_m^n,$$

we can take the limit of both sides of (4.14) and (4.15) as h tends to infinity thus getting

$$(5.2) \quad \sum_{j=0}^{\infty} x^j U_{nj+s} = \frac{(-1)^{s-1} x U_{n-s} - U_s}{(-1)^{n-1} x^2 + V_n x - 1},$$

$$(5.3) \quad \sum_{j=0}^{\infty} x^j V_{nj+s} = \frac{(-1)^s x V_{n-s} - V_s}{(-1)^{n-1} x^2 + V_n x - 1}.$$

5.1 Use of Certain Functions of xM_k^n

Following [6], we consider the power series expansion of $\exp(xM_k^n)$ [7],

$$(5.4) \quad Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!}$$

and the closed form expressions of the entries y_{ij} of Y derivable from (2.29)-(2.31) by letting f be the exponential function. Equating y_{ij} and the corresponding entry of Y on the right-hand side of (5.4), from (2.19) we obtain the identities

$$(5.5) \quad \sum_{j=0}^{\infty} \frac{x^j U_{jn+k}}{j!} = [\alpha_m^k \exp(x\alpha_m^n) - \beta_m^k \exp(x\beta_m^n)]/\Delta_m,$$

$$(5.6) \quad \sum_{j=0}^{\infty} \frac{x^j U_{jn}}{j!} = [\exp(x\alpha_m^n) - \exp(x\beta_m^n)]/\Delta_m,$$

$$(5.7) \quad \sum_{j=0}^{\infty} \frac{x^j U_{jn-k}}{j!} = (-1)^{k-1} [\alpha_m^k \exp(x\beta_m^n) - \beta_m^k \exp(x\alpha_m^n)]/\Delta_m,$$

which, by using the identity $(-1)^{k-1} \alpha_m^{-k} = -\beta_m^k$ [see (2.4)], can be summarized as

$$(5.8) \quad \sum_{j=0}^{\infty} \frac{x^j U_{jn+s}}{j!} = [\alpha_m^s \exp(x\alpha_m^n) - \beta_m^s \exp(x\beta_m^n)]/\Delta_m,$$

where n is a nonnegative integer and s is an arbitrary integer.

From (5.8), (2.7), and (2.3) we can readily derive

$$(5.9) \quad \begin{aligned} \sum_{j=0}^{\infty} \frac{x^j V_{jn+s}}{j!} &= [\alpha_m^{s-1} \exp(x\alpha_m^n) (1 + \alpha_m^2) - \beta_m^{s-1} \exp(x\beta_m^n) (1 + \beta_m^2)]/\Delta_m \\ &= \alpha_m^{s-1} \exp(x\alpha_m^n) (\Delta_m + m)/2 - \beta_m^{s-1} \exp(x\beta_m^n) (\Delta_m - m)/2 \\ &= \alpha_m^s \exp(x\alpha_m^n) + \beta_m^s \exp(x\beta_m^n). \end{aligned}$$

By considering power series expansions [1], [16], [7] of other functions of the matrix xM_k^n , the above presented technique allows us to evaluate a very

large amount of infinite series involving numbers U_n and V_n . We confine ourselves to showing an example derived from the expansion of $\tan^{-1}y$ (see [1] and [7, p. 113]).

Under the restriction

$$(5.10) \quad -1/\alpha_m^n \leq x \leq 1/\alpha_m^n,$$

we have

$$(5.11) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j-1} U_{n(2j-1)+s}}{2j-1} = [\alpha_m^s \tan^{-1}(x\alpha_m^n) - \beta_m^s \tan^{-1}(x\beta_m^n)]/\Delta_m.$$

6. Conclusion

The identities established in this paper represent only a small sample of the possibilities available to us. We believe that the Cholesky decomposition matrix M_k is a useful tool for discovering many more identities. Further investigations into the properties of matrices of this type are warranted.

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